Lexicographic Choice under Variable Capacity Constraints*

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Abstract

A (capacity-constrained) choice problem consists of a set of alternatives and a capacity. A (capacity-constrained) choice rule, at each choice problem, chooses from the alternatives without exceeding the capacity. A choice rule is *lexicographic* if there exists a list of priority orderings over potential alternatives such that at each choice problem, the set of chosen alternatives is obtainable by choosing the highest ranked alternative according to the first priority ordering, then choosing the highest ranked alternative among the remaining alternatives according to the second priority ordering, and proceeding similarly until the capacity is full or no alternative is left. Lexicographic choice rules have been useful in designing allocation mechanisms for school choice to achieve diversity. We provide a characterization of lexicographic choice rules. We discuss some implications for the Boston school choice system. We also provide a characterization of deferred acceptance mechanisms that operate based on a lexicographic choice structure.

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1 Introduction

A (capacity-constrained) choice problem consists of a choice set (a set of alternatives) and a capacity. A (capacity-constrained) choice rule, at each choice problem, chooses some alternatives from the choice set without exceeding the capacity. Choice rules are essential in the analysis of resource allocation problems where some objects, each of which has a certain capacity, is to be allocated among agents. A well-known example is the school choice problem in which each school has a certain number of seats to be allocated among students. Although student preferences are elicited from the students, endowing each school with a choice rule is a part of the design process.

Starting with the seminal study by Abdulkadiroğlu and Sönmez (2003), the school choice literature has widely focused on problems where the choice rule of a school only needs to be responsive to a given priority ordering over students, in which case the choice rule to be used is uniquely determined: the *responsive choice rule*.[1] However, when there are other concerns such as achieving a diverse student body or affirmative action, which choice rule to use is non-trivial.

We follow the axiomatic approach to formulate general principles (axioms) that apply to choice rules. Each axiom carries with it a consistency requirement or specifies a desirable procedural aspect of a choice rule. These axioms illuminate characteristics of choice rules that may be relevant for the problem, yet may not be evident from the procedural formulations of the choice rules. We consider the set of axioms characterizing a choice rule as a justification for using that choice rule in a choice problem. As for the applications, such as school choice, axiomatic characterizations of choice rules pave the way for the schools to choose an appropriate rule that fits their policy desiderata expressed in the form of choice axioms.[2]

We consider the following three properties of choice rules that have already been studied in the axiomatic literature.

Acceptance: An alternative is rejected from a choice set at a capacity only if the capacity is full.

[1] In Section 2.2, we discuss responsive choice rules.

[2] Echenique and Yenmez (2015) also follow an axiomatic approach and characterize several choice rules for a school that wants to achieve diversity.

Gross substitutes: If an alternative is chosen from a choice set at a capacity, then it is also chosen from any subset of the choice set that contains the alternative, at the same capacity.

Monotonicity: If an alternative is chosen from a choice set at a capacity, then it is also chosen from the same choice set at any higher capacity.

We introduce a property that requires consistency of the following capacity-wise revealed preference relation: an alternative is *revealed preferred* to another alternative at a capacity if there is a choice set from which the former alternative is chosen over the latter, whereas with one less capacity both alternatives are rejected from the choice set. We say that a choice rule satisfies the *capacity-wise weak axiom of revealed preference (CWARP)* if the revealed preference relation is asymmetric at each capacity.

CWARP is a counterpart of the well-known *weak axiom of revealed preference* (WARP) in the standard revealed preference framework (Samuelson, 1938), where there is no capacity parameter, and a choice rule chooses exactly one alternative from each choice set. In the standard framework, an alternative is said to be revealed preferred to another alternative if there is a choice set at which the former alternative is chosen over the latter. *WARP* requires the revealed preference relation to be asymmetric, which in a sense requires consistency of the choice behavior in responding to changes in the choice set. In our framework, the preference is revealed not only through the choice at a choice set, but also through a change in the capacity. Hence, *CWARP* requires consistency of the choice behavior in responding to changes in the choice set together with changes in the capacity.

We show that a choice rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP* if and only if it is *lexicographic*: there is a list of priority orderings over alternatives such that at each problem, the set of chosen alternatives is obtainable by choosing the highest ranked alternative according to the first priority, then choosing the highest ranked alternative among the remaining alternatives according to the second priority, and proceeding similarly until the capacity is full or no alternative is left (Theorem 1). We also provide an alternative characterization of lexicographic choice rules with another property that we introduce, called *rejection-monotonicity*. Consider a choice problem and the set of rejected alternatives for that problem. In case of an increase in the capacity, *rejection-monotonicity* requires that the new alternatives that will be chosen (if any) should not depend on the already accepted alternatives. In other words, if the set of rejected alternatives are the same for two choice sets at a capacity, then at any higher capacity, the set of accepted alternatives that were formerly rejected should be the same for the two choice sets. *CWARP* together with *acceptance* and *monotonicity* implies *rejection-monotonicity*. As a corollary to our Theorem 1, we show that a choice rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and *rejection-monotonicity* if and only if it is *lexicographic*.

In order to achieve a diverse student body, many school districts have been implementing affirmative action policies, such as in Boston, Chicago, and Jefferson County. The affirmative action policies that are in use in several school districts reveal that a natural way to achieve diversity is to allow students' priorities to vary across a school's seats, and to let each school choose students in a lexicographic fashion based on a predetermined ordering of the seats. An example is the Boston school district which aims to give priority to neighborhood applicants for half of each school's seats. To achieve this goal, Boston school district has been using a deferred acceptance mechanism based on a choice structure, where each school is endowed with a "capacity-wise lexicographic" choice rule, that is, at each capacity, the choice rule lexicographically operates based on a list containing as many priority orderings as the capacity, yet the lists for different capacity levels do not have to be related in any way.[3] Dur et al. (2016) provides an analysis of how the order of the priority orderings in the choice rule of a school may affect the outcome in the Boston school choice context. In Section 3, we consider a class of capacity-wise lexicographic choice rules discussed in Dur et al. (2016) that are relevant for the design of the Boston school choice system and show that our analysis enables us to single out a rule from the four plausible candidates.

Besides providing a first axiomatic foundation for lexicographic choice rules, our study contributes to the literature on allocation mechanisms that are based on lexicographic choice structures. To illustrate this contribution, in Section 4, we consider the variable-capacity object allocation model. In that model, Ehlers and Klaus (2016) characterize deferred acceptance mechanisms where each object has a choice rule that satisfies *acceptance*, *gross substitutes*, and *monotonicity*.[4] We introduce a novel property for allocation mechanisms,

[3]See Dur et al. (2016) for a detailed discussion of Boston's school choice mechanism.

[4]Kojima and Manea (2010) consider a setup where the capacity of each school is fixed, and characterize deferred acceptance mechanisms where each school has a choice rule that satisfies *acceptance* and *gross substitutes*.

called *demand-monotonicity*, which is motivated by and intimately related to the *rejection-monotonicity* of choice rules. As a corollary to the characterization result by Ehlers and Klaus (2016), we provide a characterization of lexicographic deferred acceptance mechanisms (Corollary 4).[5]

The paper is organized as follows. In Section 2, we introduce and characterize the lexicographic choice rules, and also provide a characterization of the responsive choice rules. In Section 3, we discuss some implications for the Boston school choice system. In Section 4, we highlight an implication of our choice theoretical analysis for the resource allocation framework: we provide a characterization of deferred acceptance mechanisms that operate based on a lexicographic choice structure. In Section 5, we conclude by discussing the main features of our analysis.

2 Capacity-Constrained Choice

Let A be a nonempty finite set of n alternatives and let $\mathcal{A}$ denote the set of all *nonempty* subsets of A. A (capacity-constrained) **choice rule** $C : \mathcal{A} \times \{1, \ldots, n\} \to \mathcal{A}$ associates with each pair $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$ of a choice set S and a capacity q, a set of choices $C(S, q) \subseteq S$ such that $|C(S, q)| \leq q$. Given a choice rule C, we denote the set of rejected alternatives at a problem (S, q) by $R(S, q) = S \setminus C(S, q)$.

2.1 Lexicographic Choice

A **priority ordering** $\succ$ is a complete, transitive, and anti-symmetric binary relation over A. A **priority profile** $\pi = (\succ_1, \ldots, \succ_n)$ is an ordered list of n priority orderings. Let Π denote the set of all priority profiles.

A choice rule C is **lexicographic for a priority profile** $(\succ_1, \ldots, \succ_n) \in \Pi$ if for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$, $C(S, q)$ is obtained by choosing the highest $\succ_1$-priority alternative in S, then choosing the highest $\succ_2$-priority alternative among the remaining

[5] Kominers and Sönmez (2016) study lexicographic deferred acceptance mechanisms in a more general matching with contracts framework. Their main focus is on stability and incentive properties of such mechanisms.

alternatives, and so on until q alternatives are chosen or no alternative is left. A choice rule is **lexicographic** if there exists a priority profile for which the choice rule is lexicographic.

Remark 1. Note that, if a choice rule is lexicographic for a priority profile $\pi = (\succ_1, \ldots, \succ_n)$, then it is lexicographic for any other priority profile that is obtained from π by replacing $\succ_n$ with an arbitrary priority ordering. In that sense, the last priority ordering is redundant.

We consider four properties of choice rules. The following three properties are already known in the literature.

Acceptance: An alternative is rejected from a choice set at a capacity only if the capacity is full. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$,

$$|C(S, q)| = \min\{|S|, q\}.$$

Gross substitutes: If an alternative is chosen from a choice set at a capacity, then it is also chosen from any subset of the choice set that contains the alternative, at the same capacity. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$ and each pair $a, b \in S$ such that $a \neq b$,

$$\text{if } a \in C(S, q), \text{ then } a \in C(S \backslash \{b\}, q).$$

Monotonicity: If an alternative is chosen from a choice set at a capacity, then it is also chosen from the same choice set at any higher capacity. Formally, for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n-1\}$,

$$C(S, q) \subseteq C(S, q+1).$$

Consider the following capacity-wise revealed preference relation. An alternative $a \in A$ is revealed to be preferred to an alternative $b \in A$ at a capacity $q > 1$ if there is a problem with capacity $q-1$ for which a and b are both rejected and a is chosen over b when the capacity is q. That is, a is **revealed to be preferred** to b at q if there exists $S \in \mathcal{A}$ such that $a, b \notin C(S, q-1)$, $a \in C(S, q)$, and $b \notin C(S, q)$. We introduce the following property which requires for each capacity the revealed preference relation be asymmetric.

Capacity-wise weak axiom of revealed preference (CWARP): For each capacity $q > 1$ and each pair $a, b \in A$, if a is revealed to be preferred to b at q, then b is not revealed to be preferred to a at q.

Remark 2. The following is an alternative definition of *CWARP*, which is formulated in line with the common formulations of WARP-type revealed preference relations in the literature.

An alternative definition of CWARP: For each capacity $q > 1$, each pair $S, T \in \mathcal{A}$ and each pair $a, b \in S \cap T$ such that $[C(S, q-1) \cup C(T, q-1)] \cap \{a, b\} = \emptyset$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(T, q) \setminus C(S, q), \text{ then } a \in C(T, q).$$

Next, we introduce another property which is implied by *CWARP* together with *acceptance* and *monotonicity*. We invoke the property in the proof of our main result. Moreover, we believe that the property also has a stand-alone normative appeal. The property, similar to *monotonicity*, considers the impact of an increase in the capacity.

Consider a problem and the set of rejected alternatives for that problem. Suppose that the capacity increases. The property requires that which alternatives among the currently rejected alternatives will be chosen (if any) should not depend on the currently accepted alternatives. In other words, if the set of rejected alternatives are the same for two choice sets, then at any higher capacity, the set of initially rejected alternatives that become accepted should be the same for the two choice sets.

Rejection-Monotonicity: For each $S, S' \in \mathcal{A}$ and each $q \in \{1, \ldots, n-1\}$,

$$\text{if } R(S, q) = R(S', q), \text{ then } C(S, q+1) \cap R(S, q) = C(S', q+1) \cap R(S', q).$$

Lemma 1. *If a choice rule satisfies acceptance, monotonicity, and CWARP, then it satisfies rejection-monotonicity.*

Proof. Let C be a choice rule. Suppose that C satisfies *acceptance* and *monotonicity*, but violates *rejection-monotonicity*. By violation of *rejection-monotonicity*, there are $S, S' \in \mathcal{A}$ and $q \in \{1, \ldots, n-1\}$ such that $R(S, q) = R(S', q)$, but $C(S, q+1) \cap R(S, q) \neq C(S', q+1) \cap R(S', q)$. By *monotonicity*, $R(S, q+1) \subseteq R(S, q)$ and $R(S', q+1) \subseteq R(S', q)$. By *acceptance*, $|R(S, q+1)| = |R(S', q+1)|$. Then, there exist $a, b \in R(S, q) = R(S', q)$ such that $a \in C(S, q+1)$, $b \notin C(S, q+1)$, $b \in C(S', q+1)$, and $a \notin C(S', q+1)$. But then, a is revealed preferred to b and vice versa, implying that C violates *CWARP*. $\square$

Theorem 1. *A choice rule is lexicographic if and only if it satisfies acceptance, gross substitutes, monotonicity, and the capacity-wise weak axiom of revealed preference.*[6]

Proof. Let C be a lexicographic choice rule. Clearly, C satisfies *acceptance* and *monotonicity*, and it is already known from the literature that C satisfies *gross substitutes* (Chambers and Yenmez, 2016). To see that it satisfies *CWARP*, let $S, S' \in \mathcal{A}$, $a, b \in A$, and $q \in \{2, \ldots, n\}$ be such that a is revealed preferred to b at q. Then, there is $S \in \mathcal{A}$ such that $a, b \in R(S, q-1)$, $a \in C(S, q)$, and $b \in R(S, q)$. But then, $a \succ_q b$. If also b is revealed preferred to a at q, then by similar arguments we have $b \succ_q a$, contradicting that $\succ_q$ is antisymmetric. Thus, the revealed preference relation is asymmetric and C satisfies *CWARP*.

Let C be a choice rule satisfying *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP*. We first construct a priority profile $(\succ_1, \ldots, \succ_n) \in \Pi$ and then show that C is lexicographic for that priority profile. For each $i, j \in \{1, \ldots, n\}$, let a_{ij} denote the j'th ranked alternative in $\succ_i$ (for instance, a_{i1} is the highest $\succ_i$-priority alternative).

To construct $\succ_1$, first set $\{a_{11}\} = C(A, 1)$. For each $j \in \{2, \ldots, n\}$, set $\{a_{1j}\} = C(A \backslash \{a_{11}, \ldots, a_{1(j-1)}\}, 1)$. To construct $\succ_2$, consider $C(A, 2)$. By *acceptance*, $|C(A, 2)| = 2$. Since $a_{11} \in C(A, 1)$, by *monotonicity*, $a_{11} \in C(A, 2)$. Set $\{a_{21}\} = C(A, 2) \setminus \{a_{11}\}$. For each $j \in \{2, \ldots, n-1\}$, set $\{a_{2j}\} = C(A \setminus \{a_{21}, a_{22}, \ldots, a_{2(j-1)}\}, 2) \setminus \{a_{11}\}$. Set $a_{2n} = a_{11}$.

The rest of the priority profile is constructed recursively as follows. For each $i \in \{3, \ldots, n\}$, first set $\{a_{i1}\} = C(A, i) \setminus \{a_{11}, a_{21}, \ldots, a_{(i-1)1}\}$ (Note that by *monotonicity*, $\{a_{11}, a_{21}, \ldots, a_{(i-1)1}\} \subseteq C(A, i)$ and by *acceptance*, $|C(A, i)| = i$). For each $j \in \{2, \ldots, n - i + 1\}$, set $\{a_{2j}\} = C(A \setminus \{a_{i1}, a_{i2}, \ldots, a_{i(j-1)}\}, i) \setminus \{a_{11}, a_{21}, \ldots, a_{(i-1)1}\}$. Note that there are $i - 1$ rankings yet to be set in $\succ_i$, which are $\{a_{i(n-i+2)}, \ldots, a_{in}\}$. For each $j \in \{n - i + 2, \ldots, n\}$, set $a_{ij} = a_{(j+i-n-1)1}$ (which assigns the alternatives $a_{11}, \ldots, a_{(i-1)1}$ to the rankings $a_{i(n-i+2)}, \ldots, a_{in}$, respectively).

Now, let $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$. Let b_1 denote the highest $\succ_1$-priority alternative in S, b_2 denote the highest $\succ_2$-priority alternative among the remaining alternatives, and so on up to $b_{\min\{|S|,q\}}$. We show that $C(S, q) = \{b_1, \ldots, b_{\min\{|S|,q\}}\}$. If $\min\{|S|, q\} = |S|$, then by *acceptance*, $C(S, q) = \{b_1, \ldots, b_{|S|}\}$. Suppose that $|S| > q$.

[6]Independence of the characterizing properties is shown in Appendix A.

The rest of the proof is by induction: we first show that $b_1 \in C(S,q)$; then, for an arbitrary $i \in \{2,\ldots,q\}$, assuming that $b_1,\ldots,b_{i-1} \in C(S,q)$, we show that $b_i \in C(S,q)$. Let $b_1 = a_{1j}$ for some $j \in \{1,\ldots,n\}$. By the construction of $\succ_1$, $b_1 \in C(A \setminus \{a_{11},\ldots,a_{1(j-1)}\},1)$. Then, by *gross substitutes* and *monotonicity*, $b_1 \in C(S,q)$.

Let $i \in \{2,\ldots,n\}$. Assuming that $b_1,\ldots,b_{i-1} \in C(S,q)$, we show that $b_i \in C(S,q)$. Let S' be the choice set obtained from S by replacing b_1 with a_{11} (note that nothing changes if $b_1 = a_{11}$), replacing b_2 with a_{21}, ..., and replacing b_{i-1} with $a_{(i-1)1}$. That is, $S' = (S \setminus \{b_1,\ldots,b_{i-1}\}) \cup \{a_{11},\ldots,a_{(i-1)1}\}$. Let $q' = i-1$. Note that $\{b_1,\ldots,b_{i-1}\} = C(S,q')$, because otherwise, by *acceptance*, there is $a \in S$ such that $a \in C(S,q')$ and $a \notin C(S,q)$, which is a violation of *monotonicity*. Also, by the construction of the priority profile and by *gross substitutes*, $\{a_{11},\ldots,a_{(i-1)1}\} = C(S',q')$. Note that $R(S,q') = R(S',q')$. By Lemma 1, C satisfies *rejection-monotonicity*. Then, by *monotonicity* and *rejection-monotonicity*, we have $R(S,q) = R(S',q)$. Since $b_i \in C(S',q)$ by the construction of the priority profile and by *gross substitutes*, we also have $b_i \in C(S,q)$. □

Corollary 1. *A choice rule is lexicographic if and only if it satisfies acceptance, gross substitutes, monotonicity, and rejection-monotonicity.*

Proof. A lexicographic choice rule satisfies *acceptance*, *gross substitutes*, and *monotonicity* by Theorem 1, and also satisfies *rejection-monotonicity* by Lemma 1. To see the other direction, note that in the proof of Theorem 1, we invoked *CWARP* only to claim that *rejection-monotonicity* is satisfied, and therefore the same proof for the if part is valid when we replace *CWARP* with *rejection-monotonicity*. □

A choice rule C can be lexicographic for two different priority profiles. Even we can say that the priority profile by which a choice rule is representable as a lexicographic choice rule is never unique. However, if C is lexicographic for two different priority profiles $\succ$ and $\succ'$, then for each pair of alternatives $a,b \in A$, if $a \succ_q b$ and $b \succ'_q a$ for some $q \in \{1,\ldots,n\}$, then either a or b must be chosen from any choice set (particularly from A) at any lower capacity. That is a or b is chosen irrespective of its relative ranking at the q-priority ordering.

To state this observation formally, for each priority ordering $\succ$ on A and for each choice set $S \in \mathcal{A}$, let $\succ_i|_S$ stand for the restriction of $\succ_i$ to S. Let $A_1 = A$, and for each $q \in \{2,\ldots,n\}$, let $A_q = A \setminus C(A,q-1)$.

Proposition 1. *If a choice rule C is lexicographic for a priority profile $(\succ_1, \ldots, \succ_n)$, then C is lexicographic for another priority profile $(\succ'_1, \ldots, \succ'_n)$ if and only if $\succ_1 = \succ'_1$ and for each $q \in \{1, \ldots, n\}$, $\succ_q |_{A_q} = \succ'_q |_{A_q}$.*

Proof. In the proof of Theorem 1, $\succeq_q$ that is constructed for each $q \in \{1, \ldots, n\}$ is such that for each choice set $S \in \mathcal{A}$, $max(S \setminus C(S, q-1), \succ_q) = C(S, q) \setminus C(A, q-1)$. Now, for each $q \in \{1, \ldots, n\}$, let $\succ^*_q = \succ_q |_{A_q}$, and $\mathcal{A}_q$ stand for the collection of all nonempty subsets of A_q. Next, define the choice function $c_q : \mathcal{A}_q \to A_q$ such that for each choice set $S \in \mathcal{A}_q$, $c_q(S) = \max\{S \setminus C(A, q-1), \succ^*_q\}$. Since C satisfies *gross substitutes*, c_q also satisfies *gross substitutes*. It follows that the preference relation $\succ^*_q$ that satisfies $c_q(S) = \max\{S \setminus C(A, q-1), \succ^*_q\}$ is unique. □

2.2 Responsive Choice

A choice rule C is **responsive for a priority ordering** $\succ$ if for each $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$, $C(S, q)$ is obtained by choosing the highest $\succ$-priority alternatives in S until q alternatives are chosen or no alternative is left. Note that C is responsive for a $\succ$ if and only if it is lexicographic for the n-tuple $(\succ, \ldots, \succ)$.

Chambers and Yenmez (2016) introduced the following property and showed that the property, together with *acceptance*, characterizes responsive choice rules for a given capacity.

Weakened weak axiom of revealed preference (WWARP): For each $S, S' \in \mathcal{A}$, $q \in \{1, \ldots, n\}$, and each pair $a, b \in S \cap S'$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(S', q) \setminus C(S, q), \text{ then } a \in C(S', q).$$

The following extension of WWARP, together with *acceptance*, characterizes responsive choice rules in the variable-capacity setup.

Capacity-wise weakened weak axiom of revealed preference (CWWARP): For each $S, S' \in \mathcal{A}$, $q, q' \in \{1, \ldots, n\}$, and each pair $a, b \in S$ such that $a, b \in S \cap S'$,

$$\text{if } a \in C(S, q) \text{ and } b \in C(S', q') \setminus C(S, q), \text{ then } a \in C(S', q').$$

Theorem 2. *A choice rule is responsive if and only if it satisfies acceptance and the capacity-wise weakened weak axiom of revealed preference.*

Proof. It is clear that a responsive choice rule satisfies *acceptance* and *CWWARP*. Let C be a choice rule satisfying *acceptance* and *CWWARP*. Clearly, *CWWARP* implies *WWARP*, and therefore by Chambers and Yenmez (2016), for each $q \in \{1, \ldots, n\}$, there is a priority ordering $\succ^q$ such that for each $S \in \mathcal{A}$, $C(S, q)$ is obtained by choosing the highest $\succ^q$-priority alternatives until the capacity q is reached or no alternative is left.

Let $(S, q) \in \mathcal{A} \times \{1, \ldots, n\}$. If $|S| \leq q$, then by *acceptance*, $C(S, q) = S$. Suppose that $|S| > q$. First note that $C(S, q-1) \subseteq C(S, q)$, since otherwise, by *acceptance*, there is a pair $a, b \in S$ such that $a \in C(S, q-1) \setminus C(S, q)$ and $b \in C(S, q) \setminus C(S, q-1)$, which contradicts *CWWARP*. Now, consider any pair $a, b \in R(S, q-1)$ such that $a \in C(S, q)$ and $b \notin C(S, q)$. By *CWWARP*, for any $S' \in \mathcal{A}$, b is not chosen over a at (S', q), implying that a has $\succ^q$-priority over b. But then, for each $S \in \mathcal{A}$, $C(S, q)$ is obtained by choosing the highest $\succ^{q-1}$-priority alternatives until the capacity q is reached or no alternative is left. Since we started with an arbitrary $q \in \{1, \ldots, n\}$, C is a choice rule that is responsive to $\succ^1$. $\square$

Given *acceptance*, to better highlight the gap between *WWARP* and *CWWARP*, we introduce the following property. The property requires that the choice from a choice set at a given capacity should not change if the choice is made in two steps: first, choosing at a lower capacity, and then choosing from among the remaining alternatives at the remaining capacity.

Composition up: For each $S \in \mathcal{A}$ and $q, q' \in \{1, \ldots, n\}$ such that $q' > q$,

$$C(S, q') = C(S, q) \cup C(S \setminus C(S, q), q' - q).$$

Proposition 2. *Let C be a choice rule satisfying acceptance. The choice rule C satisfies the capacity-wise weakened weak axiom of revealed preference if and only if it satisfies the weakened weak axiom of revealed preference and composition up.*

Proof. Suppose that C satisfies *CWWARP*. Then, it clearly satisfies *WWARP*. Suppose that $S \in \mathcal{A}$ and $q, q' \in \{1, \ldots, n\}$ are such that $q' > q$ and

$$C(S, q') \neq C(S, q) \cup C(S \setminus C(S, q), q' - q).$$

Then, by *acceptance*, either there exist $a, b \in A$ such that

$$a \in C(S, q) \setminus C(S', q') \text{ and } b \in C(S', q') \setminus C(S, q)$$

or there exist $a, b \in A$ such that

$$a \in C(S \setminus C(S, q), q' - q) \setminus C(S', q') \text{ and } b \in C(S', q') \setminus C(S \setminus C(S, q), q' - q).$$

In either case, we have a contradiction to *CWWARP*. Hence, C satisfies *composition up*.

Suppose that C satisfies *WWARP* and *composition up*. Let $S, S' \in \mathcal{A}$, $q, q' \in \{1, \ldots, n\}$, and $a, b \in A$ be such that $a, b \in S \cap S'$, $a \in C(S, q)$, and $b \in C(S', q') \setminus C(S, q)$. Suppose that $a \notin C(S', q')$. If $q = q'$, we have a contradiction to *WWARP*. Suppose, without loss of generality, that $q' > q$. Since $a \in C(S, q)$, by *composition up*, $a \in C(S, q')$. Since $b \notin C(S, q)$, by *composition up*, $b \notin C(S, q')$. But then, $a \in C(S, q')$, $b \in C(S', q') \setminus C(S, q')$, and $a \notin C(S', q')$, which is a contradiction to *WWARP*. Hence, C satisfies *CWARP*. $\square$

Corollary 2. *A choice rule is responsive if and only if it satisfies acceptance, weakened weak axiom of revealed preference, and composition up.*

The following example shows that, without *acceptance*, *CWWARP* does not imply *composition up*.

Example 1. *Let $A = \{a, b, c\}$. Let $\succ$ be a priority ordering. Let C be the choice rule such that, for each problem (S, q), $C(S, q)$ is a singleton consisting of the $\succ$-maximal alternative in S if $q \in \{1, 3\}$; and C coincides with the choice rule that is responsive for $\succ$ when $q = 2$. Note that C violates acceptance since it chooses a single alternative from any choice set when the capacity is 3. Moreover, C clearly satisfies CWWARP but violates composition up.*

3 Implications for School Choice in Boston

In the Boston school choice system, for each school there are two different priority orderings: a *walk-zone priority ordering*, which gives priority to the school's neighborhood students over all the other students, and an *open priority ordering* which does not give priority to any student for being a neighborhood student. The Boston school district aims to assign half of the seats of each school based on the walk-zone priority ordering and the other half based on the open priority ordering.

To better understand what the Boston school district wants to achieve and how it can be achieved, let us consider the following class of choice rules that is larger than the class of lexicographic choice rules. We say that a choice rule is **capacity-wise lexicographic** if, at each capacity, the rule operates based on a list containing as many priority orderings as the capacity. Unlike a lexicographic choice rule, the lists for different capacity levels are not necessarily related.

Now, the Boston school district's objective can be achieved with a capacity-wise lexicographic choice rule such that, at each capacity, the associated list consists of only the walk-zone priority ordering and the open priority ordering, and the absolute difference between the numbers of walk-zone and open priority orderings in the list is at most one. We formalize this property as follows.

Let $\succ^w$ and $\succ^o$ be walk-zone and open priority orderings. We say that a capacity-wise lexicographic choice rule satisfies the **Boston requirement for** $(\succ^w,\succ^o)$ if for each capacity q, the associated list of priority orderings $(\succ_1, \ldots, \succ_q)$ is such that

i. for each $l \in \{1, \ldots, q\}$, $\succ_l \in \{\succ^w, \succ^o\}$,
ii. difference between the number of $\succ^w$-priorities and $\succ^o$-priorities is at most one, i.e. $\left|\sum_{i=1}^q 1_{\succ^w}(\succ_i) - \sum_{i=1}^q 1_{\succ^o}(\succ_i)\right| \leq 1.$[7]

Now, it turns out that the following class of capacity-wise lexicographic choice rules are the only rules satisfying our set of properties together with the Boston requirement for $(\succ^w,\succ^o)$.

Proposition 3. *A capacity-wise lexicographic choice rule satisfies acceptance, gross substitutes, monotonicity, the capacity-wise weak axiom of revealed preference, and the Boston*

[7] $1_x(y)$ is the indicator function which has the value 1 if $x = y$ and 0 otherwise.

requirement for $(\succ^w,\succ^o)$ *if and only if it is lexicographic for a priority profile* $(\succ_1,\ldots,\succ_n)$ *such that*

i. for each $l\in\{1,\ldots,n\}$, $\succ_l\in\{\succ^w,\succ^o\}$,

ii. for each l *that is odd,* $\succ_l=\succ^w$ *if and only if* $\succ_{l+1}=\succ^o$.

Proof. By Theorem 1, a choice rule satisfying the properties must be lexicographic. The rest is straightforward. □

Dur et al. (2016) analyses the School Choice problem in Boston and four plausible choice rules stand out from their analysis, one of which is currently in use in Boston (Open-Walk choice rule). Dur et al. (2016) compare the below four choice rules in terms of how much they are biased for or against the neighbourhood students. We will compare the four choice rules with respect to our set of properties.

1. *Walk-Open Choice Rule:* At each capacity, the first half of the priority orderings in the list are the walk-zone priority ordering and the last half are the open priority ordering.
2. *Open-Walk Choice Rule:* At each capacity, the first half of the priority orderings in the list are the open priority ordering and the last half are the walk-zone priority ordering.
3. *Rotating Choice Rule:* At each capacity, the first priority ordering in the list is the walk-zone priority ordering, the second is the open priority ordering, the third is the walk-zone priority ordering, and so on.
4. *Compromise Choice Rule:* At each capacity, the first quarter of the priority orderings in the list are the walk-zone priority ordering, the following half of the priority orderings in the list are the open priority ordering, and the last quarter are again the walk-zone priority ordering.

To be precise, let us introduce the following procedures to accommodate the cases where the capacity is not divisible by two or four.

- *Walk-Open Choice Rule:* If the capacity q is an odd number, the first $\frac{q+1}{2}$ are the walk-zone priority ordering.
- *Open-Walk Choice Rule:* If the capacity q is an odd number, the first $\frac{q+1}{2}$ are the open priority ordering.

- *Compromise Choice Rule:* If the capacity q is not divisible by four, let $q = q' + k$ for some $k \in \{1, 2, 3\}$. If $k = 1$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2}$ orderings be the open priority ordering, and the last $\frac{q'}{4}$ orderings be the open priority ordering. If $k = 2$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2} + 1$ orderings be the open priority ordering, and the last $\frac{q'}{4}$ orderings be the open priority ordering. If $k = 3$, let the first $\frac{q'}{4} + 1$ orderings be the open priority ordering, the following $\frac{q'}{2} + 1$ orderings be the open priority ordering, and the last $\frac{q'}{4} + 1$ orderings be the open priority ordering.

Note that all of the above rules satisfy the Boston requirement for $(\succ^w, \succ^o)$. It is clear from Proposition 3 that the Rotating Choice Rule satisfies *acceptance*, *gross substitutes*, *monotonicity*, and the *CWARP*. We show that the other three rules violate *CWARP* while all the four rules satisfy *monotonicity*.

We first show that each one of the four rules satisfies *monotonicity*. In fact, we show that a larger class of choice rules satisfies *monotonicity*.

Let $\pi = (\succ_1, \ldots, \succ_q)$ and $\pi' = (\succ'_1, \ldots, \succ'_{q+1})$ be priority lists of size q and $q+1$, respectively. We say that π' is **obtained by insertion** from π if there exists $k \in \{1, \ldots, q+1\}$ such that $\succ'_l = \succ_l$ for each $l < k$, and $\succ'_l = \succ_{l-1}$ for each $l > k$. Note that when π' is obtained by insertion from π, a new priority ordering is inserted into the list of priority orderings in π, by keeping relative order of the other priority orderings in the list the same. It is possible that the new ordering is inserted to the very beginning or to the very end of the list.

Proposition 4. *Let C be a capacity-wise lexicographic choice rule. The choice rule C is monotonic if for each $q \in \{2, \ldots, n\}$, the priority list for q is obtained by insertion from the priority list for $q - 1$.*

Proof. Let $(S, q) \in \mathcal{A} \times \{1, \ldots, n-1\}$. Let $\pi = (\succ_1, \ldots, \succ_q)$ be the list for capacity q. Let $a \in C(S, q)$. Suppose that, in the lexicographic choice procedure, a is chosen at the t'th step, i.e. a is chosen based on $\succ_t$.

Let $\pi' = (\succ'_1, \ldots, \succ'_{q+1})$ be the list for capacity $q + 1$. Note that π' is obtained by insertion from π. Let $k \in \{1, \ldots, q+1\}$ be such that $\succ'_l = \succ_l$ for each $l < k$, and $\succ'_l = \succ_{l-1}$ for each $l > k$.

Now, consider the problem $(S, q+1)$. If $t < k$, clearly a is still chosen at the t'th step of the lexicographic choice procedure and thus $a \in C(S, q+1)$. Suppose that $t \geq k$. The rest of the proof is by induction. First, suppose that $t = k$. Note that at Step k of the choice procedure for the problem $(S, q+1)$, the choice is made based on the inserted priority ordering and at Step $k+1$, the choice is made based on $\succ_t$. Then, a is either chosen at Step k, or at Step $k+1$, the set of remaining alternatives is a subset of the set of remaining alternatives at Step t of the choice procedure for (S, q) where a is chosen, in which case a is still chosen. Thus, $a \in C(S, q+1)$.

Now, suppose that $t > k$ and each alternative that is chosen at a step $t' < t$ of the choice procedure at (S, q) is also chosen at $(S, q+1)$. Then, a is either chosen before step $t+1$ of the choice procedure for $(S, q+1)$, or at Step $t+1$, the set of remaining alternatives is a subset of the set of remaining alternatives at Step t of the choice procedure for (S, q) where a is chosen, in which case a is still chosen. Thus, $a \in C(S, q+1)$. $\square$

It is easy to see that each of the four choice rules satisfies the insertion property, which yields the following corollary.

Corollary 3. *Each one of the four rules satisfies acceptance, gross substitutes, and monotonicity.*

Proof. Each rule is capacity-wise lexicographic (lexicographic for a given capacity) and therefore satisfies *acceptance* and *gross substitutes*. *Monotonicity* follows by Proposition 4. $\square$

Proposition 5. *Among the four rules, only the rotating choice rule satisfies the capacity-wise weak axiom of revealed preference and only the rotating choice rule is lexicographic.*

Proof. Consider $(\succ_1, \ldots, \succ_n) \in \Pi$ such that the first priority ordering in the list is $\succ^w$, the second is $\succ^o$, the third is $\succ^w$, and so on. The rotating choice rule is clearly lexicographic for $(\succ_1, \ldots, \succ_n)$. Moreover, by Theorem 1, it satisfies *CWARP*. We will show that each of the other three choice rules violates *CWARP*.

Walk-Open Choice Rule: Let $A = \{a, b, c, d, e\}$. Let $\succ^w$ be defined as $a \succ^w e \succ^w b \succ^w c \succ^w d$ and $\succ^o$ be defined as $d \succ^o e \succ^o c \succ^o b \succ^o a$. Note that $C(\{a, b, c, d, 2\}) = \{a, d\}$ and $C(\{a, b, c, d, 3\}) = \{a, b, d\}$, and therefore b is revealed preferred to c at $q = 3$.

Moreover, $C(\{a,b,c,e,2\}) = \{a,e\}$ and $C(\{a,b,c,e,3\}) = \{a,c,e\}$, and therefore c is revealed preferred to b at $q = 3$, implying that C violates *CWARP*.

Open-Walk Choice Rule: Can be shown by interchanging the orderings for $\succ^w$ and $\succ^o$ in the previous example.

Compromise Choice Rule: Let $A = \{a,b,c,d,x,y\}$. Let $\succ^w$ be defined as $a \succ^w b \succ^w c \succ^w d \succ^w x \succ^w y$ and $\succ_o$ be defined as $b \succ^o c \succ^o y \succ^o x \succ^o d$. Note that $C(\{a,b,c,x,y,3\}) = \{a,b,c\}$ and $C(\{a,b,c,x,y,4\}) = \{a,b,c,x\}$, and therefore x is revealed preferred to y at $q = 4$. Moreover, $C(\{a,b,d,x,y,3\}) = \{a,b,d\}$ and $C(\{a,b,d,x,y,4\}) = \{a,b,d,y\}$, and therefore y is revealed preferred to x at $q = 4$, implying that C violates *CWARP*.

□

Remark 3. Note that the particular procedures we introduced to accommodate the cases where the capacity is not divisible by two or four are not crucial for the proof of Proposition 5. For the other procedures (for example, for the walk-open choice rule, the extra priority when the capacity is odd can alternatively be set to be the open priority ordering), the examples in the proof can simply be modified to show that *CWARP* is still violated.

Our analysis shows that if *CWARP* is deemed desirable, then the rotating choice rule should be selected since it is the only choice rule among the four plausible alternatives that satisfies *CWARP*.

4 Implications for Resource Allocation

Let N denote a finite set of agents, $|N| = n \geq 2$. Let $\mathcal{A}$ be the collection of all nonempty subsets of N. Let O denote a finite set of objects. Each agent $i \in N$ has a complete, transitive, and anti-symmetric preference relation R_i over $O \cup \{\emptyset\}$, where $\emptyset$ is the null object representing the option of receiving no object (or receiving an outside option). Given $x, y \in O \cup \{\emptyset\}$, $x \; R_i \; y$ means that either $x = y$ or $x \neq y$ and agent i prefers x to y. If agent i prefers x to y, we write $x \; P_i \; y$. Let $\mathcal{R}$ denote the set of all preference relations over $O \cup \{\emptyset\}$, and $\mathcal{R}^N$ the set of all preference profiles $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

An allocation problem with capacity constraints, or simply a **problem**, consists of a profile $R \in \mathcal{R}^N$ and a capacity vector $q = (q_x)_{x \in O \cup \{\emptyset\}}$ such that for each object $x \in O$, $q_x \in \{0, 1, \ldots, n\}$ and $q_\emptyset = n$ so that the null object has enough capacity to accommodate all agents. Let $\mathcal{P}$ denote the set of all problems. Given a problem $(R, q) \in \mathcal{P}$, an object x is **available** at the problem if $q_x > 0$.

Given a problem $(R, q) \in \mathcal{P}$, an allocation assigns to each agent exactly one object in $O \cup \{\emptyset\}$ taking capacity constraints into account. Formally, an **allocation** at (R, q) is a list $a = (a_i)_{i \in N}$ such that for each $i \in N$, $a_i \in O \cup \{\emptyset\}$ and no object $x \in O \cup \{\emptyset\}$ is assigned to more than q_x agents. Let $M(R, q)$ denote the set of all allocations at (R, q).

Given an allocation $a = (a_i)_{i \in N}$, a preference profile R, and an object $x \in O \cup \{\emptyset\}$, let $D_x(a, R) = \{i \in N : x \; P_i \; a_i\}$ denote the **demand** for x at (a, R), which is the set of agents who prefer x to their assigned object.

4.1 Lexicographic Deferred Acceptance Mechanisms

A **mechanism** is a function $\varphi : \mathcal{P} \to \mathcal{A}$ such that for each allocation problem $(R, q) \in \mathcal{P}$, $\varphi(R, q) \in M(R, q)$. For mechanisms, we introduce a new property, called *demand-monotonicity*. To introduce demand-monotonicity, consider a problem in which there is only one available object. Next, suppose that the capacity of the object is increased. Now, some of the agents who initially did not receive the object may receive it, that is, some agents may receive the object due to the capacity increase. Demand monotonicity requires that the set of agents who receive the object due to the capacity increase does not depend on the set of agents who initially receive the object. In other words, for two problems with a common capacity, if the demands for the only available object are the same, then whenever the capacity of the object increases, the sets of agents who receive the object due to the capacity increase should be the same for the two problems.

Formally, for each $x \in O$, let 1_x be the capacity profile which has 1 unit of x and nothing else. A mechanism φ satisfies **demand-monotonicity** if for each pair of problems (R, q) and (R', q) and each object $x \in O$, if for each $y \in O \setminus \{x\}$, $q_y = 0$ and $D_x(\varphi(R, q), R) = D_x(\varphi(R', q), R')$, then $D_x(\varphi(R, q + 1_x), R) = D_x(\varphi(R', q + 1_x), R')$.

A **lexicographic choice structure** $\mathcal{C} = (C_x)_{x \in O}$ associates each object $x \in O$ with a lexicographic choice rule $C_x : \mathcal{A} \times \{1, \ldots, n\} \to \mathcal{A}$. Next, we present **the lexicographic**

deferred acceptance algorithm based on $\mathcal{C}$. For each problem $(R,q) \in \mathcal{P}$, the algorithm runs as follows:

Step 1: Each agent applies to his favorite object in O. Each object $x \in O$ such that $q_x > 0$ temporarily accepts the applicants in $\mathcal{C}_x(S_x, q_x)$ where S_x is the set of agents who applied to x, and rejects all the other applicants. Each object $x \in C$ such that $q_x = 0$ rejects all applicants.

Step $r \geq 2$: Each applicant who was rejected at step $r-1$ applies to his next favorite object in O. For each object $x \in C$, let $S_{x,r}$ be the set consisting of the agents who applied to x at step r and the agents who were temporarily accepted by x at Step $r-1$. Each object $x \in O$ such that $q_x > 0$ accepts the applicants in $\mathcal{C}_x(S_{x,r}, q_x)$ and rejects all the other applicants. Each object $x \in O$ such that $q_x^r = 0$ rejects all applicants.

The algorithm terminates when each agent is accepted by an object. The allocation where each agent is assigned the object that he was accepted by at the end of the algorithm is called the C-lexicographic Deferred Acceptance allocation at (R,q), denoted by $DA^{\mathcal{C}}(R,q)$.

Lexicographic deferred acceptance mechanisms: A mechanism φ is a *lexicographic deferred acceptance mechanism* if there exists a lexicographic choice structure $\mathcal{C}$ such that for each $(R,q) \in \mathcal{P}$, $\varphi(R,q) = DA^{\mathcal{C}}(R,q)$.

Proposition 6. *Each lexicographic deferred acceptance mechanism satisfies demand-monotonicity.*

Proof. Let $\mathcal{C} = (C_x)_{x \in O}$ be a lexicographic choice structure. Let $(R,q),(R',q) \in \mathcal{P}$ and $x \in O$ be such that for each $y \in O \setminus \{x\}$, $q_y = 0$ and $D_x(DA^{\mathcal{C}}(R,q),R) = D_x(DA^{\mathcal{C}}(R',q),R') = T$. Let C_x be lexicographic for the priority profile $(\succ_1,\ldots,\succ_n) \in \Pi$. Let $S(R)$ and $S(R')$ be the sets of agents who prefer x to $\emptyset$ at R and at R', respectively. It is easy to see that $DA^{\mathcal{C}}(R,q) = C_x(S(R))$, $DA^{\mathcal{C}}(R',q) = C_x(S(R'))$, and $T = S(R) \setminus C_x(S(R)) = S(R') \setminus C_x(S(R'))$. Let $i \in T$ be the agent who is highest ranked according to $\succ_{q_x+1}$ in T. Clearly, $DA^{\mathcal{C}}(R,q') = DA^{\mathcal{C}}(R,q) \cup \{i\}$ and $DA^{\mathcal{C}}(R',q') = DA^{\mathcal{C}}(R',q) \cup \{i\}$. Hence, $D_x(DA^{\mathcal{C}}(R,q'),R) = D_x(DA^{\mathcal{C}}(R',q'),R') = T \setminus \{i\}$. □

Ehlers and Klaus (2016), in their Theorem 3, characterize deferred acceptance mechanisms based on a choice structure satisfying *acceptance*, *gross substitutes*, and *monotonicity*, with the following properties of mechanisms: unavailable-type-invariance (if the positions of the unavailable types are shuffled at a profile, then the allocation should not change); weak non-wastefulness (no agent receives the null object while he prefers an object that is not exhausted to the null object), resource-monotonicity (increasing the capacities of some objects does not hurt any agent), truncation-invariance (if an agent truncates his preference relation in such a way that his allotment remains acceptable under the truncated preference relation, then the allocation should not change), and strategy-proofness (no agent can benefit by misreporting his preferences).[8]

Theorem 3. *(Ehlers and Klaus, 2016) A mechanism is a deferred acceptance mechanism based on a choice structure satisfying acceptance, gross substitutes, and monotonicity if and only if it satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.*

Corollary 4. *A mechanism is a lexicographic deferred acceptance mechanism if and only if it satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, strategy-proofness, and demand-monotonicity.*

Proof. The following notation will be helpful. For each $x \in O$, let R^x be a preference relation such that x is top-ranked and $\emptyset$ is second-ranked. For each $S \in \mathcal{A}$ that is nonempty, let R^x_S be a preference profile such that for each $i \in S$, $R_i = R^x$, and for each $j \notin S$, R_j top-ranks $\emptyset$. For each $x \in O$ and $l \in \{0, \ldots, n\}$, let l_x denote the capacity profile where x has capacity l and every other object has capacity zero.

Let φ be a mechanism satisfying the properties in the statement of the theorem. Let $\mathcal{C} = (C_x)_{x \in O}$ be defined as follows. For each $x \in O$, $S \in \mathcal{A}$, and $l \in \{0, \ldots, n\}$, $C_x(S, l) = \{i \in S : \varphi_i(R^x_S, l_x) = x\}$.

In their proof of Theorem 3, (Ehlers and Klaus, 2016) show that if φ satisfies *unavailable-type-invariance*, *weak non-wastefulness*, *resource-monotonicity*, *truncation-invariance*, and *strategy-proofness*, then for each $x \in O$, C_x satisfies *acceptance*, *gross substitutes*, and *monotonicity*. Moreover, φ is a deferred acceptance mechanism based on $\mathcal{C}$. It is easy to see that, since φ satisfies *demand-monotonicity*, for each $x \in O$, C_x satisfies *rejection-monotonicity*. Thus, $\mathcal{C}$ is a lexicographic choice structure and φ is a lexicographic deferred acceptance mechanism.

[8] See Ehlers and Klaus (2016) for the formal definitions of the properties.

Let φ be a lexicographic deferred acceptance mechanism. *Demand-monotonicity* follows from Proposition 6. The other properties follow from Theorem 3 of Ehlers and Klaus (2016). $\square$

Remark 4. We give an example of a mechanism which satisfies all the properties in the statement of Corollary 4 except for *demand-monotonicity*, and therefore which is not a lexicographic deferred acceptance mechanism. The mechanism in the example is a deferred acceptance mechanism based on a choice structure such that the choice rule of each object is a walk-open choice rule. The example uses some arguments from the proof of Proposition 5, where it was shown that the walk-open choice rule violates *CWARP*.

Example 2. *Let $N = \{a, b, c, d, e\}$ and let O be a finite set of objects. Let $\succ^w$ be defined as $a \succ^w e \succ^w b \succ^w c \succ^w d$ and $\succ^o$ be defined as $d \succ^o e \succ^o c \succ^o b \succ^o a$. Let $(C_x)_{x\in O}$ be the choice structure such that for each object $x \in O$, C_x is the walk-open choice rule based on $(\succ^w, \succ^o)$. Let φ be the deferred acceptance mechanism based on the choice structure $(C_x)_{x\in O}$.*

Since for each $x \in O$, C_x satisfies acceptance, gross substitutes, and monotonicity, by Theorem 3, φ satisfies unavailable-type-invariance, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.

Let $x \in O$. Let q be such that $q_x = 2$ and for each $y \in O \setminus \{x\}$, $q_y = 0$. Let R be such that x is preferred to $\emptyset$ for all the agents except for e. Note that $D_x(\varphi(R, q), R) = \{b, c\}$ since $C_x(\{a, b, c, d, 2\}) = \{a, d\}$. Let R' be such that x is preferred to $\emptyset$ for all the agents except for d. Note that $D_x(\varphi(R', q), R') = \{b, c\}$ since $C_x(\{a, b, c, e, 2\}) = \{a, e\}$. Thus, $D_x(\varphi(R, q), R) = D_x(\varphi(R', q), R')$.

Now, note that $D_x(\varphi(R, q + 1_x), R) = \{c\}$ since $C_x(\{a, b, c, d, 3\}) = \{a, b, d\}$ and $D_x(\varphi(R', q + 1_x), R') = \{b\}$ since $C_x(\{a, b, c, e, 3\}) = \{a, c, e\}$. Hence, φ violates demand-monotonicity.

Remark 5. A property that is stronger than *demand-monotonicity* is the following. A mechanism φ satisfies **strong demand-monotonicity** if for each pair of problems (R, q) and (R', q) and each object $x \in O$, if $D_x(\varphi(R, q), R) = D_x(\varphi(R', q), R')$, then $D_x(\varphi(R, q + 1_x), R) = D_x(\varphi(R', q + 1_x), R')$. Clearly, *strong demand-monotonicity* implies *demand-monotonicity*. The following example shows a lexicographic deferred acceptance mechanism (in fact, a *classical* deferred acceptance mechanism based on a priority profile, which is lexicographic for a priority profile where all the priority orderings are the same) that violates *strong demand-monotonicity*.

Example 3. *Let* $A = \{1,2,3\}$*. Let* $O = \{a,b,c\}$*. Let* $\succ_a$ *be defined as* $1 \succ_a 2 \succ_a 3$*,* $\succ_b$ *be defined as* $2 \succ_a 3 \succ_a 1$*, and* $\succ_c$ *be defined as* $1 \succ_a 2 \succ_a 3$*. Let* $\mathcal{C} = (C_x)_{x \in O}$ *be a lexicographic choice structure such that for each* $x \in O$*,* C_x *is lexicographic for the priority profile* $(\succ_x, \succ_x, \succ_x)$*. Note that* DA^C *is a classical deferred acceptance mechanism based on a priority profile. Let the preference profiles* R *and* R' *be as depicted below.*

R_1	R_2	R_3
a	a	b
b	b	a
c	c	c
$\emptyset$	$\emptyset$	$\emptyset$

R'_1	R'_2	R'_3
a	a	a
b	b	c
c	c	b
$\emptyset$	$\emptyset$	$\emptyset$

Let $q = (q_a, q_b, q_c) = (1,1,1)$ *and* $q' = (q'_a, q'_b, q'_c) = (2,1,1)$*. Note that* $D_a(DA^{\mathcal{C}}(R,q),R) = D_a(DA^{\mathcal{C}}(R',q),R') = \{2,3\}$*. However,* $D_a(DA^{\mathcal{C}}(R,q'),R) = \{\emptyset\}$ *and* $D_a(DA^{\mathcal{C}}(R',q'),R') = \{3\}$*. Thus,* $DA^{\mathcal{C}}$ *is a lexicographic deferred acceptance mechanism but violates strong demand-monotonicity.*

5 Conclusion

Our formulation of a choice rule and the properties that we consider take into account that the capacity may vary. Such choice rules as well as properties that take into account that resources may vary (a well-known such property is *resource-monotonicity*, introduced by Chun and Thomson (1988)) have already been studied in the resource allocation literature (see, for example, Doğan and Klaus (2016), Ehlers and Klaus (2014), and Ehlers and Klaus (2016)). When designing choice rules especially for resource allocation purposes, such as in school choice, a designer may be interested in how the choice rule responds to changes in capacity. In that framework, our Theorem 1 shows that *acceptance*, *gross substitutes*, *monotonicity*, and *CWARP*, which may be of interest to a designer, are altogether satisfied only by lexicographic choice rules.

A lexicographic choice rule in our setup operates based on a unique list containing as many priority orderings as the maximum possible capacity. Alternatively, one could consider capacity-wise lexicographic choice rules that we have defined in Section 3, which, at each capacity, operate based on a list containing as many priority orderings as the capacity, yet the lists for different capacity levels do not have to be related in any way. A characterization of capacity-wise lexicographic choice rules, or in other words characterizing lexicographic choice rules for a fixed capacity, is also an important step in the analysis of lexicographic choice rules, which we do not answer in our paper.

We have shown that, if a designer wants to use a rule that is capacity-wise lexicographic, and if in addition finds *monotonicity* and *CWARP* desirable, than the lists for different capacity levels must be related in a very particular way: the list associated with any capacity level q must consist of the first q priority orderings in the list associated with the maximum possible capacity. In applications, this finding may be helpful to select among possible capacity-wise lexicographic choice rules, as we have illustrated in Section 3.

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Appendix

A Independence of Properties in Theorem 1

Violating only acceptance: Let $A = \{a, b, c\}$. Let $\succ$ be a priority ordering. Let C be the choice rule such that, for each problem (S, q), $C(S, q)$ is a singleton consisting of the $\succ$-maximal alternative in S. Note that C violates *acceptance* and clearly satisfies *gross substitutes*. Since the choice does not vary with capacity, C also satisfies *monotonicity* and *CWARP*.

Violating only gross substitutes: Let $A = \{a, b, c\}$. Let $\succ$ and $\succ'$ be defined as $a \succ b \succ c$ and $b \succ' a \succ' c$. Let the choice rule C be defined as follows. For each problem (S, q), $C(S, q)$ consists of the $\succ$-maximal alternative in S if $q = 1$ and $c \in S$; otherwise, $C(S, q)$ coincides with the choice rule that is responsive for $\succ_2$. Note that C satisfies *acceptance*.

Since $a \in C(\{a, b, c\}, 1) = \{a\}$ and $a \notin C(\{a, b\}, 1) = \{b\}$, C violates *gross substitutes*. To see that C satisfies *monotonicity*, first note that there is a unique choice set S such that $C(S, 2) \neq \emptyset$, which is $S = \{a, b, c\}$. Therefore, the only possibility to violate monotonicity is to have $x \in A$ such that $x \in C(\{a,b,c\},1)$ and $x \notin C(\{a,b,c\},2)$. Since $C(\{a,b,c\},1) = \{a\}$ and $C(\{a,b,c\},2) = \{a,b\}$, C satisfies *monotonicity*. To see that C satisfies *CWARP*, note the revealed preference relation at $q = 2$ consists of a unique pair: b is revealed preferred to c.

Violating only monotonicity: Let $A = \{a,b,c\}$. Let $\succ$ be defined as $a \succ b \succ c$. Let the choice rule C be defined as follows. For each problem (S,q), $C(S,q)$ consists of the $\succ$-maximal alternative in S if $q = 1$; $C(S,2) = S$ if $|S| = 2$; and $C(\{a,b,c\},2) = \{b,c\}$. Note that C satisfies *acceptance*.

Since $a \in C(\{a,b,c\},1)$ and $a \notin C(\{a,b,c\},2)$, C violates monotonicity. For $q = 1$, C satisfies *gross substitutes*, since C maximizes $\succ$; for $q \in \{2,3\}$, C clearly satisfies *gross substitutes*. To see that C satisfies *CWARP*, note that the revealed preference relation is empty at $q = 2$, since $C(\{a,b,c\},1) = \{a\}$ and $C(\{a,b,c\},2) = \{b,c\}$.

Violating only CWARP: Note that three of the four rules that we have discussed in Section 3 satisfy all the properties but *CWARP*.