The Government Spending Multiplier in a (Mis-)Managed Liquidity Trap

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March 14, 2017

Abstract

I study the impact of a government spending shock in a New Keynesian model when monetary policy is set optimally. In this framework, the economy is at the Zero Lower Bound but expectations are well managed by the Central Bank. As such, the multiplier effect of government spending increases on expected inflation is close to zero while the one on output can be larger than one. This is consistent with recent empirical evidence on the effects of the 2009 American Recovery and Reinvestment Act.

Keywords: Zero lower bound, New Keynesian, Government spending multiplier.

JEL Classification: E31, E32, E52, E62

Acknowledgments: I thank A.Tischbirek for comments on an early version of this manuscript. Some of the codes used in this paper are based on those written for Bouakez, Guillard, and Roulleau-Pasdeloup [2015]. I thank my co-authors for allowing me to use material from our joint work.
1 Introduction

After the effects of the American Recovery and Reinvestment Act of 2009 have passed and the dust has settled on the policy debate, there is still a lively debate among academics about the effectiveness of fiscal policy. Mostly after it had been enacted, the fiscal stimulus plan was hailed as effective by many academics because it was enacted at a time when the economy was in a liquidity trap with the Fed Funds rate close to zero. In the seminal contributions of Eggertsson [2010] and Christiano et al. [2011], such a policy was shown to be efficient in stimulating output in a depressed economy.

Because higher government spending would generate more employment and lower the level of slack in the economy, it would generate higher inflation. In turn, higher inflation would reduce the real interest rate and prompt an increase in aggregate demand, thereby boosting a depressed economy. This transmission mechanism is at the heart of the New Keynesian model with sticky prices. Along with the workhorse New Keynesian model itself, much doubt has been cast about the empirical relevance of the tight link between inflation and recession/recovery.

In particular, such a model will predict that the economy falls in a liquidity trap after a negative aggregate demand shock generates a large deflation. While the U.S. only experienced a mild deflation in 2009, in the following years inflation was hovering below the stated objective of 2% as the economy was still depressed. This has prompted a new wave of research trying to understand if the New Keynesian model could be amended to fit the facts.¹

On the fiscal policy front, Dupor and Li [2015] show that the American Recovery and Reinvestment Act did not spur a rise in expected inflation. From the standard New Keynesian model, an absence of inflation after a government spending shock in a liquidity trap would mean that the multiplier effect of government spending is likely to be way smaller than what can be found in Eggertsson [2010] and Christiano et al. [2011].

The objective of this paper is to reconcile a baseline New Keynesian model with observed empirical evidence; namely that (i) the deflation in the early stages of the recession was mild and short-lived and (ii) the inflationary effects of higher government spending were also pretty mild. To do so, I will show that in this model, there is no such thing as an invariant "government spending

¹See for example Ball and Mazumder [2011], Hall [2011], Gordon [2013], Del-Negro et al. [2014], Christiano et al. [2015] and Coibion and Gorodnichenko [2015].
multiplier at the zero lower bound”. It is well known that, outside a liquidity trap, the stimulative properties of government spending depend a lot on the stance of monetary policy. But this statement is also true in a liquidity trap. If the central bank is able to commit and the economy is hit by a large negative aggregate demand shock, it will be optimal to set the interest rate to zero for an extended period of time. While the economy will formally be at the zero lower bound, the central bank is able to commit and promises credibly to generate a boom in output and inflation when the natural interest rate becomes positive. As such, expected inflation is well anchored when the economy is in a liquidity trap.

The model that I develop in this paper has two main implications that are consistent with recent empirical evidence. First, the model is also able to replicate the fact that inflation did fall initially by a small amount and remained mostly positive in the subsequent quarters (point (i)). As such, the absence of a long-lasting Great Depression style deflation is no prima facie evidence that the New Keynesian model is wrong. In sharp contrast with the early stages of the Great Depression, it could mean that the Fed did a good job in managing expectations. In any case, the model in which monetary policy is optimal produces a path for inflation (initial deflationary blip, then slightly positive inflation) that resembles very much actual U.S. experience after the Great Recession. As such, it makes for a better environment to study the impact of a government spending shock compared to a model with a Central Bank that follows a Taylor rule.2

Then, I show that an increase in government spending when the central bank is behaving optimally has much less effect on expected inflation (point (ii)) than when the central bank is simply following a Taylor rule. Another result is that while the effect on expected inflation is smaller, this does not mean that the multiplier effect of government spending is negligible. In fact, while the effect on inflation will be very different in the two cases, the effect on output will be rather close. It follows that it is not enough to show that the American Recovery and Reinvestment Act did not generate expected inflation to conclude that such a policy is misguided and did not have large effects on output, at least in the short run. Theses results are robust to a variety of robustness and sensitivity exercises. If one looks at the medium to long run through the cumulative multiplier, the dampened effect on inflation is again robust. The effect on output is also qualitatively similar, except in the model with private capital and investment adjustment costs. In sum, a fiscal stimulus while the Central Bank is actively managing expectations will be able to stimulate the

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2In recent work, Cochrane [2013] also argues that the Taylor rule might not be the best tool to study monetary policy.
economy in the short run without having a noticeable effect on inflation. A bit of caution is however warranted regarding the long-run stimulative effects of such as stimulus.

This paper is related to two recent contributions that focus mainly on the information structure and deliver similar properties for government spending increases in a liquidity trap. Wiederholt [2015] shows that in a New Keynesian model with information frictions on the households' side, an increase in government spending reveals to some households that the economy is in the bad state, thereby muting the increase in inflation. His model is also able to replicate the low observed deflation during the Great Recession as households underestimate the probability to be in the bad state on average. In a sticky information setup, Kiley [2016] shows that government spending does not have any effect on the long run price level. After an initial increase in inflation, higher government spending will be associated with deflation as the price level returns to its initial value. The present paper is then a complement to these contributions as it shows that similar features can arise as a result of intended policies.

The paper is structured as follows. In section 2, I develop a standard New Keynesian model and describe both the decentralized-Taylor rule and Ramsey allocations. In section 3, I study the impact and cumulative multiplier effects of government spending in a liquidity trap for these two allocations. Finally, in section 4 I perform a number of robustness and sensitivity exercises. The conclusion follows.

2 The Model

In this section I develop a baseline New Keynesian model with wasteful government spending. As the presence of government spending in the utility function does not influence the dynamics of the model (as long as it is additively separable to private consumption, which is the common assumption in the literature), I leave it out for simplicity. This is also justified on the ground that the focus of this paper is mainly on the positive aspect of government spending, and not the normative one.

This section will be divided in two subsections. In the first one, I present the decentralized equilibrium in which monetary policy is given by a standard Taylor rule. This is the case where the liquidity trap will be mis-managed because the Central Bank is not able to commit and inflation expectations will be
deflationary. In the second subsection, I will assume that the nominal interest rate is set by a Ramsey planner. In this case, the Central Bank is able to commit and the liquidity trap will be well managed. In this setup, inflation expectations will be well anchored. In both setups, government spending is driven by a simple auto-regressive exogenous process.

2.1 Decentralized Equilibrium

There is a representative consumer that gets utility from consumption $C_t$ and disutility from working $N_t$, supplies his labor services to monopolistically competitive firms and earns a real wage in return as well as profits coming from monopolistically competitive firms. He pays lump-sum taxes, has access to one period riskless nominal bonds which yield a nominal interest rate of $R_t$. Importantly, the period utility function is disturbed by a ‘preference shock’ $\xi_t$. Formally, the household’s maximization program is the following:

$$E_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} U(C_{t+s}, N_{t+s}),$$  

subject to the budget constraint:

$$P_t C_t + T_t + \frac{B_t}{1 + R_t} \leq W_t N_t + D_t + B_{t-1}.$$

Optimal choice with respect to consumption and holding of bonds is given by the following Euler equation:

$$U_{C_t} = \beta E_t \frac{\xi_{t+1}}{\xi_t} \frac{U_{C_{t+1}}}{1 + \pi_{t+1}} (1 + R_t),$$

where $U_{C_t}$ is the marginal utility one gets from consuming one more unit of the final good and $\pi_t$ is the CPI inflation rate. Because of the autoregressive structure of $\xi_t$, if there is a negative shock at time $t$, then the ratio $\frac{\xi_{t+1}}{\xi_t}$ will increase, giving more incentives to save and consume less to the household. If this shock is large enough, the Zero Lower Bound will be a binding constraint.

There is a continuum of mass 1 of monopolistically competitive firms. They rent labor services from the representative household and operate a constant return technology using only labor. They face a quadratic cost when adjusting their prices as in Rotemberg [1982], which is proportional to final output.
Firm $z$ chooses its individual price $P_t(z)$ to maximize the following objective function:

$$
\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} \frac{U_{C,t+s}}{U_{C,t}} \left\{ \frac{(1+\tau) P_t(z)}{P_t} Y_t(z) - \frac{W_t}{P_t} N_t(z) - \frac{\psi}{2} \left( \frac{P_t(z)}{P_{t-1}(z)} - 1 \right)^2 \right\},
$$

where $\tau = 1/(\theta - 1)$ and $Y_t$ is aggregate output. Using the fact that every firm is identical and will choose the same price in equilibrium, the first order condition with respect to $P_t(z)$ reduces to the following New Keynesian Phillips Curve:

$$
\psi \pi_t (1 + \pi_t) = \beta \mathbb{E}_t \xi_{t+1} \frac{U_{C,t+1}}{U_{C,t}} \frac{N_{t+1}}{N_t} \psi \pi_{t+1} (1 + \pi_{t+1}) - \frac{\theta U_{N,t}}{U_{C,t}} - \theta,
$$

where $\theta$ is the elasticity of substitution across varieties and $\psi$ governs the size of the price adjustment cost. I have assumed that the subsidy $\tau$ optimally corrects for the absence of a markup on leisure at the steady state. The production function is simply

$$
Y_t = N_t.
$$

In equilibrium, combining the budget constraints of the household and the government (Ricardian equivalence holds and the budget is balanced each period), using the definition of firm’s profits one gets the resource constraint of this economy:

$$
N_t \left( 1 - \frac{\psi}{2} \pi_t^2 \right) = C_t + G_t.
$$

Finally, the model is closed by specifying a rule for the setting of the nominal interest rate. Following much of the literature, I assume that it follows a standard Taylor rule of the form:

$$
1 + R_t = \max \left\{ 1, \frac{1 + \phi_R R^n_t}{\beta} (1 + \pi_t)^{\phi_{\pi}} \left( \frac{Y_t}{Y^n_t} \right)^{\phi_Y} \right\},
$$

where $1/\beta$ is the steady state nominal rate, $R^n_t$ (resp. $Y^n_t$) is the natural rate of interest (resp. output) in this economy, i.e the real interest rate (resp. output level) that would prevail under flexible prices. If $\phi_R = 0$, we have a standard Taylor rule. If otherwise $\phi_R = 1$, then the Central Bank targets the natural rate of interest. Such an interest rule will be useful to highlight how the Ramsey policymaker is able to improve on an interest rule that tries to match the natural rate each period.\(^3\)

\(^3\)Cúrdia et al. [2015] show that a model in which the central bank targets the natural rate of
Government consumption expenses and the preference shock are given by the following $AR(1)$ processes:

\[
G_t = G_{t-1}^\rho G^{1-\rho} \exp(\epsilon_t^\xi),
\]

\[
\xi_t = \xi_{t-1}^{\rho} \exp(\epsilon_t^\xi),
\]

\[
\epsilon_t^j \sim N(0, \sigma_{j,t}^2) \text{ for } j \in \{\xi, g\}
\]

where $G$ is the steady state value of government expenses. I log-linearize all the model equations around the steady state of this economy.

As is by now well known,\(^4\) if the preference shock is large enough, it will generate a negative response of inflation and output large enough to send the economy in a liquidity trap. Once the economy is there, government spending can have a large multiplier effect for two reasons. First, whether or not the economy is in a liquidity trap higher government spending pushes the real wage up due to the increase in labor demand. This raises the marginal cost of the firms and, as a consequence, generates inflation. Secondly, because the shock is persistent, expected inflation also increases after an increase in government spending. Since the nominal rate is stuck at zero, such an increase in expected inflation decreases the expected real interest rate. From the Euler equation, it follows that households want to consume more, which in turn raises inflation, and so and so on.

As is shown in Eggertsson [2010], expected inflation plays a very important role there. Because firms anticipate that inflation is going to be persistently low after the preference shock they decrease their price even more on impact; this aggravates the recession. At the end of the day, government spending does what monetary policy is not able to do anymore: generate expected inflation. What is important here is that, when stuck at the zero lower bound, monetary policy is completely passive. However, what we saw during the current Great Recession in the U.S. is that the successive Chairs of the Federal Reserve were not merely sitting idle. They made announcements to try to influence inflation expectations in the private sector all along.

According to the typology developed in Filardo and Hofmann [2014], the Federal Reserve engaged in open-ended forward guidance\(^5\) from December 2008 interest is better able to fit the data than one with a standard Taylor rule.

\(^4\)See Eggertsson [2010] and Christiano et al. [2011] for two well known examples.

\(^5\)"the Committee anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate for some time". From the FOMC minutes, 16 Dec 2008 as cited in Filardo and Hofmann [2014]
to July 2011. It then switched to calendar-based forward guidance\(^6\) until November 2012. Finally it turned to threshold-based forward guidance\(^7\) until the end of 2016.

As such, being in a liquidity trap is not equivalent with monetary policy being unable to control inflation expectations. As will be clear in the next subsection, it can be optimal for the Ramsey planner to stay at the zero lower bound for some period after the natural interest rate has become positive.

### 2.2 Ramsey Equilibrium

In this subsection, I drop the Taylor rule and assume that the Central Bank has the ability to fully commit to a certain path for the nominal interest rate.\(^8\) Admittedly, an actual Central Bank is unlikely to exhibit such a high degree of commitment. Therefore, this assumption should be viewed as a simple way of capturing the fact that the central bank may have some power to influence the private sector’s expectations during a liquidity trap. The Ramsey planner solves the following maximization program, which I express directly as a

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\(^6\)“the Committee [...] currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015”. From the FOMC minutes, 13 Sep 2012 as cited in Filardo and Hofmann [2014]

\(^7\)“The Committee now anticipates, based on its assessment of these factors, that it likely will be appropriate to maintain the current target range for the federal funds rate well past the time that the unemployment rate declines below 6-1/2 percent”. From the FOMC minutes, 18 Dec 2013 as cited in Filardo and Hofmann [2014]

\(^8\)There is a large literature investigating how government spending should be set optimally over the cycle, see for instance Woodford [2011], Werning [2011], Schmidt [2013], Nakata [2013], Bilbiie et al. [2014], and Nakata [2015]. In related work, Bouakez, Guillard, and Roulleau-Pasdeloup [2015] develop a more general setup in which both public consumption and investment can be set optimally over the cycle. Such a framework however prevents the analysis of discretionary increases in government spending, which is the focus of this paper.
Lagrangian:

\[ L_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t) \right. \\
+ \phi_{1,t} \left[ \mathbb{E}_0 \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1 + \pi_{t+1}} (1 + R_t) - U_{C,t} \right] \\
+ \phi_{2,t} \left[ \beta \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} \frac{Y_{t+1} \psi R_{t+1}}{U_{C,t}} \right. \\
\left. \left. - \psi \pi_{t+1} (1 + \pi_{t+1}) - \psi \pi_t (1 + \pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta \right] \\
+ \phi_{3,t} \left[ N_t \left( 1 - \frac{\psi}{2} \pi_t^2 \right) - C_t \right] \\
+ \phi_{4,t} \left[ R_t - 0 \right] \right\} \]

The equilibrium is then given by equations (3)-(6) together with the first order conditions with respect to \( C_t, N_t, \pi_t \) and \( R_t \) and the complementary slackness condition coming from the Zero Lower Bound constraint. The latter are quite complicated, so they are reported in the appendix.

2.3 Calibration and solution method

The utility function follows Christiano et al. [2011] and is given by:

\[ U(C_t, N_t) = \frac{(C_t^\gamma (1 - N_t)^{1-\gamma})^{1-\sigma}}{1 - \sigma}, \quad \sigma > 1 \] (7)

The model is calibrated at a quarterly frequency. The parameter \( \sigma \) is equal to 2 and \( \gamma \) is set so that \( N = 1/3 \) at steady state. This gives \( \gamma = 0.2857 \). The elasticity of substitution across goods is set to \( \theta = 6 \), which gives a steady state markup of 20\%. Given these parameters, the price adjustment cost parameter is fixed so that it corresponds to a Calvo probability of 0.75 in log linear terms. This gives \( \psi = 90 \). The discount factor is set to \( \beta = 0.99 \). For the Taylor rule, I set \( \phi_{\pi} = 1.5 \) and \( \phi_y = 0.25 \) which are standard values. Finally, I set \( \rho_{\xi} = 0.9 \) for the preference shock so that the economy spends some time at the zero lower bound. Since increases in government spending are usually short-lived, I set \( \rho_g = 0.7 \) for the government spending shock. The share of government spending

\[ \text{In Albertini et al. [2014], we show that with this particular value, the spending increase essentially vanishes after 2 years, which is roughly consistent with the duration of the American Recovery and Reinvestment Act provisions. I will later on study the sensitivity of the results} \]
spending at steady state is calibrated to $g = G/Y = 0.2$ to match the post-WWII average in the U.S.

To solve the model with the occasionally binding constraint, I use the piecewise linear algorithm developed by Guerrieri and Iacoviello [2014]. It consists in estimating two different policy rules: one for the regime when the zero lower bound is binding and one for when it is not. The algorithm allows for endogenous exit of the zero lower bound. If a government spending shock is large enough, it can potentially get the economy out of the liquidity trap.\textsuperscript{10} This is more general than the two state markov process that is usually assumed in this type of framework. Furthermore, Guerrieri and Iacoviello [2014] show that their algorithm delivers approximation errors that are on par with state-of-the-art global projection methods. I compute the Euler equation residuals for the simulations with the baseline model —see Figures 12 and 13 in Appendix 6.1. As expected, the larger the preference shock, the longer the ZLB episode and the higher the residuals become. In my simulations, I make sure that the maximum absolute value of Euler equation residuals is lower than 1\% (in terms of private consumption).

To visualize what optimal policy looks like after a large preference shock, I show the path of the main endogenous variables in this framework and compare it to the decentralized allocation in Figure 1. I calibrate the size of the shock so that the trough in output matches the one observed in the U.S. during the Great Recession, \textit{i.e.} 5.65\%.

The main take-away from Figure 1 is that expected inflation is rather well managed under the optimal policy. As in Eggertsson and Woodford [2003], the central bank commits to keeping the nominal interest rate at its zero lower bound even after the natural interest rate becomes positive again.\textsuperscript{11} In other words, monetary policy is too accommodative during this period, which generates a boom in consumption and inflation as soon as the natural rate of interest becomes positive. This percolates back to the first periods as agents anticipate higher inflation, and thus cut down less on consumption. Because the fall in aggregate demand is dampened, so is the fall in inflation. It is thus not effective to follow blindly the natural rate of interest in this setup.\textsuperscript{12} With this in

\textsuperscript{10}In fact, for my baseline setup this is what happens in the decentralized equilibrium. In the Ramsey equilibrium however, for an increase of government spending of up to 100\%, the ZLB duration is unchanged. These results are available upon request.

\textsuperscript{11}See also Jung et al. [2005] and Adam and Billi [2006].

\textsuperscript{12}If the Central Bank follows a standard Taylor rule, then the nominal interest rate will be pinned at zero for a longer time period than the natural rate is negative. This comes from the fact that this rule generates deflationary expectations which worsen the recession and in-
Figure 1: Ramsey optimal monetary policy and Taylor Rule after a negative preference shock

Notes: Private consumption, inflation and output gap are expressed in percent deviations from steady state level. The nominal interest rate is in level.
mind, I will now use a more standard Taylor rule in the rest of the text.

What is important to note here is that whereas the economy is in a liquidity trap in both allocations in the first 9 periods, what happens is very different. Wu and Xia [2016] derive a shadow policy rate for the Federal Reserve and show that it conveys additional information about the real stance of monetary policy with respect to the Effective Federal Funds Rate. In particular, this shadow rate shows that monetary policy was quite accommodative despite being stuck at the ZLB. This is evidence that one has to go beyond the fact that the nominal rate is zero to gauge the real stance of monetary policy. I reproduce their shadow rate for the period 2007 to 2015 in appendix Figure 9. Since effects of increasing government spending at the zero lower bound rely heavily on inflation expectations, these are poised to be different in the two allocations. As a result, there will be no such thing as ‘the government spending multiplier in a liquidity trap’. As will become clear, even for the same duration of the liquidity trap, government spending is likely to have different effects in both allocations. I take up this issue in the next section.

3 Government spending multipliers in (mis-)managed liquidity traps

The goal here is to compare the reaction of expected inflation and output to a government spending shock in a liquidity trap in the two setups. I proceed as follows: after a preference shock in period 1 (that may or may not be big enough to send the economy in a liquidity trap), government spending increases in period 2. Both of these shocks are surprises and are not anticipated by the Ramsey planner. For increasing values of the initial preference shock shock, I compute the following two objects:

\[
\frac{dY_t}{dG_t} = \frac{(1 - g)\dot{z}_t + g}{g} \quad \text{and} \quad \frac{dE_t \hat{\pi}_{t+1}}{dG_t} = 4 \cdot \frac{\Delta E_t \hat{\pi}_{t+1}}{G},
\]

where hatted variables denote log-deviations from their steady state values and \(\dot{z}_t\) is the difference between the scenario with and without an increase in government spending—which is normalized to 1—for each variable \(z\). Note increases the duration of the trap. The difference with the optimal rule is then quite blurry in this setup, as both keep the nominal rate low for an extended period of time but for very different reasons.

13 The data has been downloaded from the author’s website. Swanson and Williams [2014] also show that interest rates for longer maturities were still reactive until the end of 2011.
that in the second expression the fraction is divided by the level of government spending at steady state. The impact multiplier on inflation is multiplied by 4 to give a number in annualized percent terms. Finally, as stated before I assume that the Central Bank does not target the natural rate, i.e. $\phi_R = 0$. To make sure that I do not compare apples and oranges, I plot the value of the impact multiplier as a function of the duration of the liquidity trap in each simulation in Figure 2.

Note that the multiplier effect on expected inflation is much higher in the decentralized case than for the Ramsey one.\footnote{As previously stated, Euler equation errors grow with the severity of the recession. This is especially the case in the decentralized equilibrium. Since I restrict the magnitude of these errors, the schedule of multiplier effects is truncated in the decentralized equilibrium.} For this simulation, the impact
multiplier effect on inflation is always positive in the decentralized equilib-
rium and (mostly) negative in the Ramsey equilibrium. This comes from the
fact that expected inflation depends a lot on the fact that the Central Bank is
able to commit in the Ramsey equilibrium. Most of the intuition behind the
results in Figure 2 comes from the Euler equation. I take the log linear version
of equation (3) and iterate it forward\(^{15}\) to get:

\[
c_t = -\frac{U_C}{CU_{CC}} \mathbb{E}_t \sum_{k=0}^{\infty} \left( i_{t+k} - \pi_{t+k+1} + \log(\beta) + (\rho - 1)\xi_{t+k+1} \right) - \frac{NU_{CN}}{CU_{CC}} n_t \tag{8}
\]

where lowercase letters denote log deviations from steady state. The constant
term before the real interest rate is positive while the one in front of hours
is negative. From this equation, there are two ways for an increase in gov-
ernment spending to crowd in private consumption: (i) generate an increase
in inflation which, if associated with a zero interest rate will generate a de-
crease in the long-run real interest rate or (ii) generate an increase in hours
worked/decrease in leisure. Provided that leisure and private consumption
are complements (which is the case for the baseline calibration), leisure will
have to increase to go back to its steady state level and so private consumption
will increase in tandem.

In the decentralized equilibrium with a Taylor rule, both effects work to in-
crease the multiplier effect of government spending. The longer is the liquid-
ity trap, the larger the decrease in the long run real interest rate and the more
private consumption is crowded in: the first term in equation (8) becomes
positive. In turn, the large increase in aggregate demand on impact calls for
a large increase in labor demand. Because of the complementarity between
consumption and leisure, the initial increase in hours worked will amplify the
initial effect on private consumption: the second term in equation (8) becomes
positive. In the Ramsey equilibrium, the low effect of government spending
on inflation (more on that shortly) means that the effect that goes through the
real interest rate will be dampened: the first term in equation (8) never be-
comes positive.\(^{16}\) Eventually, the impact multiplier effect on consumption will
be positive (i.e. the output multiplier will be higher than 1) because of the in-
crease in hours worked that brings about an expected increase in leisure.

Because government spending still persists after the zero lower bound has
been lifted, the nominal rate should increase more in comparison to the case
where government spending is kept constant. This increases the real inter-

\(^{15}\)See the appendix for a detailed derivation.

\(^{16}\)In fact, the long-run real interest rate never decreases, even for a liquidity trap of more
than 4 years.
est rate and puts negative pressure on aggregate demand (it decreases $c_t$ from equation (8)) and thus inflation while the economy is in the liquidity trap.\textsuperscript{17} Since expected inflation depends a lot on monetary policy when the latter is optimally set, the short lived increase in inflation after government spending kicks in is quickly dominated by a relative decrease in inflation because of what happens when the economy gets out of the ZLB. This deflationary effect lasts for a major part of the time where the economy is at the ZLB so that higher government spending initially generates less expected inflation (this effect dissipates for a longer ZLB spell). This is in sharp contrast to what happens in the decentralized allocation. In this case, expectations are deflationary so that the increase in government spending makes a difference. This sets off a virtuous spiral whereby an increase in government spending decreases the expected real interest rate and crowds in consumption.

To sum up, whether monetary policy is set optimally or not makes a difference both qualitatively and quantitatively. With optimal monetary policy, higher government spending is associated with a lower impact multiplier on output. Also, expected inflation decreases on average after a government spending shock in the Ramsey equilibrium, while it increase in the decentralized one. Finally, in both setups the effectiveness of fiscal policy as a stimulative tool depends on the expected length of the liquidity trap. According to the Blue Chips forecasts, during the implementation of the ARRA the median expected length of the ZLB episode was around one year.\textsuperscript{18} While this is a rather short ZLB spell, from Figure 2 one can see that taking into account forward guidance drives the output multiplier effect from higher than 1 to lower than 1. Even for this small duration, the effect on expected inflation is also very different across setups.

Finally, the difference between multipliers might be a direct consequence of the fact that the Ramsey policy is more efficient in mitigating the initial output gap. Therefore, both multiplier effects could be lower in this setup simply because the crisis is less severe, even for the same duration of the liquidity trap in both scenarios. To show that this is actually not the case, I plot in Figure 3 the impact multiplier effects on output and inflation as a function of the initial output gap after the preference shock.\textsuperscript{19} What clearly stands out is that both

\textsuperscript{17}In a related setup in which monetary policy is given by a Taylor rule, Woodford [2011] shows analytically that if the government spending increase persists after the ZLB has stopped binding, this will reduce the multiplier effect at the ZLB in a non-linear way.

\textsuperscript{18}See Figure 4 in Swanson and Williams [2014]. In a recent paper, Hallett et al. [2016] suggest that the correct estimated ZLB duration as of 2009 is closer to 9 quarters.

\textsuperscript{19}In Figure 11 in the Appendix, I plot the impact multiplier effects as a function of the size of the initial shock. The patterns that emerge are very similar to the ones in Figure 3.
Figure 3: Impact Output and Inflation multipliers as a function of initial output gap

Note: A number of 4 in the x-axis means that the initial drop in output is equal to 4%.
impact multipliers display the same pattern as before.20

The impact multiplier might not convey all the information however. To consider all the dynamics in the response, I also consider the cumulative multiplier. Indeed, Uhlig [2010] puts the emphasis on the long run effects of government spending which can undermine the stimulative properties of the fiscal package taken as a whole. I compute the cumulative multiplier effects on output and inflation as follows:

\[ M_{y,h} = \frac{1}{g} \sum_{t=1}^{h} \left( 1 - g \right) \Delta \hat{c}_t + g \cdot \Delta \hat{g}_t, \]
\[ M_{\pi,h} = 4 \cdot \frac{1}{G} \sum_{t=1}^{h} \Delta \hat{c}_t \Delta \hat{\pi}_t, \]

where \( h \) is the horizon for the computation. I choose \( h = 200 \) for the simulation since all variables have returned to their steady state value after this time. I plot the cumulative multipliers as a function of the duration of the ZLB in Figure 4.

As before, the cumulative multiplier effect of government spending in the decentralized case is higher than 1. That is not the case for the Ramsey allocation, which is consistent with the fact that the maximum impact multiplier is slightly higher than 1 at its maximum.

At the end of the day, the model in which monetary policy is run by a Ramsey planner seems to be more consistent with recent empirical evidence. On top of that, there are additional benefits associated with choosing this specification to study the impact of government spending policies. First, the model has a unique rational expectations equilibrium even at the Zero Lower Bound. In the model with a Taylor rule, the central bank loses the capacity to influence inflation altogether and this opens the door to sunspot driven equilibria. In the Ramsey equilibrium, the policymaker can still influence inflation expectations in a liquidity trap.21 At least under my baseline calibration, this is enough to rule out sunspot fluctuations.22

20 As there is no liquidity trap in the decentralized equilibrium for an output gap lower than roughly 1%, the multiplier effect is constant. In contrast with Christiano et al. [2011], the impact multiplier effect on output is lower than 1 in this situation. In my setup, monetary policy reacts to the output gap and this magnifies the increase in the nominal and real interest rates after a government spending shock.

21 In a related paper, Armenter [2016] shows that the policymaker is generally not able to promise a given target for inflation at the ZLB. However, in his setup the policymaker is not able to commit as it is here, which is why I obtain a different result.

22 While a systematic inquiry of the determinacy properties of the Ramsey equilibrium at the ZLB is beyond the scope of this paper, for all the calibrations employed in the baseline results and robustness tests there is a unique rational expectations equilibrium.
Figure 4: Cumulative Output and Inflation multipliers
Then, the Ramsey allocation gives a much more reasonable description of the Great Recession. To clarify this, I plot in Figure 5 the path of the Ramsey equilibrium with and without an increase in government spending. The preference shock is calibrated so that the economy experiences a trough at \(-5.65\%\) for output as in the recent U.S. Great Recession. The government spending shock is calibrated so that it represents 1.58\% of steady state GDP to mimic the actual size of the recent American Recovery and Reinvestment Act of 2009.

![Figure 5: The Great Recession With and Without Fiscal Stimulus (Ramsey Equilibrium)](image)

The main take-away from Figure 5 is that for such a simple model, it does a remarkably good job at describing the behavior of the main macroeconomic variables during the Great Recession. As in the data, the model economy experiences an initial drop in inflation. Subsequently, inflation turns positive and converges to its long run level of zero. Before it does so, there is a short deflationary blip. Output is initially depressed but recovers quickly, overshoots its steady state level and converges towards the latter from below. The nominal interest rate stays at its lower bound for 4 and a half years before jumping
With this in mind, the contribution of the fiscal stimulus is as follows: consistent with the results presented so far, the increase in government spending does not really matter for inflation dynamics. Inflation is somewhat larger with the fiscal stimulus initially, but the largest deviation actually coincides with inflation being lower with the fiscal stimulus. The reason for that is the same which have been described to explain the negative multiplier on expected inflation in the Ramsey equilibrium. Regarding the path of output however, the fiscal stimulus makes a clear contribution: output is noticeably lower without the discretionary increase in government spending during the early stages of the recession.

In Figure 10 (in the appendix), I conduct the same exercise under the assumption that monetary policy is given by the Taylor rule described earlier. In this specification, two features are somewhat at odds with what actually happened during the Great Recession. First, the economy experiences a large deflation of roughly $-6\%$ in annual terms. Then, the liquidity trap lasts only for two years, that is a full two and a half years short of the length in the Ramsey equilibrium. With this in mind, one can see that fiscal policy still dampens the fall in output, but it has a larger (albeit rather small in absolute terms) impact on inflation compared to the Ramsey equilibrium.

4 Robustness and Sensitivity

4.1 Robustness Analysis

4.1.1 Utility Function

In this section I study some variations of the model to show that the results do not overly depend on the exact specification that I chose in the last section. I begin with the utility function. The fact that hours worked and private consumption were assumed to be complements has important implications for the size of the multiplier (see Monacelli and Perotti [2008] and Bilbiie [2011]). In particular, in this setup such a feature works to increase the multiplier effect on output, which can be higher than 1 even in normal times (see Christiano et al. [2011]). To show that this is not crucial, I repeat the exercise of the last
section with a separable utility function:

$$U(C, N_t) = \log(C_t) - N_t^\varphi.$$ 

This utility function is very commonly used in the literature. With this functional form, $1/\varphi$ is the Frisch elasticity of labor supply. Consistent with much of the literature, I set $\varphi = 2$ so that the Frisch elasticity of labor supply is equal to 0.5. I plot both the cumulative and impact multipliers as a function of the duration of the liquidity trap in Figure 6:

What stands out of Figure 6 is that the results are qualitatively similar with the ones from last section. The impact multiplier effect in the Ramsey case is never larger than 1 however. Indeed, with separable preferences the second
term in equation (8) vanishes so that private consumption is never crowded in. In addition, an increase in government spending when monetary policy is carried out optimally generate much less expected inflation. In this respect, the gap between the Ramsey and decentralized allocation is even more pronounced: the maximum impact on expected inflation in the decentralized case is as high as around 9% in annual terms, whereas in the Ramsey allocation it is still negative.

The intuition behind this large multiplier on expected inflation is the following. In the model with non-separable labor supply the Frisch elasticity of labor supply is given by

$$\frac{\partial n_t}{\partial w_t} = \left( \frac{N \cdot U_{NN}}{U_N} - \frac{N \cdot U_{CN}}{U_C} \right)^{-1},$$

which is equal to 2 in my baseline calibration. This means that the elasticity of labor supply is much lower in the model with separable utility. As a consequence, for a given increase in labor demand to be met by labor supply in this model, a larger increase in the real wage is required. This will trigger a larger increase in current and expected future marginal costs, which in turn gives rise to the larger multiplier on both output and expected inflation in a liquidity trap.

4.1.2 Taylor Rule

Another functional form that potentially has an impact on the result is the Taylor rule in the decentralized case. In the form that I have adopted up until now, the Central bank is assumed to be reactive in that it reacts to current variations in inflation and output. However, most papers that try to fit a Taylor rule to the data assume some internal persistence: the nominal interest rate today depends on its last period value. Accordingly, I repeat the experiments of the last section with the following Taylor rule:

$$1 + R_t = \max \left\{ 1, (1 + R_t-1)^{\rho_R} \left[ \frac{1 + \phi_R R_t^n}{\beta} (1 + \pi_t)^{\phi_\pi} \left( \frac{Y_t^n}{Y_t^n} \right)^{\phi_Y} \right]^{1-\rho_R} \right\}.$$  

To the extent that $\rho_R > 0$, the current interest rate will depend on its past realization. I set a standard value of $\rho_R = 0.7$ and report the results of this experiment in Figure 7.
What stands out of Figure 7 is that the results are also the same qualitatively. It is still the case that finding a low reaction of inflation in the data is not an indication that a government spending stimulus is not working.

4.1.3 A Model with Private Capital

In this subsection, I keep the augmented Taylor Rule and assume additionally that private capital is needed to produce the final good. Further, the accumulation of private capital is subject to an investment adjustment cost in the

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23 The decentralized as well as the Ramsey allocations are described in full detail in the appendix.
same vein as Christiano et al. [2005]:

\[ K_{t+1} = (1 - \delta)K_t + \left( 1 - S \left( \frac{I_t}{I_{t-1}} \right) \right) I_t, \]

where \( K_t \) is the stock of capital at the beginning of period \( t \) and \( I_t \) is private investment. Since the model is linearized, what matters ultimately is the second derivative of the adjustment cost function with respect to its argument. Following Christiano et al. [2005], I set \( S''(1) = 2.5 \) in the simulations.

I plot the results regarding the impact and cumulative multipliers of government spending as a function of the duration of the ZLB episode in Figure 8. As before, the impact multiplier on inflation is much smaller for the Ramsey allocation. The same can be said for the impact output multiplier, to a lesser extent. It should be noted that the magnitude of the impact on inflation for the decentralized case is much smaller than before. This is due to the fact that real marginal costs in this economy depend on the marginal product of capital. As a consequence, they react less on impact, which translates into the behavior of inflation. The same pattern holds for the cumulative multipliers, with a larger difference for the output multiplier in this case.

The reason is the following: as a discretionary increase in government spending generate less inflation, this puts upwards pressure on the real interest rate. The arbitrage condition between private capital and government bonds means that (modulo variations in Tobin’s Q) both should give the same real return. Because the marginal productivity of capital is decreasing, a higher return means a lower level of capital, which is achieved through lower investment. In the decentralized equilibrium, the increase in (expected) inflation due to government spending mitigates this effect so that the cumulative multiplier is much higher. In the Ramsey allocation on the other hand, the decline in private investment undoes most of the stimulative effects of government spending in the medium to long run.
At the end of the day, the inclusion of private capital calls for a qualification of the results obtained with simpler models. Indeed, a low cumulative response of expected inflation does not coincide with a sizable cumulative output response in the Ramsey allocation. As such, a fiscal stimulus is able to boost the economy in a short term fashion without generating an increase in expected inflation.

4.2 Sensitivity Analysis

All the results that I have presented so far are based on the calibration that I have outlined earlier. While mostly standards, the choice of parameters is not
innocuous in driving the results. With this in mind, I will study the sensitivity of the gap between multipliers in the two allocations with respect to a range of parameters.

As is made clear in Eggertsson [2010] and Christiano et al. [2011], the more flexible the price the larger the recession for a given preference shock and the larger the government spending multiplier. Therefore, I will show how my results change if I make prices more sticky. Indeed, we know that the multiplier effect will increase (especially in the decentralized equilibrium) with more flexible price, so I want to see whether the model with prices that are more sticky will still yield a meaningful difference between the Ramsey and decentralized allocation. My baseline calibration is $\psi = 90$, which corresponds to a Calvo probability to change prices of 0.8. In the alternative calibration that I study now I assume that $\psi = 413$, which corresponds to a Calvo probability of 0.9. With this calibration, the corresponding average price duration doubles from 5 to 10 quarters.

Dupaigne and Fève [2016] show that the persistence of the government spending shock is important in shaping the magnitude of the government spending multiplier, especially in a model with private capital. Accordingly, I study an alternative calibration in which $\rho_g = 0.9$.

In the model with separable utility function, the extent to which households experience labor disutility is crucial in determining the elasticity of inflation to variations in aggregate demand. There is no clear consensus regarding the value of the Frisch elasticity $1/\varphi$, so I study an alternative calibration in which $\varphi = 5$ (see Galí [2010]). This yields an elasticity of labor supply of 0.2 which is in line with a large body of empirical papers that find low values for this elasticity.

Finally, as I have highlighted earlier, what happens once the economy gets out of the liquidity trap is important in shaping the dynamics inside the liquidity trap. Therefore, what type of rule the Central Bank will revert to is important in this respect. To investigate this, I assume that the Central Bank targets a measure of the natural rate of interest.

For each of those alternative parameter values, I compute the multiplier effects of a government spending increase on expected inflation and output for all the model variants that I have studied in the paper. I then compute the difference between the impact multiplier effect in the Ramsey and decentralized allocations for a given length of the liquidity trap episode. In the model, agents have rational expectations and know the "true" duration of the liquidity trap episode, which corresponds to the announcement by the Central Bank.
The closest empirical counterpart for this is when, in the middle of 2011, the Federal Reserve announced that it intended to keep the nominal rate at zero until mid-2013. Accordingly, I set the duration of the ZLB spell to 2 years, or 8 quarters.

The main result of the preceding sections is that the gap in multiplier effects is more important for expected inflation than for output, especially at short horizons. Accordingly, I report the ratio of these multipliers gaps for expected inflation over the one for output in Table 1.

Table 1: Gap Between Multipliers in Ramsey and Decentralized Allocations

<table>
<thead>
<tr>
<th></th>
<th>$\psi = 413$</th>
<th>$\psi = 5$</th>
<th>$\rho = 0.9$</th>
<th>$\phi_R = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Imp</td>
<td>Cum</td>
<td>Imp</td>
<td>Cum</td>
</tr>
<tr>
<td>Baseline</td>
<td>1.26</td>
<td>0.17</td>
<td>3.55</td>
<td>1</td>
</tr>
<tr>
<td>Separable Utility</td>
<td>3.04</td>
<td>0.44</td>
<td>9.83</td>
<td>2.18</td>
</tr>
<tr>
<td>Taylor Rule</td>
<td>1.19</td>
<td>0.14</td>
<td>3.51</td>
<td>1</td>
</tr>
<tr>
<td>Private Capital</td>
<td>0.5</td>
<td>0.01</td>
<td>1.06</td>
<td>0.06</td>
</tr>
</tbody>
</table>

An entry that is higher than 1 means that the difference in multiplier effects between the two regimes is higher for expected inflation than for output. This is the case in most of the configurations. As before, this effect is much lower when considering cumulative multipliers, especially when the model features private capital with adjustment costs.

5 Conclusion

All ZLB episodes are not born equal. The dynamics of an economy at the ZLB are very dependent on whether the Central Bank is still actively managing expectations of the private sector. In this paper, I have formalized this by assuming that the monetary authority acted like a Ramsey planner. When this is the case, the economy will not experience sustained deflation in a deep recession and an increase in government spending will not generate a large increase in inflation.

As a consequence, the recent findings in Dupor and Li [2015] that the Amer-
ican Recovery and Reinvestment Act of 2009 did not generate a large increase in expected inflation cannot be taken as _prima facie_ evidence that the New Keynesian transmission channel is wrong. Furthermore, such evidence is not necessarily inconsistent with the fact that government spending policy is ineffective at the ZLB. In the framework I have used in this paper, the multiplier effect on output when monetary policy is optimal can be higher than 1 if the recession is deep enough, while inflation still does not increase by a comparatively large amount.
References


6 Appendix

6.1 Figures

Figure 9: Shadow Interest Rate

Data has been downloaded from Jing Cynthia Wu’s website and reproduces the shadow interest rate for the Federal Reserve computed in Wu and Xia [2016].
Figure 10: The Great Recession With and Without Fiscal Stimulus in the Decentralized Equilibrium
Figure 11: Impact Output and Inflation multipliers as a function of initial preference shock

Note: A number of 0.1 in the x-axis means that $\beta$ initially increases by 10%.
The Euler equation errors are expressed in terms of steady state private consumption. More formally, I compute $U_{C,t-\beta E_t} \frac{\xi_{t+1}}{\xi_t} + \frac{\xi_{t+1}}{1+R_t}$ for each period after both a government spending and preference shock of varying size. What I report here is the maximum of the absolute value of this object. A value of 0.01 in the $z$-axis means that the error amounts to 1% of steady state consumption.
The Euler equation errors are expressed in terms of steady state private consumption. More formally, I compute 

$$U_{C_{t-j}, \beta E_{t+1}} \frac{U_{C_{t+1}}}{C_{t+1}(1+R_{t})}$$

for each period after both a government spending and preference shock of varying size. What I report here is the maximum of the absolute value of this object. A value of 0.01 in the z-axis means that the error amounts to 1% of steady state consumption.
6.2 Model without Private Capital

6.2.1 Decentralized Equilibrium

The household’s maximization program is the following:

$$E_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} U (C_{t+s}, N_{t+s})$$, (9)

subject to the budget constraint:

$$P_t C_t + T_t + \frac{B_t}{1 + R_t} \leq W_t N_t + D_t + B_{t-1}.$$ (10)

Optimal choice for private consumption, labor supply and the holdings of bonds yield:

$$U_{C,t} = \beta \xi_{t+1} \frac{U_{C,t+1}}{1 + \tau_{t+1}} (1 + R_t)$$ (11)

$$\frac{W_t}{P_t} = -\frac{U_{N,t}}{U_{C,t}}$$ (12)

where $\tau_t = \frac{P_t}{P_{t-1}} - 1$. The firms choose their individual price $P_t(z)$ to maximize the following objective function:

$$E_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} \frac{U_{C,t+s}}{U_{C,t}} \left\{ \frac{(1 + \tau) P_t(z)}{P_t} Y_t(z) - \frac{W_t}{P_t} N_t(z) - \frac{\psi}{2} \left( \frac{P_t(z)}{P_{t-1}(z)} - 1 \right)^2 N_t \right\},$$

where $\tau = 1/(\theta - 1)$. Using equation (12) and the fact that every firm is identical and will choose the same price, the first order condition with respect to $P_t(z)$ is given by:

$$\psi \pi_t (1 + \pi_t) = \beta \xi_{t+1} \frac{U_{C,t+1}}{U_{C,t}} \frac{N_{t+1}}{N_t} \psi \pi_{t+1} (1 + \pi_{t+1}) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta.$$ (13)

Using the definition of profits, plugging it in the household’s budget constraint I obtain the resource constraint of this economy:

$$N_t \left( 1 - \frac{\psi}{2 \pi_t^2} \right) = C_t + G_t.$$ (14)

Finally, the model is closed by specifying a rule for the setting of the nominal interest rate. Following much of the literature, I assume that it follows a
standard Taylor rule of the form:

\[
1 + R_t = \max \left\{ 1, R_t^n (1 + \pi_t)^{\phi_r} N_t^{\phi_Y} \right\}.
\] (15)

6.2.2 Steady State

The share of government spending at steady state is denoted by:

\[
\tilde{g} = \frac{G}{Y}
\]

Given the functional form for the utility function, equation (12) can be re-written as:

\[
F_N = \frac{1 - \gamma}{\gamma} \frac{C}{1 - N}
\]

\[\Leftrightarrow 1 = \frac{1 - \gamma}{\gamma} \cdot \frac{N}{1 - N} (1 - \tilde{g})\]

\[\Leftrightarrow N = \frac{\gamma}{\gamma + (1 - \gamma)(1 - \tilde{g})}\]

where we have used the resource constraint for the second equation. In practice, I choose \(\gamma\) so that \(N = 1/3\) at steady state. In this case, I have

\[
\gamma = \frac{(1 - \tilde{g})}{2 + (1 - \tilde{g})}
\]

The resource constraint can be re-written as:

\[C = (1 - \tilde{g})N\]
6.2.3 Ramsey Equilibrium

I reproduce the Ramsey program for convenience:

\[
\mathcal{L}_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U_C (C_t, N_t) + \phi_{1,t} \left[ \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} (1 + I_t) - U_{C,t} \right] + \phi_{2,t} \left[ \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1} N_{t+1}}{U_{C,t} N_t} \psi \pi_{t+1} (1 + \pi_{t+1}) - \psi \pi_t (1 + \pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta \right] + \phi_{3,t} \left[ N_t \left( 1 - \frac{\psi}{2} \pi_t^2 \right) - C_t - G_t \right] + \phi_{4,t} \left[ R_t - 0 \right] \right\}
\]

The First Order Conditions are given by the following set of equations:

\[
\frac{\partial \mathcal{L}_0}{\partial C_t} = 0 \Leftrightarrow \xi_t U_C = \phi_{1,t} U_{C,t} + \phi_{2,t} \frac{1}{F_{N,t}} \left[ \frac{U_{C,t} U_{C,t} - U_{C,t} U_{N,t}}{U_{C,t}^2} \right] + \phi_{3,t} 
\]

(16)

\[
+ \beta \psi \mathbb{E}_t \phi_{2,t} \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1} U_{C,t+1} N_{t+1}}{U_{C,t} N_t} d(\pi_{t+1}) 
\]

(17)

\[
- \frac{\xi_t}{\xi_{t-1}} U_{C,t} \left\{ \phi_{1,t-1} \frac{1 + R_{t-1}}{1 + \pi_t} + \psi \phi_{2,t-1} \frac{N_t}{N_{t-1}} d(\pi_t) \right\} 
\]

\[
\frac{\partial \mathcal{L}_0}{\partial N_t} = 0 \Leftrightarrow \xi_t U_{N,t} = \phi_{2,t} \left\{ \frac{U_{N,t} U_{C,t} - U_{C,t} U_{N,t}}{U_{C,t}^2} \right\} + \phi_{1,t} U_{C,t} 
\]

(18)

\[
- \phi_{3,t} \left( 1 - \frac{\pi_t^2}{2} \right) - \frac{\xi_t}{\xi_{t-1}} U_{C,t} \left\{ \phi_{1,t-1} \frac{1 + R_{t-1}}{1 + \pi_t} + \psi \phi_{2,t-1} \frac{N_t}{N_{t-1}} d(\pi_t) \right\} 
\]

\[
\frac{\partial \mathcal{L}_0}{\partial \pi_t} = 0 \Leftrightarrow \phi_{3,t} N_t \pi_t + \phi_{2,t} d'(\pi_t) = \frac{\xi_t}{\xi_{t-1}} U_{C,t} \left\{ \phi_{2,t-1} \frac{N_t}{N_{t-1}} d(\pi_t) - \phi_{1,t-1} \frac{1 + R_{t-1}}{\psi (1 + \pi_t)^2} \right\} 
\]

(19)

\[
\frac{\partial \mathcal{L}_0}{\partial R_t} = 0 \Leftrightarrow \phi_{4,t} = -\beta \mathbb{E}_t \phi_{1,t} \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1 + \pi_{t+1}} 
\]

(20)
To this set of equations I must add the complementary slackness condition:

\[ R_t \phi_{4,t} = 0. \]

### 6.2.4 Steady State

The steady state value of \( C \) and \( N \) is still given by the values reported before. I derive here the steady state value for the Lagrange Multipliers of the Ramsey program. I focus on the zero inflation steady state in which the zero lower bound is not binding. As such, I have \( R > 0 \) and \( \phi_4 = 0 \). From equation (20), I get that

\[ \phi_1 = 0. \]

From equation (16) and the fact that \( \phi_2 = \phi_1 = 0 \), I get:

\[ \phi_3 = U_C. \]

### 6.2.5 Log linear approximation outside ZLB

I first focus on the equations that are also present in the decentralized equilibrium. The Euler equation for consumption is:

\[
c_t = \mathbb{E}_t c_{t+1} + \frac{U_C}{CU_{CC}} (i_t - \mathbb{E}_t \pi_{t+1} + \log(\beta) + (\rho - 1)\hat{\xi}_t) + \frac{NU_{CN}}{CU_{CC}} (\mathbb{E}_t n_{t+1} - n_t)
\]

Iterating forward I obtain:

\[
c_t = -\frac{U_C}{CU_{CC}} \mathbb{E}_t \sum_{k=0}^{\infty} (i_{t+k} - \pi_{t+k+1} + \log(\beta) + (\rho_{\xi} - 1)\hat{\xi}_{t+k+1}) - \frac{NU_{CN}}{CU_{CC}} \mathbb{E}_t \sum_{k=0}^{\infty} (n_{t+k+1} - n_{t+k})
\]

\[
c_t = -\frac{U_C}{CU_{CC}} \mathbb{E}_t \sum_{k=0}^{\infty} (i_{t+k} - \pi_{t+k+1} + \log(\beta) + (\rho_{\xi} - 1)\hat{\xi}_{t+k+1}) - \frac{NU_{CN}}{CU_{CC}} n_t,
\]

where I have used the fact that most terms cancel out in the second sum and \( n_\infty = 0 \) after a temporary shock. The resource constraint is given by:

\[
n_t = (1 - g)c_t + g \cdot g_t.
\]

40
The log linear New Keynesian Phillips curve is:

\[ \pi_t = \beta E_t \pi_{t+1} + \kappa m_c, \]

(23)

where

\[ m_c_t = \left( \frac{NU_{NN}}{U_N} - \frac{NU_{CN}}{U_C} \right) n_t + \left( \frac{CU_{CN}}{U_N} - \frac{CU_{CC}}{U_C} \right) c_t. \]

I now focus on the log linear equilibrium conditions for the Ramsey equilibrium when the economy is outside the Zero Lower Bound. As such, from equation (20) I have

\[ \phi_{1,t} \equiv 0. \]

I use this equation to simplify the derivations from now on. I start with equation (19). When log linearized around the zero inflation steady state it gives:

\[ \phi_3 N \cdot \pi_t + \hat{\phi}_{2,t} = \hat{\phi}_{2,t-1}, \]

(24)

where a hat denotes the log linear deviation of the Lagrange multiplier from its steady state value. With this in mind, the log linear version of equation (16) is:

\[ \frac{CU_{CC}}{U_C} c_t + \frac{NU_{CN}}{U_C} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \theta \left[ \frac{U_{CN} U_C - U_{CC} U_N}{U_C^2} \right] \hat{\phi}_{2,t}. \]

Likewise, the log linear version of equation (18) is:

\[ \frac{CU_{CN}}{U_N} c_t + \frac{NU_{NN}}{U_N} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \theta \left[ \frac{U_{NN} U_C - U_{CN} U_N}{U_N U_C^2} \right] \hat{\phi}_{2,t}. \]

(25)

### 6.2.6 Log linear approximation at ZLB

I now focus on the log linear equilibrium conditions when the economy is stuck at the Zero Lower Bound. When this is the case, the complementary slackness condition is binding and I have \( \phi_{4,t} > 0 \). As a consequence, from equation (20) I have

\[ \hat{\phi}_{4,t} = -\beta \phi_3 \cdot \hat{\phi}_{1,t}. \]

\[ \text{Note that since the steady state value of } \phi_2 \text{ is zero, a first order Taylor expansion of } \phi_{2,t} \text{ yields simply } \hat{\phi}_{2,t} = \phi_{2,t}. \]
When log linearized around the zero inflation steady state (19) gives:

$$\phi_3 N \cdot \pi_t + \hat{\phi}_{2,t} = \hat{\phi}_{2,t-1} - \frac{\phi_3}{\psi \beta} \hat{\phi}_{1,t-1},$$  \hspace{1cm} (26)

where a hat denotes the log linear deviation of the Lagrange multiplier from its steady state value. Now that $\phi_1$ is no longer constant, the log linear version of equation (16) is:

$$\frac{CU_{CC}}{U_C} \hat{c}_t + \frac{NU_{CN}}{U_C} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \frac{\theta}{U_C} \left[ \frac{U_{CN}U_C - U_{CC}U_N}{U_C^2} \right] \hat{\phi}_{2,t}$$

$$+ \frac{U_{CC}}{U_C} \hat{\phi}_{1,t} - \frac{U_{CC}}{\beta U_C} \hat{\phi}_{1,t-1}. \hspace{1cm} (27)$$

Likewise, the log linear version of equation (18) is:

$$\frac{CU_{CN}}{U_N} \hat{c}_t + \frac{NU_{NN}}{U_N} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \frac{\theta}{U_N} \left[ \frac{U_{NN}U_C - U_{CN}U_N}{U_N U_C^2} \right] \hat{\phi}_{2,t}$$

$$+ \hat{\phi}_{1,t} - \frac{U_{CN}}{\beta U_N} \hat{\phi}_{1,t-1}. \hspace{1cm} (28)$$

Finally, the Euler equation for consumption becomes:

$$c_t = \mathbb{E}_t c_{t+1} + \frac{U_C}{CU_{CC}} \left( \log(\beta) - \mathbb{E}_t \pi_{t+1} + (\rho - 1) \hat{\xi}_t \right) + \frac{NU_{CN}}{CU_{CC}} (\mathbb{E}_t n_{t+1} - n_t) \hspace{1cm} (29)$$
6.3 Model with Private Capital

6.3.1 De-centralized allocation

The production function now reads:

\[ F(N_t, K_t) = N_t^a K_t^b. \]

With this modification, the equation for optimal pricing by the monopolistic firm takes the following form:

\[ \psi d(\pi_t) = \psi \beta \xi_t U_{C,t+1} F(N_{t+1}, K_{t+1}) d(\pi_{t+1}) - \theta \frac{U_{N,t}}{U_{C,t} F_{N,t}} + \theta. \]

The Lagrangian associated with the Household maximization program is the following:

\[
\mathcal{L}_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t) + \Omega_{1,t} \left[ W_t N_t + P_t r^k_t K_t + R_{t-1} B_t - P_t (C_t + I_t) - B_{t+1} - P_t Q_t X_t \right] + \Omega_{2,t} \left[ X_t + (1 - \delta) K_t + \left( 1 - S \left( \frac{I_t}{I_{t-1}} \right) \right) I_t - K_{t+1} \right] \right\},
\]

where \( Q_t \) is the price of installed capital, \( r^k_t \) is the return from capital (which will be equal to the marginal product of capital in equilibrium) and \( X_t \) is the amount of private capital that households members by from one another. In equilibrium, \( X_t \equiv 0 \) and I just include it so that I can determine the price of installed capital. I end up with the following first order conditions:
\[
\frac{\partial L_0}{\partial N_t} = 0 \iff \frac{W_t}{P_t} = -\frac{U_{N,t}}{U_{C,t}} \tag{30}
\]

\[
\frac{\partial L_0}{\partial C_t} = 0 \iff \xi_t U_{C,t} = \Omega_{1,t} P_t \tag{31}
\]

\[
\frac{\partial L_0}{\partial X_t} = 0 \iff \Omega_{1,t} P_t Q_t = \Omega_{2,t} \tag{32}
\]

\[
\frac{\partial L_0}{\partial I_t} = 0 \iff \Omega_{1,t} P_t Q_t = \Omega_{2,t} \tag{33}
\]

\[
\frac{\partial L_0}{\partial K_{t+1}} = 0 \iff U_{C,t} = \beta \xi_t \left[ 1 - \frac{1}{\bar{Y}_t} + \frac{r_{t+1}}{Q_t} \right] \tag{34}
\]

\[
\frac{\partial L_0}{\partial K_{t+1}} = 0 \iff U_{C,t} = \beta \xi_t \left[ 1 + \frac{r_{t+1}}{Q_t} \right] \tag{35}
\]

The resource constraint of the economy is now:

\[
F(N_t, K_t) \left( 1 - \frac{\psi}{2} \pi_t^2 \right) = C_t + G_t + I_t
\]

**6.3.2 Steady State**

By construction, at steady state \( S(1) = S'(1) = 0 \). From equation (34), I get \( Q = 1 \). Let me introduce the following notations for steady state ratios:

\[ \bar{i} = \frac{I}{Y}, \quad \bar{g} = \frac{G}{Y} \]

Using equation (35), I obtain:

\[
\beta F_K = 1 - \beta (1 - \delta) \implies \bar{i} = \frac{\delta K}{Y} = \frac{b \beta \delta}{1 - \beta (1 - \delta)}
\]

where I have also used the functional form of the production function to substitute for \( F_K = b \frac{Y}{K} \). Given the functional form for the utility function, equation

44
(30) can be re-written as:

\[ F_N = \frac{1 - \gamma}{\gamma} \frac{C}{1 - N} \]

\[ \Leftrightarrow a \frac{Y}{N} = \frac{1 - \gamma}{\gamma} \frac{Y}{1 - N} (1 - \tilde{g} - \tilde{i}) \]

\[ \Leftrightarrow N = \frac{a \gamma}{a \gamma + (1 - \gamma)(1 - \tilde{g} - \tilde{i})} \]

where we have used the resource constraint for the second equation. In practice, we choose \( \gamma \) so that \( N = 1/3 \) at steady state. In this case, we have

\[ \gamma = \frac{(1 - \tilde{g} - \tilde{i})}{2a + (1 - \tilde{g} - \tilde{i})} \]

The resource constraint can be re-written as:

\[ C = (1 - \tilde{g} - \tilde{i}) N^a K^b \]

\[ \Leftrightarrow C = (1 - \tilde{g} - \tilde{i}) N^a \left( \frac{1}{\delta} \frac{C}{1 - \tilde{g} - \tilde{i}} \right)^b \]

\[ \Leftrightarrow C = (1 - \tilde{g} - \tilde{i}) \left[ N^a \left( \frac{1}{\delta} \right)^b \right]^{\frac{1}{1 - \delta}} \]

Using the steady state value of \( C \) and \( N \) I can recover the steady state level output and the stock of capital:

\[ K = \frac{\tilde{i}}{\delta} \frac{C}{1 - \tilde{g} - \tilde{i}} \quad , \quad Y = N^a K^b. \]

### 6.3.3 Log-Linear Approximation

The accumulation equation for private capital is now given by:

\[ k_t = (1 - \delta)k_{t-1} + \delta i_t. \quad (36) \]

Approximated around the zero inflation steady state, the resource constraint is now:

\[ \frac{NF_N}{F} n_t + \frac{KF_k}{F} k_t = (1 - g) c_t + g \cdot g_t^c + \frac{K}{Y} (k_{t+1} - (1 - \delta) k_t). \quad (37) \]
The log linear approximation of equation (34) reads:

\[
\hat{q}_t = S''(1)(i_t - i_{t-1}) - \beta S''(1)(E_t i_{t+1} - i_t) \tag{38}
\]

Approximating equation (35), I get:

\[
\hat{U}_{C,t} = E_t \hat{U}_{C,t+1} + (\rho - 1) \xi_t - q_t + \beta (1 - \delta) E_t q_{t+1} + (1 - \beta (1 - \delta)) E_t \{mc_{t+1} + \hat{F}_{K,t+1}\} \tag{39}
\]

where I have defined:

\[
\hat{U}_{C,t} = \frac{CU_{CC}}{U_C} c_t + \frac{NU_{CN}}{U_C} n_t
\]

\[
\hat{F}_{K,t} = \frac{NF_{KN}}{F_K} n_t + \frac{KF_{KK}}{F_K} k_t.
\]

Equations (21), (22) and (23) are still valid, where real marginal cost is now defined as

\[
mc_t = \hat{U}_{N,t} - \hat{U}_{C,t} - \hat{F}_{N,t}
\]

with

\[
\hat{U}_{N,t} = \frac{NU_{NN}}{U_N} n_t + \frac{CU_{CN}}{U_C} c_t \tag{40}
\]

\[
\hat{F}_{N,t} = \frac{NF_{NN}}{F_N} n_t + \frac{KF_{NK}}{F_K} k_t. \tag{41}
\]
6.3.4 The Ramsey allocation

\[ L_0 \equiv \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left\{ \xi_t U(C_t, N_t) \right. \]

\[ + \Gamma_{1,t} \left[ \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1 + \pi_{t+1}} (1 + R_t) - U_{C,t} \right] \]

\[ + \Gamma_{2,t} \left[ \psi \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} F(N_{t+1}, K_{t+1}) d(\pi_{t+1}) - \psi d(\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} \frac{1}{F_{N,t}} - \theta \right] \]

\[ + \Gamma_{3,t} \left[ F(N_t, K_t) \left( 1 - \frac{\psi}{2} \frac{\pi_t^2}{2} \right) - C_t - G_t - I_t \right] \]

\[ + \Gamma_{4,t} \left[ (1 - \delta) K_t + (1 - S_{P,t}) I_t - K_{t+1} \right] \]

\[ + \Gamma_{5,t} \left[ \beta \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} Q_{t+1} U_{C,t+1} S'_{P,t+1} \tilde{I}_{t+1}^2 + Q_t U_{C,t} \left( 1 - S_{P,t} - S'_{P,t} \tilde{I}_t \right) - U_{C,t} \right] \]

\[ + \Gamma_{7,t} \left[ \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{Q_t} \left( (1 - \delta) Q_{t+1} - \frac{U_{N,t+1}}{U_{C,t+1}} F_{K,t+1} \right) - U_{C,t} \right] \]

\[ + \Gamma_{7,t} \left[ R_t - 0 \right] \right\}, \]

where I have defined

\[ \tilde{I}_t = \frac{I_t}{I_{t-1}}, \quad S_{P,t} = S_P \left( \tilde{I}_t \right), \quad S'_{P,t} = S'_P \left( \tilde{I}_t \right), \quad S''_{P,t} = S''_P \left( \tilde{I}_t \right) \]

After some algebra, the FOC with respect to \( I_t \) yields:

\[ \Gamma_{3,t} I_t = \Gamma_{5,t} I_t \left( 1 - S_{P,t} - S'_{P,t} \tilde{I}_t \right) + Q_t U_{C,t} S_{P,t} \left( \Gamma_{6,t} \frac{\tilde{I}_t}{\xi_{t-1}} I_t - \Gamma_{6,t} \right) \]

\[ + \beta \mathbb{E}_t Q_{t+1} U_{C,t+1} \tilde{S}_{P,t+1} \left( \Gamma_{6,t+1} - \Gamma_{6,t} \frac{\tilde{I}_{t+1}}{\xi_t} I_{t+1} \right) + \beta \mathbb{E}_t \Gamma_{5,t+1} S'_{P,t+1} \tilde{I}_{t+1} I_{t+1}, \]

where I have defined

\[ \tilde{S}_{P,t} \equiv 2 \cdot S'_{P,t} \tilde{I}_t + S''_{P,t} \tilde{I}_t^2. \]
The FOC with respect to private consumption now reads:

\[
\xi_t U_{C,t} = \Gamma_{3,t} + U_{CC,t} \left[ \Gamma_{1,t} + \Gamma_{6,t} + \Gamma_{5,t} \left( 1 - Q_t \left( 1 - S_{P,t} - S'_{P,t} I_t \right) \right) \right]
\]

\[
+ \theta \Gamma_{2,t} \left[ \frac{1}{F_{N,t}} \left( \frac{U_{CN,t} U_{C,t} - U_{CC,t} U_{N,t}}{U_{C,t}^2} \right) + \psi \beta \Gamma_{2,t} F_{C,t} \frac{\xi_t + 1}{\xi_t} \frac{U_{C,t+1} Y_{t+1}}{U_{C,t}^2} d(\pi_{t+1}) U_{CC,t} \right]
\]

\[
- U_{CC,t} \left[ \Gamma_{1,t-1} \left( 1 + R_{t-1} \right) + \psi \Gamma_{2,t-1} \frac{Y_t}{U_{C,t-1}} d(\pi_t) + \Gamma_{5,t-1} S'_{P,t} I_t^2 Q_t \right]
\]

\[
- U_{CC,t} \left[ (1 - \delta) Q_t - \frac{F_{K,t} U_{CN,t}}{F_{N,t} U_{CC,t}} \right]
\]

(43)

where I have defined

\[
Y_t = F(N_t, K_t).
\]

Similarly, the FOC with respect to hours worked now reads:

\[
\xi_t U_{N,t} = \Gamma_{1,t} U_{CN,t} + \theta \Gamma_{2,t} \left[ \frac{1}{F_{N,t}} \left( \frac{U_{NN,t} U_{C,t} - U_{CN,t} U_{N,t}}{U_{C,t}^2} \right) + \frac{U_{N,t} F_{NN,t}}{U_{C,t}^2} \right] - \Gamma_{3,t} F_{N,t} \left( 1 - \frac{\pi_t^2}{2} \right)
\]

\[
+ \left[ \Gamma_{6,t} + \Gamma_{5,t} \left( 1 - Q_t \left( 1 - S_{P,t} - S'_{P,t} I_t \right) \right) \right] U_{CN,t} + \psi \beta \Gamma_{2,t} F_{C,t} \xi_t + 1 \frac{U_{C,t+1} Y_{t+1}}{U_{C,t}^2} d(\pi_{t+1}) U_{CN,t}
\]

\[
- \xi_t \left[ \Gamma_{1,t-1} \left( 1 + R_{t-1} \right) + \psi \Gamma_{2,t-1} \frac{Y_t}{U_{C,t-1}} d(\pi_t) + \Gamma_{5,t-1} S'_{P,t} I_t^2 Q_t \right]
\]

\[
- \xi_t \left[ \frac{F_{K,t} U_{CN,t}}{F_{N,t}} - \frac{U_{NN,t} U_{C,t} - U_{CN,t} U_{N,t}}{U_{C,t}^2} \right]
\]

\[
+ \psi F_{N,t} \left[ \frac{\xi_t + 1}{\xi_t} \frac{U_{C,t+1} Y_{t+1}}{U_{C,t}^2} d(\pi_{t+1}) - \frac{\xi_t}{\xi_t - 1} \frac{U_{C,t}}{U_{C,t-1} Y_{t-1}} d(\pi_t) \right]
\]

(44)

The FOC with respect to private capital is:

\[
\Gamma_{4,t} = \beta (1 - \delta) E_t \Gamma_{4,t+1} + \beta E_t \left[ \Gamma_{3,t+1} F_{K,t+1} \left( 1 - \frac{\pi_t^2}{2} \right) + \theta \Gamma_{2,t+1} \frac{U_{N,t+1} F_{N,t+1}}{U_{C,t+1}^2} \right]
\]

\[
+ \beta E_t \xi_t + 1 \frac{U_{C,t+1}}{\xi_t} \left[ \psi \Gamma_{2,t} \frac{F_{K,t} U_{C,t} - d(\pi_{t+1})}{U_{C,t}} \right] - \frac{\xi_t + 2}{\xi_t + 1} \frac{U_{C,t+2} Y_{t+1} F_{K,t+1}}{U_{C,t+1}^2} d(\pi_{t+2})
\]

(45)

(46)
The FOC with respect to the price of installed private capital $Q_t$ yields:

$$U_{C,t} \Gamma_{5,t} \left(1 - S_{P,t} - S_{P,t} \dot{I}_t\right) = -\frac{\xi_t}{\xi_{t-1}} U_{C,t} \left[ \Gamma_{5,t-1} S'_{P,t} \dot{I}_t^2 + \Gamma_{6,t-1} \frac{1 - \delta}{Q_{t-1}} \right]$$

$$+ \beta \Gamma_{6,t} E_{\dot{t}} \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{Q_t^2} \left[ (1 - \delta) Q_{t+1} - \frac{U_{N,t+1} F_{K,t+1}}{U_{C,t+1} F_{N,t+1}} \right]$$

(47)

Finally, the FOC with respect to inflation and the nominal interest rate are the same as before.

6.3.5 Steady State

From equations (42) and (44), I have that

$$\Gamma_4 = \Gamma_3 = U_C.$$

Evaluating equations (46) and (47) at steady state and using the identity $\beta F_K = 1 - \beta (1 - \delta)$, I obtain:

$$\Gamma_6 = \Gamma_5 = 0.$$

6.3.6 Log linear approximation

Approximating the bond Euler Equation around the zero inflation steady state, I get:

$$\hat{\xi}_t + \dot{U}_{C,t} = \hat{\Gamma}_{3,t} + \frac{U_{CC}}{U_C} \left( \hat{\Gamma}_{1,t} + \hat{\Gamma}_{6,t} \right) - \frac{\theta}{U_C} \frac{U_{CN} U_C - U_{CC} U_N}{U_N U_C} \hat{\Gamma}_{2,t}$$

$$- \frac{U_{CC}}{U_C} \left( 1 - \delta - \frac{F_K U_{CN}}{F_N U_{CC}} \right) \hat{\Gamma}_{6,t-1} - \frac{U_{CC}}{\beta U_C} \hat{\Gamma}_{1,t-1},$$

(48)
where we have used the fact that $F_N \cdot U_C = -U_N$ at steady state. Similarly, for hours worked I get:

$$
\dot{U}_{N,t} + \hat{\xi}_t = \hat{\Gamma}_{3,t} + \hat{\Gamma}_{N,t} + \frac{\theta}{U_N F_N} \left\{ \frac{U_{NN} U_C - U_{CN} U_N}{U_N^2} + F_{NN} \right\} \hat{\Gamma}_{2,t} + \frac{U_{CN}}{U_N} \left[ \hat{\Gamma}_{1,t} + \hat{\Gamma}_{6,t} - \beta^{-1} \hat{\Gamma}_{1,t-1} \right]
$$

$$
- \left\{ (1 - \delta) \frac{U_{CN}}{U_N} - \frac{F_K U_{NN}}{F_N} U_N - \frac{F_K F_N - F_K F_{NN}}{F_N^2} \right\} \hat{\Gamma}_{6,t-1}
$$

The first order condition with respect to private capital (46) is now:

$$
\hat{\Gamma}_{4,t} = \beta (1 - \delta) \hat{E}_{4,t+1} + (1 - \beta (1 - \delta)) \hat{E}_{4,t} \left[ \hat{\Gamma}_{3,t+1} + \hat{\Gamma}_{K,t+1} + \theta \frac{F_{NK}}{U_N F_K} \hat{\Gamma}_{2,t+1} \right]
$$

$$
+ \beta \frac{F_{KK} F_N - F_K F_{NK}}{F_N} \hat{\Gamma}_{6,t}
$$

Using the fact that $Q = 1$ and $\beta F_K = 1 - \beta (1 - \delta)$, the log linear approximation of equation (42) can be written as follows:

$$
\hat{\Gamma}_{3,t} = \hat{\Gamma}_{4,t} - S''(1) \left( (1 + \beta) i_t - i_{t-1} - \beta \hat{E}_t \hat{E}_{t+1} \right) + \frac{S''(1)}{I} \left( \Gamma_{5,t-1} + \beta \hat{E}_t \Gamma_{5,t+1} - (1 + \beta) \Gamma_{5,t} \right),
$$

where $I$ is the steady state value of the level of private investment. Similarly, the log linear approximation of equation (47) gives:

$$
\hat{\Gamma}_{5,t} = \hat{\Gamma}_{6,t} - (1 - \delta) \hat{\Gamma}_{6,t-1}.
$$