Ex-Ante Stable Lotteries

Jan Christoph Schlegel*
Faculty of Business and Economics, University of Lausanne, Switzerland
jschlege@unil.ch

Abstract

We study the allocation of indivisible objects (e.g. school seats) to agents by lotteries. Agents have preferences over different objects and have different priorities at different objects. The priorities can contain indifferences, some agents may have the same priority at some object. A lottery is ex-ante stable if there does not exist an agent-object pair such that we can increase the probability of matching this pair at the expense of agents, who have lower priority at the object, and of objects which are less preferred by the agent.

As a first result, we show that this fairness condition is very demanding: Only few agent-object pairs have a positive probability of being matched. The number of pairs in the support depends on how many indifferences in the priorities the lottery exploits. In the extreme case where no object is matched with positive probability to two equal priority agents, the lottery is almost degenerate. Otherwise, the size of the support is completely determined by the size of the lowest priority classes of which agents are matched to the respective objects. We interpret our result as an impossibility result. With ex-ante stability one cannot go much beyond randomly breaking ties and implementing a (deterministically) stable matching with respect to the broken ties.

As a second result, we derive a new characterization of the set of lotteries that can be decentralized by a pseudo-market with priority-specific pricing as introduced by He et al. (2015). These allocations coincide with the ex-ante stable lotteries that do not admit a strong stable improvement cycle.

JEL-classification: C78, D47
Keywords: Matching; School Choice; Lotteries; Ex-Ante Stability; Pseudo-Markets

*I thank Battal Doğan, Bettina Klaus, Panos Protopapas and participants of the Sixth World Congress of the Game Theory Society for useful comments on a previous version of the paper. I gratefully acknowledge financial support by the Swiss National Science Foundation (SNSF) under project 100018-150086.

1 Introduction

The assignment of students to schools (Abdulkadiroglu and Sönmez, 2003) is one of the major applications of matching theory. A school choice mechanism assigns students to schools taking into account the preferences of students and priorities of the students at the different schools. Thick priorities are a generic problem in school choice. Students are prioritized according to very coarse criteria (based e.g. on proximity, or having a sibling in the school) such that many students have the same priority for a seat at a school. One can sometimes not avoid to treat students differently ex-post even though they have the same priorities and preferences. However, ex-ante, some form of fairness can be restored by the use of lotteries. This has motivated researchers to study the problem of designing fair school choice lotteries.

A minimal ex-ante fairness requirement for random assignments under priorities is that the lottery should respect the priorities. One way of formalizing this requirement is the following: A student i has *ex-ante justified envy* if there is a school s where a lower priority student j has a positive probability of receiving a seat and i would rather have a seat in s than at some other school at which he has a positive probability of receiving a seat. In this case, it would be natural to eliminate the justified envy, i.e. changing the probability shares such that i has a higher chance of receiving a seat at school s at the expense of the lower ranked student j. *Ex-ante stability* requires that there is no ex-ante justified envy. In the school choice set-up, ex-ante stability has been introduced by Kesten and Ünver (2015). For the classical marriage model the condition was first considered by Roth et al. (1993). He et al. (2015) define an appealing class of mechanisms that implement ex-ante stable lotteries. These mechanisms generalize the pseudo-market mechanisms of Hylland and Zeckhauser (1979) by allowing for priority-specific pricing (agents with different priorities are offered different prices).

Even though ex-ante stability is, in a sense, a minimal ex-ante fairness requirement, it is demanding. In an environment with strict priorities (no ties) and where each school has one seat to allocate, it follows from an earlier result by Roth et al. (1993) that each student has a positive probability of receiving a seat at, at most, two schools. In other words, an ex-ante stable assignment is almost deterministic. We generalize this result to the more general set-up with quotas and ties. With strict priorities, we show that an ex-ante stable lottery is almost degenerate, since

- each student has a positive probability of receiving a seat at, at most, *two* distinct schools.

- For each school all but possibly *one* seat are assigned deterministically. For this one seat that is assigned by a lottery, two students have a positive probability of receiving it.

With ties in the priorities, ex-ante stability is naturally less demanding. However, ex-ante stability imposes a lot of structure on the lottery. We show that the size of the support of an ex-ante stable lottery (the number of pairs being matched with positive probability) is determined by the number of ties the lottery "uses" (i.e. how many agents who have equal priority at some object are matched with positive probability to that object). More precisely, we show that for each ex-ante stable lottery the size of the support is determined by the size of the "cut-off" priority classes: Here, cut-off priority classes are the lowest priority classes at an object, such that an agent of that priority class gets that object with positive probability.

Finally, we can relate our result to properties of the class of mechanisms defined by He et al. (2015). In their mechanism, lottery shares are allocated through a competitive market. Agents have a budget with which they can buy probability shares. For each object there is a cut-off priority class. All agents within this priority class face the same finite price for shares in the object type. All agents ranked below the cut-off cannot buy the object type, i.e. they face an infinite price, agents ranked above the cut-off can obtain shares in the object type for free. Since prices reflect the priorities, lotteries implemented by the pseudo-market mechanism are ex-ante stable. With our result, we can identify the cut-off priority classes that define the prices, as the cut-off priority classes that determine the size of the support. Moreover, we provide two results that can be interpreted as (constrained) welfare theorems for pseudo-market equilibria with priority-specific prizing: First, we show that the equilibrium random assignments do not admit strong stable improvement cycles (a strong version of a notion introduced by Erdil and Ergin (2008) for deterministic matchings). Second, we show that ex-ante stable random assignments that do not admit a strong stable improvement cycle are exactly those that can be decentralized by a price equilibrium with priority-specific pricing. Thus, we obtain a characterization of the class of equilibrium random assignments in terms of mild efficiency and fairness conditions.[1]

The proofs in this paper use the graph representation of assignment problems due to Balinski and Ratier (1997). As far as we know, this repre-

[1] He et al. (2015) provide another characterization in terms of ex-ante stability and the fairness requirement of "equal claims."

sentation has not been used so far in the study of lotteries.[2] We think that our results demonstrate the usefulness of this particular representation for the study of random assignments with priorities.

2 Model

There is a set of n agents N and a set of m object types M. A generic agent is denoted by i and a generic object type by j. Of each object type j, there is a finite number of copies $q_j \in \mathbb{N}$. We assume that there are as many objects as agents, $\sum_{j \in M} q_j = n$. Each agent i has strict preferences P_i over different types of objects. Each object type j has a strict priority ranking $\succ_j$ of agents. Later in Subsections 3.2 and 3.3 we will also consider the case where object types have indifferences in their priorities.

A **deterministic assignment** is a mapping $\mu : N \to M$ such that for each $j \in M$ we have $|\mu^{-1}(j)| = q_j$. A **random assignment** is a probability distribution over deterministic assignments. By the Birkhoff-von Neumann Theorem, each random assignment corresponds to a bi-stochastic matrix and vice versa each such matrix corresponds to a random assignment (see Kojima and Manea (2010) for a proof in the set-up that we consider.) Thus each random assignment is represented by a matrix $\Pi = (\pi_{ij}) \in \mathbb{R}^{N \times M}$ such that

$$0 \leq \pi_{ij} \leq 1, \quad \sum_{j \in M} \pi_{ij} = 1, \quad \sum_{i \in N} \pi_{ij} = q_j,$$

where π_{ij} is the probability that agent i is matched to an object of type j. The **support** of Π is the set of all non-zero entries of the matrix Π, i.e.

$$\text{supp}(\Pi) := \{ij \in N \times M : \pi_{ij} \neq 0\}.$$

We say that agent i is **fractionally matched** to object type j if there is a positive probability of the pair being matched but they are not matched for sure, i.e. $0 < \pi_{ij} < 1$. A random assignment represented by the matrix $\Pi = (\pi_{ij})$ is **ex-ante blocked** by agent i and object type j if there is some agent $i' \neq i$ with $\pi_{i'j} > 0$ and $i \succ_j i'$ and some object type j' with $\pi_{ij'} > 0$ and $j \, P_i \, j'$. A random assignment is **ex-ante stable** if it is not blocked by any agent-object type pair.

[2] Echenique et al. (2013) use the graph representation to prove a result about the testable implications of matching theory.

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

P_1	P_2	P_3
2	1	3
1	2	2
3	3	1

$\succeq_1$	$\succeq_2$	$\succeq_3$
2	1	1
1	2	2
3	3	3

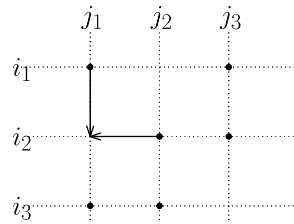

Figure 1: The matrix represents a random assignment. Preferences and priorities are as in the table in the middle. The assignment has several blocking pairs, for example the pair $(2,1)$: In the corresponding directed graph, there is a horizontal arc from $(2,2)$ to $(2,1)$ and a vertical arc from $(1,1)$ to $(2,1)$. Since $\pi_{1,1} > 0$ and $\pi_{2,2} > 0$, agent 2 and object type 1 ex-ante block the assignment.

2.1 Graph representation

Next, we introduce the graph representation of Balinski and Ratier (1997). In the following, a directed graph Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite set of vertices and $E(\Gamma)$ is a set of ordered pairs of vertices called arcs. For a preference profile $P = (P_i)_{i\in N}$ and priority profile $\succ = (\succ_j)_{j\in M}$ we construct a directed graph Γ as follows: The vertices are the agent-object type pairs,

$$V(\Gamma) = N \times M.$$

There are two kind of arcs. A **horizontal arc** connects two vertices ij and ij' that contain the same agent. A **vertical arc** connects two vertices ij and $i'j$ that contain the same object type. The direction of the arc is determined by the preferences respectively priorities. A horizontal arc points to the more preferred object type according to the agent's preferences. A vertical arc points to the agent with higher priority in the object type's priority. Thus

$$(ij, i'j') \in E(\Gamma) \Leftrightarrow (i = i', j' \, P_i \, j \text{ or } j = j', i' \succ_j i).$$

Immediately from the definition of ex-ante stability we obtain the following necessary condition for ex-ante stability (see Figure 1).

Lemma 1. *If $\Pi = (\pi_{ij})$ is ex-ante stable, then there cannot exist both a horizontal arc (ij', ij) and a vertical arc $(i'j, ij)$ with $\pi_{ij'}, \pi_{i'j} > 0$ both pointing to ij.*

Let Π be an assignment matrix. We denote by $\Gamma(\Pi)$ the subgraph of Γ that is induced by the fractionally matched pairs, i.e. we have $V(\Gamma(\Pi)) =$

$\{ij : 1 > \pi_{ij} > 0\}$ and

$$(ij, i'j') \in E(\Gamma(\Pi)) \Leftrightarrow ((ij, i'j') \in E(\Gamma) \text{ and } ij, i'j' \in V(\Gamma(\Pi)).$$

3 Results

3.1 Strict Priorities

We are ready to state and prove the main results for the case with strict priorities. First we show that if Π represents an ex-ante stable assignment, then it has small support.

Proposition 1. *If priorities are strict, then for each ex-ante stable assignment* Π *we have*

$$|supp(\Pi)| \leq n + m.$$

Proof. We prove the proposition by a double counting argument. Let $U \subseteq V(\Gamma)$ be the set of vertices that have an incoming horizontal arc in Γ. Observe that this is the same set as the set of vertices that have an incoming horizontal arc in $\Gamma(\Pi)$. For each $i \in N$, let $M_i(\Pi) \subseteq M$ be the set of object types j such that ij has an incoming horizontal arc in $\Gamma(\Pi)$. By definition, we have $|U| = \sum_{i \in N} |M_i(\Pi)|$. Let $i \in N$. Either i is deterministically matched or he is fractionally matched to multiple object types. In the first case, we have $M_i(\Pi) = \emptyset$. In the second case, let $j \in M_i(\Pi)$ be the least preferred object type (according to i's preferences) among the object types that are fractionally matched to i under Π. Since j is i's least preferred object type to which he is matched, there is for each such object type $j' \neq j$ a horizontal arc pointing from ij to ij'. Thus, in either case, $|\text{supp}(\Pi_i)| - 1 = |M_i(\Pi)|$ where $\text{supp}(\Pi_i)$ is the support of the i-row of Π. Summing over N we obtain

$$\text{supp}(\Pi) - n \leq \sum_{i \in N} |M_i(\Pi)| = |U|. \tag{1}$$

Next we bound $|U|$ from above. Let $j \in M$. Suppose there is a horizontal arc in $\Gamma(\Pi)$ pointing to ij and another horizontal arc in $\Gamma(\Pi)$ pointing to $i'j$. If there were a vertical arc pointing from ij to $i'j$, we would have a contradiction to Lemma 1 and vice versa if there were a vertical arc pointing from $i'j$ to ij, we would also have a contradiction to Lemma 1. Thus for each j there is at most one agent i such that ij has an incoming horizontal arc. Thus $|U| \leq m$. Combining this inequality with Inequality 1, we obtain the desired result. □

$$
\begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
0 & \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\
\frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Figure 2: A bi-stochastic matrix representing a random assignment. The object types corresponding to the second and to the third column have two copies. The other object types have one copy. The agent i' corresponding to the first row is fractionally matched to three object types. The minimal bi-stochastic matrix Π' containing i' is the 3×3-matrix shaded in red. Note that $|\text{supp}(\Pi')| = 7 > 3+3 = |N'|+|M'|$. Thus Π' is not ex-ante stable and therefore Π is not ex-ante stable.

It follows rather immediately from the bound on the support that ex-ante stable assignments under strict priorities are almost degenerate.

Corollary 1. *If priorities are strict, then for each ex-ante stable assignment the following holds:*

1. *Each agent i is either deterministically matched or there are two object types that are fractionally matched to i.*
2. *For each object type j there are at most two agents that are fractionally matched to j.*

Proof. Suppose there is an ex-ante stable Π such that some agent i' is fractionally matched to at least three object types. Consider a minimal bi-stochastic sub-matrix $\Pi' \subseteq \Pi$ containing i', i.e. a minimal (in terms of number of rows and columns) matrix $(\pi_{ij})_{(i,j)\in N'\times M'}$ with $i' \in N' \subseteq N$ and $M' \subseteq M$ such that

1. $\sum_{j\in M'} \pi_{ij} = 1$ for any $i \in N'$,
2. $q'_j := \sum_{i\in N'} \pi_{ij} \in \mathbb{N}$ for any $j \in M'$,

3. $\sum_{j \in M'} q'_j = |N'|$.

By minimality, Π' contains only fractionally matched agents (see Figure 2 for an example). Thus every agent in Π' is fractionally matched to two or more object types. Moreover, i is fractionally matched to at least three object types. Thus $|\text{supp}(\Pi')| \geq 3 + 2 \cdot (|N'| - 1) > 2 \cdot |N'| \geq |N'| + |M'|$ and, by Proposition 1, Π' is not ex-ante stable. But each blocking pair of Π' is also a blocking pair of Π. Therefore, Π is not ex-ante stable contradicting our assumption. A symmetric argument shows the second part of the corollary. □

3.2 Thick Priorities

Now we consider the more general case where priorities can be weak. For each object type j we have a weak (reflexive, complete and transitive) priority order $\succeq_j$ of the agents. We let $i \sim_j i'$ if and only if $i \succeq_j i'$ and $i' \succeq_j i$. We let $i \succ_j i'$ if and only if $i \succeq_j i'$ but not $i' \succeq_j i$. The priorities $\succeq_j$ of an object type j partition N in equivalence classes of equal priority agents, i.e. in equivalence classes with respect to $\sim_j$. We call these equivalence classes **priority classes** and denote them by $I_j^1, I_j^2 \ldots, I_j^\ell$ with indices increasing with priority. We use the notation $i \succ_j I_j^k$ to indicate that i has higher priority at j than the agents in the priority class I_j^k. The definition of ex-ante stability remains the same as before, in particular, the object type j in the blocking pair must strictly prioritize i over i' in order to ex-ante block.

We now generalize Proposition 1 to the case with thick priority classes. We define for each random assignment Π and priority profile $\succeq = (\succeq_j)_{j \in M}$, **cut-off priority classes** as follows: For each $j \in M$ let $I_j(\Pi) \subseteq N$ be the lowest priority class that contains an agent that is matched to j under Π. Formally, we let $c := \min\{c \in \{1, \ldots, \ell\} : \exists i \in I_j^c \text{ with } \pi_{ij} > 0\}$ and define $I_j(\Pi) := I_j^c$. With this notation we obtain the following upper bound on the size of the support.

Theorem 1. *If* Π *is ex-ante stable then*

$$|supp(\Pi)| \leq n + \sum_{j \in M} |I_j(\Pi)|.$$

Proof. Again we use the graph representation as introduced in Section 2. We model indifferences in priorities by undirected edges, i.e. unordered pairs of vertices. Now there are two kind of vertical edges: Vertical arcs pointing from a vertex ij to $i'j$ such that $i' \succ_j i$ and **neutral** vertical edges connecting vertices ij and $i'j$ such that $i \sim_j i'$. Neutral edges do not have a direction. Note that Lemma 1 remains to hold.

Again we use a double counting argument. As before, let $U \subseteq V(\Gamma)$ be the set of vertices with an incoming horizontal arc. The same argument as in in the proof of Proposition 1, shows that

$$|\mathrm{supp}(\Pi)| - n \leq |U|. \tag{2}$$

Next we bound $|U|$ from above in terms of the sizes of cut-off classes. For each $j \in M$, let $N_j(\Pi) \subseteq N$ be the set of agents i such that ij has an incoming horizontal arc in $\Gamma(\Pi)$. By definition, we have $|U| = \sum_{j \in M} |N_j(\Pi)|$. Let $j \in M$ and suppose $i, i' \in N_j(\Pi)$, i.e. suppose there is a horizontal arc in $\Gamma(\Pi)$ pointing to ij and another horizontal arc in $\Gamma(\Pi)$ pointing to $i'j$. If there were a vertical arc pointing from ij to $i'j$ we would have a contradiction to Lemma 1 and vice versa if there were a vertical arc pointing from $i'j$ to ij we would also have a contradiction to Lemma 1. Thus, in this case ij and $i'j$ are connected by a neutral edge. Thus, each $i \in N_j(\Pi)$ is in the same indifference class at j. Moreover, by Lemma 1, for each $i \in N_j(\Pi)$ there is no vertical arc from a $i'j$ with $\pi_{i'j} > 0$ pointing to ij. Therefore $N_j(\Pi) \subseteq I_j(\Pi)$. Thus for each $j \in M$ we have $|N_j(\Pi)| \leq |I_j(\Pi)|$. Summing over M we obtain

$$|U| = \sum_{j \in M} |N_j(\Pi)| \leq \sum_{j \in M} |I_j(\Pi)|.$$

Combining this inequality with Inequality 2, we obtain the theorem. $\square$

Note that Theorem 1 generalizes Proposition 1. If the profile $\succeq$ is strict then for each $j \in M$ we have $I_j(\Pi) = 1$. Therefore, the second term on the right hand side of the inequality is m. The following example illustrates Theorem 1.

Example 1. Consider five agents, five object types, each with a single copy ($q_j = 1$ for each object type), and the following preferences and priorities.

P_1	P_2	P_3	P_4	P_5	$\succeq_1$	$\succeq_2$	$\succeq_3$	$\succeq_4$	$\succeq_5$
1	1	1	2	5	$1,2,3$	5	$3,5$	4	1
4	2	2	5	3	$\vdots$	$2,3,4$	$\vdots$	1	2
5	5	3	4	2		$\vdots$		$\vdots$	3
2	3	5	3	1					$4,5$
3	4	4	1	4					$\vdots$

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$

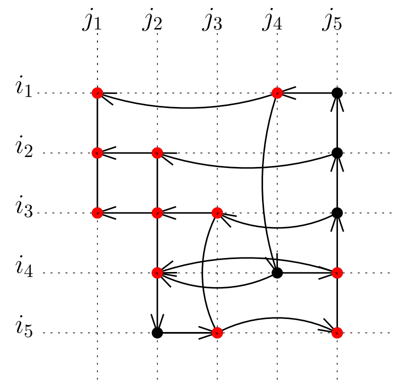

Figure 3: The red vertices have an incoming horizontal arc. In this example: $\sum_{j\in M} |I_j(\Pi)| = 3+3+2+1+2 = 11$. Thus, the right-hand side of the inequality in the theorem is $5+11 = 16$. In this example the bound is sharp and the inequality holds with equality: $|\mathrm{supp}(\Pi)| = 16$.

In Figure 3, we consider a random assignment that is ex-ante stable for the above preferences and priorities. In this assignment, the upper bound on the size of the support holds with equality.

3.3 Pseudo-Market Mechanisms and Stable-Improvement Cycles

In the last section, we relate our results to the pseudo-markets with priority specific pricing of He et al. (2015). In their paper, a random assignment is generated by a pseudo-market of probability shares. Each agent has a budget of tokens and can "buy" probability shares at the different object types. Agents face different prices depending on their priority. More precisely, for each object type there is a "cut-off" priority class. All agents ranked below the cut-off of an object type cannot buy shares at the object type, i.e. they face an infinite price, all agents in the cut-off class face the same finite price and all agents ranked above the cut-off class can get shares at the object type for free. By construction, the mechanisms always implement an ex-ante stable random assignment. It is then straightforward to see that the cut-off priority classes that define the prices correspond exactly to the cut-off priority classes from Section 3.2 that bound the size of the support of an ex-ante stable random assignment. This observation explains the particular structure of the mechanisms of He et al. (2015). The fact that for each object type there are exactly three different prices that are defined by a cut-off class, can be explained by the properties of the support that we have derived in Section 3.2. In particular, these properties follow entirely from "ordinal" considerations and are not related to cardinal utility or the exact probabilities with which object types are allocated. Only the ranking of object types by the agents and the support of the random assignment play a role.

Using the combinatorial techniques introduced in the previous sections, we prove an additional result about the pseudo-market equilibria. We provide a characterization of the equilibria as the set of all ex-ante stable random assignments that satisfy a weak form of the absence of stable improvement cycles à la Erdil and Ergin (2008). Each market equilibrium with priority specific pricing is ex-ante stable and weakly constrained efficient. On the other hand, each ex-ante stable random assignments that is weakly constrained efficient can be decentralized as market equilibrium with priority-specific pricing. The results can be interpreted as welfare theorems: Pseudo-market equilbria under priority-specific pricing are weakly constrained efficient and weakly constrained efficient random assignments can be decentralized as market equilibria.

First, we formally define the pseudo-markets of He et al. (2015). Instead of a preference profile P, we now consider a profile $U = (u_{ij})_{i\in N, j\in M} \in \mathbb{R}_+^{N\times M}$ of **von-Neumann-Morgenstern (vNM) utilities**. We require that utilities are strict, i.e. for each $i \in N$ and $j, j' \in M$ with $j \neq j'$ we have $u_{ij} \neq u_{ij'}$. Utility profile U is consistent with preference profile P, if for each agent $i \in N$ and each pair of object types $j, j' \in M$ we have $jP_ij' \Leftrightarrow u_{ij} > u_{ij'}$. Each utility profile U is consistent with one preference profile P that we call the preference profile induced by U. A utility profile contains additional information compared to a preference profile. In addition to ranking the objects, the vNM-utilities express the rates with which agents substitute probability shares at the different object types. Each agent i has an exogenous **budget** $0 < b_i \leq 1$ of tokens for which he can buy probability shares. For a utility profile U and a vector of budgets $b = (b_i)_{i\in N}$ an **equilibrium with priority-specific pricing** is a triple (Π, I, p) consisting of an assignment Π, cut-off classes $I = (I_j)_{j\in M}$, where for each $j \in M$ the set $I_j \subseteq N$ is a priority class of $\succeq_j$, and cut-off prices $p = (p_j)_{j\in M} \in \mathbb{R}_+^m$ such that the following holds: Defining for each $i \in N$ a price vector $p_i =$

$(p_{ij})_{j\in M} \in \mathbb{R}^m_+$ where

$$p_{ij} = \begin{cases} \infty, & \text{if } I_j \succ_j i, \\ p_j, & \text{if } i \in I_j, \\ 0, & \text{if } i \succ_j I_j. \end{cases} \tag{3}$$

the random assignment Π satisfies for each $i \in N$,

$$\Pi_i = (\pi_{ij})_{j\in M} \in \left\{ \operatorname{argmax} \sum_{j\in M} u_{ij} \cdot \pi_{ij} \text{ subject to} \sum_{j\in M} p_{ij} \cdot \pi_{ij} \leq b_i \right\}. \tag{4}$$

It can be shown (He et al., 2015) that for each pair (U, b) there exists an equilibrium with priority-specific pricing. In the following, we denote by $\mathcal{E}(U, b)$ the set of all random assignments that are part of an equilibrium with priority specific-pricing with respect to U and b. By construction, each random assignment in $\mathcal{E}(U, b)$ is an ex-ante stable random assignment.

Proposition 2 (He et al., 2015)**.** *For each utility profile U and budget vector b, each $\Pi \in \mathcal{E}(U, b)$ is ex-ante stable with respect to the preference profile induced by U.*

Proof. Choose an equilibrium (Π, I, p) for (U, b) that supports the random assignment Π. Suppose Π is ex-ante blocked by i and j. Then $i \succ_j I_j$ and therefore $p_{ij} = 0$. Moreover, there is a j' with $j \, P_i \, j'$ and $\pi_{ij'} > 0$. Thus i would obtain a higher expected utility by obtaining shares in j which i can get for free rather than shares in j'. We have a contradiction. □

We now show that in addition to the fairness notion of ex-ante stability, equilibrium random assignments satisfy a natural ordinal efficiency property. A **strong (ex-ante) stable improvement cycle** for an ex-ante stable matching Π is a sequence of agents $i_1, i_2, \ldots, i_k$ and object types $j_1, j_2, \ldots, j_k$ such that for each $1 \leq \ell \leq k$ (taking indices modulo k) the following holds:

1. Agent i_ℓ is fractionally matched to j_ℓ under Π.

2. Agent i_ℓ prefers $j_{\ell+1}$ to j_ℓ.

3. Agents i_ℓ and $i_{\ell-1}$ have the same priority at j_ℓ.[3]

[3]Stable improvement cycles have originally been defined for deterministic stable matchings (Erdil and Ergin, 2008) and generalized to the probabilistic set-up by Kesten and Ünver (2015). A stable improvement is a more general notion in the following sense: In

P_1	P_2
2	1
1	2

$\succeq_1$	$\succeq_2$
$1,2$	$1,2$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

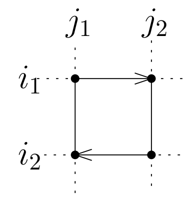

Figure 4: A minimal example of an ex-ante stable random assignment with a strong stable improvement cycle: There are two agents and two object types with capacity 1. The random assignment in the middle is ex-ante stable. The two agents and the two object types form a strong stable improvement cycle: Agent 1 prefers object type 2 to object type 1. Agent 2 prefers object type 1 to object type 2. Both agents have the same priorities at the two object types. We could achieve an ex-ante Pareto improvement by matching agent 1 to object type 2 for sure and matching agent 2 to object type 1 for sure.

In the graph-theoretic language that we have used throughout the paper, a stable improvement cycle is an alternating cycle of horizontal arcs and neutral vertical edges. Strong stable improvement cycles can be used to increase ex-ante efficiency without violating ex-ante stability, by reallocating probability shares within priority classes. This can be achieved by transferring probability shares in j_1 from i_2 to i_1, probability shares in j_2 from i_3 to i_2, and so on. Changing probability shares in this way will not change the shares of the object types allocated to the different indifference classes. It only reallocates shares within priority classes. Figure 4 gives a minimal example of an ex-ante stable random assignment with a strong stable improvement cycle.

Next, we show that equilibrium random assignment with priority specific prizing have no strong stable improvement cycles.

Proposition 3 (Constrained First Welfare Theorem)**.** *Let* $\Pi \in \mathcal{E}(U, b)$. *Then* Π *has no strong stable improvement cycle with respect to the ordinal preferences induced by* U*.*

comparison to strong stable improvement cycles, the third item in the definition is replaced by the weaker condition that i_ℓ is one of the highest priority agents that prefers j_ℓ to some of the object types that he is fractionally matched to under Π. In some sense, strong improvement cycles are the most obvious sources for an ex-ante, stability-preserving Pareto-improvement, since their resolution only reallocates probability shares within priority classes, whereas other stable improvement cycles might redistribute probability shares across priority classes.

Proof. Suppose for the sake of contradiction that there is a strong stable improvement cycle, $\{i_1, i_2, \dots, i_k\}, \{j_1, j_2, \dots, j_k\}$. Choose an equilibrium (Π, I, p) for (U, b) that supports the random assignment Π. For $1 \leq \ell \leq k$, let p_{i_ℓ} be the individual price vector for agent i_ℓ defined by Equation (3). As i_ℓ prefers $j_{\ell+1}$ to j_ℓ and buys probability shares at j_ℓ, agent i_ℓ faces a higher price at object type $i_{\ell+1}$, i.e. $p_{i_\ell, j_{\ell+1}} > p_{i_\ell, j_\ell}$. As agents i_ℓ and $i_{\ell-1}$ have the same priority at j_ℓ we have $p_{i_\ell, j_\ell} = p_{i_{\ell-1}, j_\ell}$. We obtain:

$$p_{i_1,j_1} < p_{i_1,j_2} = p_{i_2,j_2} < p_{i_2,j_3} = \ldots = p_{i_k,j_k} < p_{i_k,j_1} = p_{i_1,j_1}$$

This is a contradiction. □

It turns out that the two properties of ex-ante stability and the absence of strong stable improvement cycles characterize market equilibrium random assignments with priority-specific prizing.

Theorem 2 (Constrained Second Welfare Theorem)**.** *If Π is ex-ante stable and does not have a strong stable improvement cycle, then there exists a utility profile U consistent with the preference profile P and budget vector b, such that $\Pi \in \mathcal{E}(U, b)$.*

Proof. Let Π be ex-ante stable and not have a strong stable improvement cycle. Recall the definition of cut-off classes $I_j(\Pi)$ from Section 3.2. We use the cut-off classes as cut-off classes for the definition of the equilibrium that decentralizes Π. Thus, we let $I = (I_j(\Pi))_{j \in M}$. It remains to construct the cut-off prices p, utilities U and budgets b such that (Π, I, p) is an equilibrium under U and b. First we define the prices. In the following, for each $i \in N$ we denote the set of object types for which i is in the cut-off class by $c_i(\Pi) := \{j \in M : i \in I_j(\Pi)\}$. First we show that we can order the objects types $j_1, \dots, j_m$ such that for $k < \ell$ there is no agent i with $j_k, j_\ell \in c_i(\Pi)$ such that $j_\ell P_i j_k$ and $\pi_{ij_k} > 0$. The ranking will allow us to define cut-off prices such that $p_{j_1} > p_{j_2} > \ldots > p_{j_m}$.

To construct the ordering, we consider the graph representation of Π as defined in Section 2. As in the proof of Theorem 1, indifferences in the priorities are represented by undirected edges. Denote for each object type j by $V_j \subseteq V(\Gamma)$ the set of vertices corresponding to the cut-off class of j, i.e. $V_j := \{ij \in V(\Gamma) : i \in I_j(\Pi)\}$. Consider the subgraph H^1 of Γ with vertex set $V(H^1) = \bigcup_{j \in M} V_j$ and arc set $E(H^1) = \{(ij, ij') : ij, ij' \in V(H^1), \pi_{ij} > 0\}$. We choose as highest ranked object type j_1 an object type j_1 such that the set V_{j_1} has no outgoing arc to $V(H^1) \setminus V_{j_1}$. Such an object type exist because of the absence of strong stable improvement cycles: Start with an arbitrary object type j. If V_j has no outgoing arc to $V(H^1) \setminus V_j$, then we let $j_1 = j$. Otherwise choose an arbitrary outgoing arc to a set $V_{j'}$ with $j' \neq j$. If there is no outgoing arc from $V_{j'}$ to $V(H^1) \setminus V_{j'}$, then we let $j_1 = j'$. Otherwise we choose an arbitrary outgoing arc to a set $V_{j''}$. Iterating in this way, we either find an object type j_1 as desired or we find a strong stable improvement cycle. The latter leads to a contradiction. After having determined the maximal object type j_1 we consider the subgraph H^2 of H^1 induced by the vertices $\bigcup_{j \neq j_1} V_j$. In H^2, an analogous argument shows that there is object type j_2 such that V_{j_2} has no outgoing arc from V_{j_2} to $V(H^2) \setminus V_{j_2}$. Iterating in this way, we obtain an ordering of the object types $j_1, j_2, \ldots, j_m$. We choose any prices such that $p_{j_1} > p_{j_2} > \ldots > p_{j_m} > 0$. We define budgets $b = (b_i)_{i \in N}$ by $b_i = \sum_{j \in M} p_{ij} \cdot \pi_{ij}$.

Next we define utilities. For each $i \in N$ we define $U_i \in \mathbb{R}^M$ as follows. Let j_0 be the most preferred object type, according to P_i, among the object types that agent i can get for free, i.e. among object types for which he is ranked higher than the cut-off. We choose an arbitrary $\epsilon > 0$ and let $u_{ij_0} = \epsilon$. For each object type $j \in \text{supp}(\Pi_i) \setminus \{j_0\}$ we define

$$u_{ij} = p_j + \epsilon.$$

Before we define utilities for the remaining object types, we show that defining utilities for object types in $\text{supp}(\Pi_i)$ in this way is consistent with P_i. Suppose there are two object types $j, j' \in \text{supp}(\Pi_i)$ and jP_ij'. If both $j, j' \in c_i(\Pi)$, then by the construction of the prices, we have $p_j > p_{j'}$ and therefore $u_{ij} > u_{ij'}$ consistently with P_i. If $j' \notin c_i(\Pi)$, then we have $j' = j_0$. Thus $u_{ij} = p_j + \epsilon > \epsilon = u_{ij'}$ consistently with P_i.

It remains to define utilities for objects types in $M \setminus (\text{supp}(\Pi_f) \cup \{j_0\})$. First we consider object types for which i is in the cut-off class, i.e. object types in $c_i(\Pi) \setminus \text{supp}(\Pi_i)$. We define the utilities for these object types iteratively starting with the most preferred one, followed by the second most preferred one etc. Let $j \in c_i(\Pi) \setminus \text{supp}(\Pi_i)$ be the most preferred object type according to P_i for which we have not defined u_{ij} yet. We choose $0 < u_{ij} \leq p_j + \epsilon$ such that u_{ij} is consistent according to P_i with the already defined utilities. This is possible: As observed above, for each $j' \in c_i(\Pi)$, with jP_ij' we either have $j' \notin \text{supp}(\Pi_i)$ or $j' \in \text{supp}(\Pi_i)$ and $p_{j'} < p_j$. In the first case, we have not yet defined $u_{ij'}$. In the second case, we have $u_{ij'} = p_{j'} + \epsilon < p_j + \epsilon$. Thus we can choose $u_{ij'} < u_{ij} \leq p_j + \epsilon$. Similarly, if jP_ij_0, then $u_{ij_0} = \epsilon < p_j + \epsilon$. Thus we can choose $u_{ij_0} < u_{ij} \leq p_j + \epsilon$.

Once we have defined utilities for object types in $\text{supp}(\Pi_i) \cup c_i(\Pi)$ we may choose the utilities for the other object types in an arbitrary manner consistent with P_i.

Finally, we show that Π_i satisfies Equation (3). Observe that if i requests probability shares at a free object, then he will only do so at the most preferred one among them, which is j_0. Thus, in an optimum, i will request probability shares only in object types for which he is in the cut-off and (possibly) object type j_0. Thus, the optimization problem for agent i is equivalent to solving

$$\begin{aligned}
\max \quad & \sum_{j \in c_i(\Pi)} u_{ij} \cdot \pi_{ij} + u_{ij_0} \cdot \pi_{ij_0} \\
\text{subject to} \quad & \sum_{j \in c_i(\Pi)} p_j \cdot \pi_{ij} \leq b_i, \\
& \pi_{ij_0} = 1 - \sum_{j \in c_i(\Pi)} \pi_{ij}, \\
& 0 \leq \sum_{j \in c_i(\Pi)} \pi_{ij} \leq 1, \\
& 0 \leq \pi_{ij} \leq 1.
\end{aligned}$$

The objective function can be written as

$$\sum_{j \in c_i(\Pi) \cap \text{supp}(\Pi_i)} (p_j + \epsilon) \cdot \pi_{ij} + \sum_{j \in c_i(\Pi) \setminus \text{supp}(\Pi_i)} u_{ij} \cdot \pi_{ij} + \epsilon \cdot \pi_{ij_0}.$$

Plugging in the second constraint this simplifies to

$$\begin{aligned}
& \sum_{j \in c_i(\Pi) \cap \text{supp}(\Pi_i)} (p_j + \epsilon) \cdot \pi_{ij} + \sum_{j \in c_i(\Pi) \setminus \text{supp}(\Pi_i)} u_{ij} \cdot \pi_{ij} + \epsilon \cdot (1 - \sum_{j \in c_i(\Pi)} \pi_{ij}) \\
= & \sum_{j \in c_i(\Pi) \cap \text{supp}(\Pi_i)} p_j \cdot \pi_{ij} + \sum_{j \in c_i(\Pi) \setminus \text{supp}(\Pi_i)} (u_{ij} - \epsilon) \cdot \pi_{ij} + \epsilon.
\end{aligned}$$

The problem becomes

$$\begin{aligned}
\max \quad & \sum_{j \in c_i(\Pi) \cap \text{supp}(\Pi_i)} p_j \cdot \pi_{ij} + \sum_{j \in c_i(\Pi) \setminus \text{supp}(\Pi_i)} (u_{ij} - \epsilon) \cdot \pi_{ij} + \epsilon \\
\text{subject to} \quad & \sum_{j \in c_i(\Pi)} p_j \cdot \pi_{ij} \leq b_i, \\
& 0 \leq \sum_{j \in c_i(\Pi)} \pi_{ij} \leq 1, \\
& 0 \leq \pi_{ij} \leq 1.
\end{aligned}$$

$$\begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ 0 & \frac{1}{6} & \frac{2}{3} & 0 & \frac{1}{6} \end{pmatrix}$$

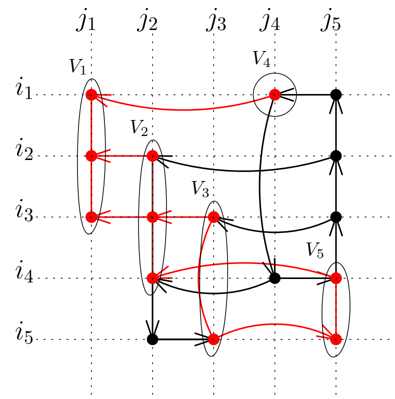

Figure 5: The red subgraph is the graph H induced by the sets $V_1, V_2, \ldots, V_5$.

By definition of the utilities, we have for each $j \in c_i(\Pi) \setminus \mathrm{supp}(\Pi_i)$ that $u_{ij} - \epsilon \leq p_j$. Thus, each random assignment where i buys shares in object types with $u_{ij} = p_j + \epsilon$, until the budget constraint binds, and then fills up his remaining capacity with the object type j_0, is an optimum of the above problem. In particular, Π_i is an optimum. □

Example 1 (cont.). For our example, cut-off prices and utilities are constructed as follows: First we rank the object types. Consider the vertex sets $V_1, \ldots, V_5$ induced by the cut-off classes of the object types. See Figure 5. Note that V_1 has no outgoing arc to any of the sets V_2, V_3, V_4 and V_5. Thus, we assign object type 1 the highest cut-off price, e.g. $p_1 = \frac{5}{4}$. Next, note that V_4 has no outgoing arc to V_2, V_3 or V_5. Thus, we assign object type 4 the second highest cut-off price, e.g. $p_4 = 1$. Next, note that V_2 has no outgoing arc to V_3 or V_5. Thus, we assign object type 2 the third highest cut-off price, e.g. $p_2 = \frac{3}{4}$. Next, note that V_5 has no outgoing arc to V_3. Thus, we assign object type 5 the fourth highest cut-off price, e.g. $p_5 = \frac{1}{2}$. Finally, assign the remaining object type 3 the lowest cut-off price, e.g. $p_3 = \frac{1}{4}$.

The budget vector is given by $b = (\frac{11}{12}, \frac{2}{3}, \frac{5}{8}, \frac{1}{3}, \frac{1}{4})$. Finally, we define utilities. We let $\epsilon_1 = 3$. For the three object types $1, 4, 5$ that are fractionally matched to agent 1, we have $u_{11} = p_1 + \epsilon_1 = 4\frac{1}{4}$, $u_{14} = p_4 + \epsilon_1 = 4$ and $u_{15} = \epsilon_1 = 3$. For the other two object types, we may choose any utilities that are consistent with the preferences for 1, e.g. $u_{12} = 2$ and $u_{13} = 1$. In a similar way, we can define the utilities for the other agents. We obtain $u_1 = (4\frac{1}{4}, 2, 1, 4, 3), u_2 = (4\frac{1}{4}, 3\frac{3}{4}, 2, 1, 3), u_3 = (3\frac{1}{4}, 2\frac{3}{4}, 2\frac{1}{4}, 1, 2), u_4 = (1, 3\frac{3}{4}, 2, 3, 3\frac{1}{2})$ and $u_5 = (2, 3, 3\frac{1}{4}, 1, 3\frac{1}{2})$.

References

Abdulkadiroglu, A. and Sönmez, T. (2003): "School Choice: A Mechanism Design Approach." *American Economic Review*, 93(3): 729–747.

Balinski, M. and Ratier, G. (1997): "Of Stable Marriages and Graphs, and Strategy and Polytopes." *SIAM Review*, 39: 575–604.

Echenique, F., Lee, S., Shum, M., and Yenmez, M. B. (2013): "The Revealed Preference Theory of Stable and Extremal Stable Matchings." *Econometrica*, 81(1): 153–171.

Erdil, A. and Ergin, H. (2008): "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice." *The American Economic Review*, 98(3): 669–689.

He, Y., Miralles, A., Pycia, M., and Yan, J. (2015): "A Pseudo-Market Approach to Allocation with Priorities." Technical report, Toulouse School of Economics (TSE).

Hylland, A. and Zeckhauser, R. (1979): "The efficient allocation of individuals to positions." *The Journal of Political Economy*, pages 293–314.

Kesten, O. and Ünver, M. U. (2015): "A theory of school-choice lotteries." *Theoretical Economics*, 10(2): 543–595. ISSN 19336837. doi: 10.3982/TE1558.

Kojima, F. and Manea, M. (2010): "Incentives in the Probabilistic Serial Mechanism." *Journal of Economic Theory*, 145(1): 106–123.

Roth, A. E., Rothblum, U. G., and Vande Vate, J. H. (1993): "Stable matchings, optimal assignments, and linear programming." *Mathematics of Operations Research*, 18(4): 803–828.