Serial Priority in Project Allocation: A Characterisation

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7 December 2017

Abstract

We consider a model in which projects are to be assigned to agents based on their preferences, and where projects have capacities, i.e., can each be assigned to a minimum and maximum number of agents. The extreme cases of our model are the social choice model (the same project is assigned to all agents) and the house allocation model (each project is assigned to at most one agent). We propose a natural extension of the dictatorial rule (social choice model) and the serial priority rule (house allocation model) to cover the intermediate cases, and call it the *strong serial priority rule*. We show that, when minimum and maximum capacities are common to all projects, a strong serial priority rule is characterised by the axioms of *strategy-proofness, group-non-bossiness, limited influence, unanimity*, and *neutrality*. Our result thus provides a bridge between the characterisations in Gibbard (1973), Satterthwaite (1975) and Svensson (1999). We also provide an independent characterisation of the serial priority rule in the house allocation model, and demonstrate some new relations between the axioms.

JEL classification: C78, D71.

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1 Introduction

We consider the problem of assigning projects to agents based on preferences, where projects have minimum and maximum capacities that must be respected. Capacity constraints are a common problem faced, for example, by firms and educational institutions. Maximum capacities are often imposed on projects for reasons of space or efficiency, and minimum capacities because of feasibility. For instance, firms might find it inefficient to put too many workers on a given project, but at the same time might also need to ensure that there are enough workers to complete it. For schools or colleges offering courses to students, a course with too few students signing up for it might be too expensive to run, while a course with too many students cannot possibly fit them all into the classroom. Such a model also covers other situations: teachers assigning group homework to students, employers assigning office spaces to staff, airlines assigning flight routes to pilots, or even police precincts assigning neighbourhood beats to officers. In each case, we can imagine situations where the projects in question cannot be assigned to too many or too few agents.

In the present work, we make the simplifying assumption that all projects have the same maximum and minimum capacities. Thus, we represent minimum and maximum capacities as simply two numbers, $\underline{q}$ and $\bar{q}$, respectively, which can each range from 1 to n, where n is the total number of agents in the model. While this restriction might rule out some applications (i.e., project-specific capacity constraints), it nevertheless retains much of what makes this model interesting. Moreover, this framework also includes some familiar special cases. For instance, when project capacities are such that each project can be assigned to at most one agent (i.e., $\underline{q} = \bar{q} = 1$), we are in what is called the *house allocation* model. At the other extreme, where society as a whole has to pick one project to be assigned to all agents together (i.e., $\underline{q} = \bar{q} = n$), we are in the *social choice* model. Our new results chiefly pertain to the cases of intermediate capacities (i.e., $1 \leq \underline{q} \leq \bar{q} \leq n$, with at least one strict inequality). However, we also show a close link between our work and some seminal results for the two restricted models above.

We are interested in finding a feasible allocation (respecting capacity constraints for all projects) for any combination of agents' preferences over projects[1]. It should be noted that we assume there are more projects than can be feasibly assigned, and so, in particular, we do not insist that every project be assigned; instead, we merely require that an assigned project respects its constraints. In particular, we are interested in Pareto-efficient[2], group-strategy-proof[3] and neutral[4] rules. If there are no minimum capacities, we have an obvious candidate: the serial priority rule (also known as the serial dictatorship). According to this rule, agents are ranked by a pre-specified order. Faced with any combination of preferences, we start with the first agent in the order and, in turn, assign agents their most-preferred project from those that are still available after earlier agents have received their assignments. This rule is widely used in practice, and is attractive in its simplicity. Firstly, the order of agents can be determined by relatively transparent and objective criteria, such as seniority or grade-point-averages. Secondly, the only restriction on assigning any project is that it should not have reached its maximum capacity.

The serial priority rule also satisfies many desirable properties. In the house allocation model, it is strategy-proof, neutral, and non-bossy[5] (Svensson, 1999). It is Pareto-efficient. It is also group-strategy-proof (Pápai, 2000a). Moreover, in the house allocation model, any Pareto-efficient allocation can be reached using a serial priority rule with a suitable ordering of agents (Abdulkadiroğlu and Sönmez, 1998). The serial priority rule is contained in a class of rules that is Pareto-efficient, individually rational, strategy-proof, weakly neutral and consistent (Sönmez and Ünver, 2010). The serial priority rule is contained in a class of (hierarchical exchange) rules that is group-strategy-proof, Pareto-efficient and reallocation-

[1] We assume that agents care only about the project they are assigned, and for instance do not care about who else might be assigned the same project.

[2] A feasible allocation is Pareto-efficient if there is no other feasible allocation that makes some agent better off (with respect to her preferences) without making any other agent worse off. A rule is Pareto-efficient if it always prescribes feasible Pareto-efficient allocations.

[3] A rule is strategy-proof if it ensures that no agent can gain from misreporting her preferences. Group-strategy-proofness extends this property to groups of agents (of any size).

[4] A rule is neutral if the allocations it prescribes do not depend on how projects are named. We provide formal definitions later.

[5] A rule is non-bossy if no agent, by changing her preferences, can change any other agent's assignment without also changing her own assignment.

proof (Pápai, 2000a). Further, in a model with endogenous information acquisition, the serial priority rule is the unique rule that is ex-ante Pareto-efficient, strategy-proof and non-bossy (Bade, 2015).

However, when we move to a model with non-trivial minimum capacities, then the serial priority rule loses many of these nice properties. In particular, it is neither Pareto-efficient nor strategy-proof (Monte and Tumennasan, 2013). The intuition behind this observation is that minimum capacities introduce a new dimension of feasibility, and so now agents may need to also consider the preferences of later agents. In particular, a project assigned to them might not be assigned to enough agents later, thus remaining infeasible. Thus an agent might not truthfully reveal her preferences (e.g., an earlier agent who worries about a project remaining infeasible might pick a 'safer' project even though it is less preferred, or a later agent who can render some project infeasible could influence earlier agents to pick something that she prefers). Similarly, this rule can be Pareto-inefficient, when coordination failures lead to projects being under-subscribed that otherwise enough agents might prefer.

These issues with the serial priority rule arise precisely because it is not designed to deal with minimum capacities. Monte and Tumennasan (2013) propose a stronger version of the serial priority rule for this model with non-trivial minimum capacities, which they call 'serial dictatorship with project closures'. In order to ensure feasibility, the set of projects available to be assigned to later agents in the ordering is constrained in two ways: firstly, agents can only be assigned projects that are not filled up by earlier agents' choices (respecting maximum capacities). This is the same as for the serial priority rule. Additionally, however, agents can only be assigned one of those projects which lead to a feasible allocation, based on earlier agents' observed assignments, and a potential combination of assignments for later agents. Thus, the rule makes only feasible assignments at every stage, and the strategic problems associated with feasibility are overcome.

Indeed, Monte and Tumennasan (2013) show that the serial dictatorship with project closures is strategy-proof and Pareto-efficient. Kamiyama (2013) extends this rule to a more general capacity constraint setting, and shows that it retains these features. In another extension, Klijn (2017) studies a general framework in which workers have preferences over not only the sets of projects they are assigned, but also their possible co-workers. The 'serial shrink project allocation' rule proposed there bears a close resemblance to the serial dictatorship with project closures, though it is used in a more general model of preferences. Klijn (2017) shows that this rule is (weakly) group-strategy-proof and Pareto-efficient. Cechlárová and Fleiner (2017) study computational and strategic properties of this rule in a more general model.

We propose a parsimonious definition of the serial dictatorship with project closures, and call it the *strong serial priority rule* ($SSPR$). We strengthen earlier results by showing that the $SSPR$ is group-strategy-proof, Pareto-efficient, and neutral. In fact, we pin the rule down exactly: our main result is that, under general (but common) capacity constraints, a project allocation rule satisfies strategy-proofness, group-non-bossiness[6], unanimity[7], neutrality and limited influence if and only if it is an $SSPR$. Along the way, we demonstrate other effects of minimum capacities: in particular, strategy-proofness, non-bossiness and neutrality do not guarantee Pareto-efficiency in the general model, despite doing so in the house allocation model (Svensson, 1999).

There is a close relation between our work and some important results for the special cases discussed earlier. In the social choice model, the famous Gibbard-Sattherthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975) states that the only strategy-proof and unanimous rule is dictatorial[8]. At the other extreme, in the house allocation model, the serial priority rule is the unique deterministic rule that is strategy proof, non-bossy and neutral (Svensson, 1999). In terms of capacity constraints, our result fills the space between. Moreover, we establish a bridge between the two extremes by showing the following:

1. In the social choice model, the $SSPR$ is equivalent to the dictatorial rule.
2. In the house allocation model, the $SSPR$ is equivalent to the serial priority rule.
3. Our axioms are implied by strategy-proofness and unanimity in the social choice model,

[6]Group-non-bossiness extends non-bossiness to groups in a natural way. If, by changing their preferences, one group of agents retain the same aggregate assignment (possibly exchanging with each other), then the assignments of agents in the other group must remain exactly the same.

[7]A rule is unanimous if it gives agents their most-preferred assignments whenever it is feasible to do so.

[8]The dictatorial rule identifies one agent who, for all profiles, gets her top-ranked project independent of the preferences of other agents (all other agents are automatically assigned this project).

and are implied by strategy-proofness, non-bossiness, and neutrality in the house allocation model[9].

At first glance, one might be surprised that a natural generalisation of the serial priority rule should require so much to pin it down, particularly when the two extreme rules require relatively little by way of axioms. However, it soon becomes clear that the presence of general capacity constraints greatly increases the set of feasible allocations compared to the extreme cases, thereby increasing the number of restrictions required to derive the same rule.

While the other axioms are well-known, limited influence is new to this paper. It can be thought of as requiring 'seniority' with respect to individual projects. To understand the property, suppose there is an instance of some agent being assigned a project that some other agent prefers to her own assignment (the first is 'more senior' than the second agent for that project). Then, limited influence requires that the junior agent not be able to displace the senior agent at any other instance where she is assigned that project. In terms of our motivating examples, it is quite natural: if some workers have preferred status at some projects, some students are more eligible for certain courses, some pilots have first call on some flight routes, etc., then they have a higher (earlier) claim on them than others. Limited influence is a powerful axiom: indeed, we use it to provide a characterisation of the serial priority rule in the house allocation model that is independent of the one in Svensson (1999),[10], thereby showing how limited influence is key to understanding serial priority rules in general. We also show how weakening or dropping this requirement in our general model allows a much wider class of rules than serial priority.

In related literature on minimum capacities, Rhee (2011) extends the unit capacity framework to one where each project can be assigned to exactly two agents. This model is related to ours, except that agents are already organised in fixed pairs. The 'serial priority over cou-

[9] The implications in the social choice model follow naturally (often trivially). For the house allocation model, however, we have to use some of the methods of the original proof in Svensson (1999) to derive them. Thus we are unable to claim these results as direct corollaries.

[10] In terms of limited influence, neutrality, and non-wastefulness. Non-wastefulness is a weak notion of Pareto-efficiency in the house allocation model. It requires that whenever there is an agent who prefers a project to her assignment, there is another agent who is assigned that project. This property does not hold in the general model with minimum capacities.

ples' rule proposed there is strategy-proof, non-bossy and neutral, but it is Pareto-inefficient if we also allow the couples in question to change. Our model includes this model, and we show that the $SSPR$ is also Pareto-efficient. Fragiadakis et al. (2015) also consider a model with minimum capacities, but the chief difference is that they assume that each project is required to be assigned, whereas we allow for some projects to not be assigned at all.

The serial priority rule also has an application to cases where agents can consume more than one object. It is the only rule that is strategy-proof, totally non-bossy and satisfies citizen sovereignty (Pápai, 2001). If agents also have quotas, it is the unique rule that is strategy-proof, Pareto-efficient, non-bossy and neutral (Pápai, 2000b; Hatfield, 2009). Moreover, it is the unique rule that is Pareto-efficient, coalitional strategy-proof and resource-monotonic (Ehlers and Klaus, 2003). For the sake of brevity, we omit the definitions of the additional axioms mentioned above.

The paper is organised as follows. In Section 2, we present the model. In Section 3 we define the strong serial priority rule, and connect it to the dictatorial rule and the serial priority rule. In Section 4 we discuss the axioms and present our new characterisation of the serial priority rule in the house allocation model. In Section 5 we present the characterisation of the strong serial priority rule. In Section 6, we study the special cases of house allocation and social choice. In Section 7, we demonstrate the independence of the axioms.

2 The Model

There is a finite set of *agents* $N = \{\dots, i, j, \dots\}$ (with $|N| = n \geq 2$) and a finite set of *projects* $Z = \{a, b, c, ...\}$. We assume an artificial 'null project' $\emptyset \in Z$, and assume that $\emptyset$ is not scarce. Each project $a \in Z$ has a minimum capacity $\underline{q}$ and maximum capacity $\bar{q}$ with $\underline{q}, \bar{q} \in \{1, \dots, n\}$ and such that $\underline{q} \leq \bar{q}$. Let m denote the maximum number of different projects that can be assigned in any feasible allocation, i.e., m is an integer such that $\underline{q}m \leq n < \underline{q}(m+1)$. We assume that there are more projects available than can be feasibly assigned to agents, i.e., that $|Z| \geq max\{3, m+1\}$[11].

An allocation associates a project with each agent. Formally, we denote an *allocation* by

[11] Our results hold with $|Z| \geq 3$, but we assume $|Z| \geq m+1$ as well for ease of exposition.

a vector $x \in Z^n$ with $x = (x_1, ..., x_n)$, where $x_i \in Z$ denotes the *assignment* of agent i in x. We call the allocation $(\emptyset, \dots, \emptyset)$ the *null allocation*. An allocation is feasible if it assigns each project either to nobody, or to a number of agents within the minimum and maximum capacity. Formally, for any $a \in Z$ and any allocation x, let the set of agents who are assigned a in x be denoted $S_a(x) = \{j \in N \mid x_j = a\}$. Then an allocation x is *feasible* if, for all $a \in Z$, $|S_a(x)| = 0$ or $\underline{q} \leq |S_a(x)| \leq \bar{q}$. Let the set of all feasible allocations be given by $A \subsetneq Z^n$.

Agents care only about the project they are assigned, and preferences over assignments are strict. Formally, agent $i \in N$ has *preferences* that are given by a linear order R_i[12] over Z. For any a, b, $a\, R_i\, b$ is interpreted as 'project a is at least as good as project b for agent i under preferences R_i'. The associated strict relation is given by P_i, such that $a\, P_i\, b$ if $a\, R_i\, b$ and $a \neq b$. For any $a, b \in Z$, $a\, P_i\, b$ means 'a is preferred by i to b under preferences R_i'. In terms of our artificial null project, we assume that each agent prefers any 'real' project to receiving the null project, i.e., $a\, P_i\, \emptyset$ for all agents $i \in N$, all projects $a \in Z$, and all preferences R_i.

A collection of preferences for all agents is called a *preference profile*, or simply a *profile*, and is denoted by $R = (R_1, ..., R_n)$. The set of all profiles is denoted by $\mathcal{R}$. In this model we usually suppress reference to $\mathcal{R}$, with the understanding that we operate on the full domain of preferences everywhere. As is the convention, we write R_{-i} for a sub-profile of preferences of all agents other than i. Similarly, for a subset of agents M, we write $R_M, R'_M, \dots$ and $R_{-M}, R'_{-M}, \dots$ to denote sub-profiles of preferences of agents in subsets M and $N \setminus M$, respectively. For an agent i, a set of projects $Y \subseteq Z$, and preferences R_i, we denote by $top(R_i, Y)$ agent's i's (unique) top-ranked project in Y according to R_i. Similarly, for a subset of agents M, we denote by $top(R_M, Y)$ the $|M|$-dimensional vector of top-ranked projects in Y of agents in M according to their preferences in R_M. Often, we simply write $top(R_i), top(R_M)$, etc., when Y is clear from the context.

A *(project allocation) rule (rule)* is a function $f : \mathcal{R} \to A$ that maps every profile to a feasible allocation. For any agent i, $f_i(R)$ is the assignment she receives at profile R according to the rule f. Similarly, for any subset of agents M, $f_M(R)$ is the vector of assignments of

[12] R_i is a binary relation that is reflexive (for all $a, a\, R_i\, a$), complete (for all $a, b, a\, R_i\, b$ or $b\, R_i\, a$), transitive (for all $a, b, c, a\, R_i\, b$ and $b\, R_i\, c$ imply $a\, R_i\, c$) and antisymmetric (for any a, b, $a\, R_i\, b$ and $b\, R_i\, a$ imply $a = b$).

agents in M at R.

3 The Strong Serial Priority Rule

The primitive of the *SSPR* is an *agent ordering*, which is a bijection $\sigma : \{1, \ldots, n\} \to N$ that maps every agent in N uniquely to an integer between 1 and n. For σ and agent $i \in N$, $\sigma^{-1}(i)$ is the position of i in σ. Let Σ be the set of all agent orderings. For $\sigma \in \Sigma$, and for any $i, j \in N$, we say that i *precedes* j *in* σ if she occupies an earlier position, i.e., $\sigma^{-1}(i) < \sigma^{-1}(j)$. For $\sigma \in \Sigma$ and for any $i \in N$, let the *set of preceding agents for* i *in* σ be $\Omega_\sigma(i) = \{j \in N | \sigma^{-1}(j) < \sigma^{-1}(i)\}$.

The Strong Serial Priority Rule

Let the capacities $\underline{q}$ and $\bar{q}$ be given. Let $s^0 = (\emptyset, \ldots, \emptyset)$ be the null allocation, and let the agent ordering $\sigma \in \Sigma$ be given. Let R be a profile. The allocation $\varphi^\sigma(R)$ is determined in rounds as follows:

Round t, $1 \leq t \leq n$:

1. **Constrained assignment set:** Let $Z_t(s^{t-1})$ be the set of all projects that agent $\sigma(t)$ can be assigned, which produce a feasible allocation for the given assignments of preceding agents for $\sigma(t)$ and some combination of assignments for other agents. Formally, define $Z_t(s^{t-1}) = \{b \in Z \mid \text{there exists } x \in A \text{ with } x_i = s_i^{t-1} \text{ for all } i \in \Omega_\sigma(\sigma(t)), \text{ and } x_{\sigma(t)} = b\}$. In particular, note that $Z_1(s^0) = Z$.

2. **Assignments:** Agent $\sigma(t)$ is assigned her most-preferred project in $Z_t(s^{t-1})$, and the allocation is updated to reflect this. Formally, set $s^t_{\sigma(t)} = top(R_{\sigma(t)}, Z_t(s^{t-1}))$, and let $s^t_j = s^{t-1}_j$ for all $j \neq \sigma(t)$.

3. **Verification:** If $t = n$, set $\varphi^\sigma(R) = s^n$, and stop. Otherwise, go to Round $t + 1$.

It is clear that the algorithm terminates in exactly n steps, and that the resulting allocation $\varphi^\sigma(R)$ is feasible. In any round t, the set $Z_t(s^{t-1})$ can be easily determined. As we proceed along the ordering σ, each agent can be assigned any project as long as (1) it has not been assigned to $\bar{q}$ agents already; and (2) m distinct projects have not already been assigned at least once. In the latter case, the agent may be assigned only from among these m projects. We give some examples.

Example 1. Let $N = \{1,2,3,4,5,6\}$, let $Z = \{a,b,c,d\}$, let $\underline{q} = 2, \bar{q} = 3$. Then $m = 3$. For $i,j \in N$, let $\sigma = (1,2,3,4,5,6)$ (so agent 1 picks first, then agent 2, and so on). For the profiles given in Figure 1, it is easy to work out the resulting allocations for the SSPR at σ (given in boxes).

Figure 1: SSPR: Examples

R						R'					
R_1	R_2	R_3	R_4	R_5	R_6	R'_1	R'_2	R'_3	R'_4	R'_5	R'_6
[a]	[b]	[a]	[c]	d	c	[a]	[b]	[d]	c	d	c
d	a	d	d	[c]	d	d	a	a	[d]	c	d
c	c	c	a	a	a	c	c	c	a	[a]	a
b	d	b	b	b	[b]	b	d	b	b	b	[b]

Agents 1, 2, 3 who appear first in the order can always be assigned their top-ranked projects, but the assignments of later agents depend on what these agents have been assigned. In particular, note that for profile R, agent 5 cannot get project d, even though it is her top-ranked project, because projects a,b,c have been assigned to at least one agent each, and there is no feasible allocation that could follow from an assignment of d to agent 5. Similarly, the only project available to agent 6 that ensures a feasible allocation is project b, even though she prefers project a and it has spare capacity. In profile R', agent 4 and later have to be assigned from among a,b,d, even though each of them prefers project c, as no project distinct from these can be assigned in a feasible allocation.

We now show that the $SSPR$ is equivalent to the serial priority rule in the house allocation model, and the dictatorial rule in the social choice model.

Serial Priority Rule

Let $\underline{q} = \bar{q} = 1$. Let s^0 be the null allocation, and let $\sigma \in \Sigma$ be given. Let R be a profile. We denote the serial priority rule by φ^{σ}_{SP}, where the allocation $\varphi^{\sigma}_{SP}(R)$ is determined in rounds as follows:

<u>Round t, $1 \leq t \leq n$:</u>

1. **Constrained assignment set:** Projects assigned to earlier agents are unavailable in Round t. That is, define $\hat{Z}_t(s^{t-1}) = (Z \setminus \{s_i^{t-1}\}_{i \in \Omega_\sigma(\sigma(t))}) \cup \{\emptyset\}$.

2. **Assignments:** Set $s^t_{\sigma(t)} = top(R_{\sigma(t)}, \hat{Z}_t(s^{t-1}))$, and set $s^t_j = s^{t-1}_j$ for all $j \neq \sigma(t)$.

3. **Verification:** If $t = n$, set $\varphi^{\sigma}_{SP}(R) = s^n$ and stop. Otherwise, go to Round $t + 1$.

When $\underline{q} = \bar{q} = 1$, it is easy to see that $Z_t(s^{t-1}) = \hat{Z}_t(s^{t-1})$ for any round t. This is because, once a project has been assigned to some agent in a round $t' < t$, and there is only one copy of it, there is no feasible allocation that allows it to be assigned to agent $\sigma(t)$ as well. Since the algorithmic procedure is the same for the two rules, it follows that the $SSPR$ is equivalent to the serial priority rule.

Dictatorial Rule

Definition **1.** Let $\underline{q} = \bar{q} = n$. A project allocation rule φ_D is the *dictatorial rule* if there exists an agent $i \in N$ (the *dictator*) such that, for all profiles R, $\varphi_D(R) = top(R_i, Z)$.

Let $\underline{q} = \bar{q} = n$, and consider the $SSPR$. In the first round, agent $\sigma(1)$ is assigned her top-ranked project in Z. Since the ordering σ is fixed, this is true at every profile. In subsequent rounds, the *only* feasible allocation remaining is that which assigns the same project to every other agent, i.e., the set $Z_t(s^{t-1})$ in any round $t > 1$ consists precisely of this project. Thus the feasible allocation produced by the $SSPR$ in this case assigns the top-ranked project of agent $\sigma(1)$ to all agents. So $\sigma(1)$ is the dictator, and it follows that the $SSPR$ is equivalent to the dictatorial rule.

4 Axioms for Project Allocation Rules

We now turn to the axioms that we require of a rule. First, strategy-proofness requires truth-telling to be a weakly dominant strategy for all agents. Formally, a rule f satisfies **strategy-proofness** if, for all profiles R, all agents $i \in N$, and all preferences R'_i, we have that $f_i(R)\, R_i\, f_i(R'_i, R_{-i})$. Similarly, a rule is group-strategy-proof if it is immune to strategic manipulations by groups. For a profile R, a set of agents $M \subseteq N$, and a sub-profile R'_M, we say that M can *manipulate at R via R'_M* if $f_i(R'_M, R_{-M})\, R_i\, f_i(R)$ for all $i \in M$, and $f_j(R'_M, R_{-M})\, P_j\, f_j(R)$ for some $j \in M$. A rule f satisfies **group-strategy-proofness** if, for any profile R, there does not exist M and R'_M such that M can manipulate at R via R'_M.

Non-bossiness requires that an agent not be able to affect other agents' outcomes via changing her preferences, without affecting her own. Formally, a rule f satisfies **non-bossiness** if, for all profiles R, all agents $i \in N$, and all preferences R'_i, we have that $[f_i(R'_i, R_{-i}) = f_i(R)] \implies [f(R'_i, R_{-i}) = f(R)]$. In the house allocation model, Pápai (2000a) shows that group-strategy-proofness is equivalent to the combination of strategy-proofness and non-bossiness. The arguments are quite general and apply to our model as well. The proof is reproduced here for completeness.

Lemma **1** (Pápai (2000a)). *A rule f is group-strategy-proof if and only if it is strategy-proof and non-bossy.*

Proof: Let f satisfy group-strategy-proofness. It is clear that it satisfies strategy-proofness. To see that f satisfies non-bossiness, consider a profile R, agents $i, j \in N$, and preferences R'_i such that $f_i(R'_i, R_{-i}) = f_i(R)$. Then $f_i(R'_i, R_{-i})\ R_i\ f_i(R)$. Suppose $f_j(R'_i, R_{-i}) \neq f_j(R)$. If $f_j(R'_i, R_{-i})\ P_j\ f_j(R)$, then $M = \{i, j\}$ can manipulate at R via (R'_i, R_j), violating group-strategy-proofness. Similarly, if $f_j(R)\ P_j\ f_j(R'_i, R_{-i})$, then M can manipulate at (R'_i, R_{-i}) via (R_i, R_j), violating group-strategy-proofness. Thus $f_j(R'_i, R_{-i}) = f_j(R)$, and f satisfies non-bossiness.

For the other direction, let f satisfy strategy-proofness and non-bossiness. Consider $M \subseteq N$, a profile R, and a subprofile R'_M, such that for all $i \in M$, $f_i(R'_M, R_{-M}) R_i f_i(R)$. Let $i \in M$, and let preferences $\hat{R}_i$ be such that $top(\hat{R}_i) = f_i(R'_M, R_{-M})$ and other projects are ranked the same as in R_i. By strategy-proofness, $f_i(\hat{R}_i, R_{-i}) = f_i(R)$. So by non-bossiness, $f(\hat{R}_i, R_{-i}) =$

$f(R)$. By a repeated argument for all agents in M, we have that $f(\hat{R}_M, R_{-M}) = f(R)$. But by strategy-proofness and non-bossiness, $f(\hat{R}_M, R_{-M}) = f(R'_M, R_{-M})$. So $f(R'_M, R_{-M}) = f(R)$, and f satisfies group-strategy-proofness. ■

Neutrality ensures that a rule treats all projects symmetrically, up to relabelling of their names. Formally, a rule f satisfies **neutrality** if, for all profiles R and all permutations[13] π of Z, we have that $f(\pi R) = \pi f(R)$. We have the following well-known result:

THEOREM **1** (Svensson (1999)). *Let $\underline{q} = \bar{q} = 1$. Then a rule f satisfies strategy-proofness, non-bossiness, and neutrality, if and only if it is a serial priority rule.*

Next, we define Pareto-efficiency. For two allocations $x, y \in Z^n$ and a profile R, we say x *Pareto-dominates* y *at* R if all agents are at least as well-off in x as at y, and some agent is strictly better off in x, i.e., if $x_i \; R_i \; y_i$ for all $i \in N$ and $x_j \; P_j \; y_j$ for some $j \in N$. A rule f satisfies **Pareto-efficiency** if, for any profile R, there is no feasible allocation $x \in A$ that Pareto-dominates $f(R)$ at R.

Svensson (1999) shows that when $\underline{q} = \bar{q} = 1$, strategy-proofness, non-bossiness and neutrality imply Pareto-efficiency. However, when $\underline{q} > 1$, this is no longer true.

PROPOSITION **1.** *Let $\underline{q} > 1$. Then strategy-proofness, non-bossiness and neutrality do not imply Pareto-efficiency.*

Proof: We give an example of a rule (adapted from Rhee (2011)) that satisfies strategy-proofness, non-bossiness and neutrality, but fails to satisfy Pareto-efficiency. Let $N = \{1, 2, 3, 4\}, Z = \{a, b, c\}$, $\underline{q} = \bar{q} = 2$, and let f be a rule that, for any profile R, makes assignments as follows:

[13] A permutation applied to a collection of objects X is a bijection $\pi : X \to X$ that associates each object in X with a unique object in X (possibly itself). In our case, we use it to mean a relabelling of projects such that a collection of projects exchange their names. For example, under π, project a may now be called project b ($\pi(a) = b$), which is now called project c ($\pi(b) = c$), which in turn is called project a ($\pi(c) = a$). The permutation π applied to a preference profile R (written as πR) or an allocation x (written as πx) permutes the projects in the preferences or the allocation according to the permutation applied to the underlying set of projects Z. It should be noted that, under our assumptions on preferences, permutations do not ever change the position of the null project, which always remains last in every agent's preference ordering.

1. $f_1(R) = top(R_1, Z)$ and $f_2(R) = f_1(R)$.

2. $f_3(R) = top(R_3, Z \setminus f_1(R))$ and $f_4(R) = f_3(R)$.

It is easy to see that f satisfies strategy-proofness, non-bossiness and neutrality. In particular, preferences of agents 2 and 4 are irrelevant to their assignments. Agent 1 always gets her top-ranked project, while agent 3 always gets her most-preferred project that is not already assigned to agent 1. However, this rule is not Pareto-efficient. For example, consider the profile in Figure 2, with assignments prescribed by f for this profile given in boxes.

Figure 2: f is not Pareto-efficient

R_1	R_2	R_3	R_4
$\boxed{a}$	c	a	c
b	$\boxed{a}$	$\boxed{b}$	$\boxed{b}$
c	b	c	a

In particular, the allocation $x = (a, c, a, c)$ Pareto-dominates $f(R)$ at R. Since x is feasible, f is not Pareto-efficient. ∎

However, the combination of strategy-proofness, non-bossiness and unanimity ensures Pareto-efficiency, where unanimity is a condition that requires that a rule should assign agents their top-ranked projects whenever the vector of top-ranked projects is itself a feasible allocation. Formally, a rule f satisfies **unanimity** if, for all profiles R, we have that $[top(R) \in A] \implies [f(R) = top(R)]$. It can be checked that the rule f defined in Proposition 1 fails to satisfy unanimity. We give a result adapted from Pápai (2001) to show that this is why it is not Pareto-efficient.

LEMMA **2.** *A rule satisfying strategy-proofness, non-bossiness and unanimity is Pareto-efficient.*

Proof: Let f satisfy strategy-proofness, non-bossiness and unanimity. Suppose f fails to satisfy Pareto-efficiency. Then there is a profile R and a feasible allocation $x \in A$ such that

$x_i \, R_i \, f_i(R)$ for all $i \in N$, and $x_j \, P_j \, f_j(R)$ for some $j \in N$. In particular, $x \neq f(R)$. Let $k \in N$ and let R'_k be preferences such that $top(R'_k) = x_k$, and all other projects are ranked the same as in R_k. Since $x_k R_k f_k(R)$, by strategy-proofness we have that $f_k(R'_k, R_{-k}) = f_k(R)$. Thus, by non-bossiness, $f(R'_k, R_{-k}) = f(R)$. By a repeated argument for all $k' \in N$ and $R'_{k'}$ with $top(R'_{k'}) = x_{k'}$ and other projects ranked the same as in $R_{k'}$, we have that $f(R') = f(R)$. However, $top(R') = x$ and $x \in A$, thus unanimity implies $f(R') = x$. Since $x \neq f(R)$, this is a contradiction. ■

When $\underline{q} = \bar{q} = n$, we have the Gibbard-Sattherthwaite Theorem.

Theorem **2** (Gibbard (1973), Satterthwaite (1975)). *Let $\underline{q} = \bar{q} = n$. Then a rule is strategy-proof and unanimous if and only if it is a dictatorial rule.*

We see that strategy-proofness, non-bossiness and neutrality guarantee the serial priority rule in the house allocation model. Also, strategy-proofness and unanimity guarantee dictatorship in the social choice model. We show in Section 5 that the $SSPR$ satisfies all four of these axioms. However, these axioms permit a larger class of rules under general capacity constraints. We now give an example of a rule that satisfies these axioms but is not an $SSPR$.

Example **2.** Let $\mathcal{N} = \{1, 2, 3, 4\}, \mathcal{Z} = \{a, b, c\}$, $\underline{q} = \bar{q} = 2$. Let f be a rule similar to the $SSPR$, but that works as follows: For a preference profile R, if $top(R_1) = top(R_2)$, then $\sigma = (1, 2, 3, 4)$. Otherwise, $\sigma = (1, 3, 2, 4)$.

Agent 1 always gets her top-ranked project, for any preference profile. Agent 2 can always guarantee the assignment of the same project as agent 1 (by ranking it first) or agent 3 otherwise. Agent 3 can always get her top-ranked project when it is distinct from the top-ranked project of agent 1. Agent 4's assignment does not depend on her preferences. We show the allocation prescribed by f for some profiles in Figure 3.

It is left to the reader to verify that f satisfies the axioms of strategy-proofness, non-bossiness, unanimity and neutrality. However, f is not an $SSPR$, as the ordering σ depends on the profile and is not fixed ex-ante. ■

We propose a new axiom, which we call limited influence. For an agent $i \in N$ and a project $a \in Z$, let $U_i(a)$ refer to the set of agents who, for some profile R, are assigned

Figure 3: Example

R				R'				R''			
R_1	R_2	R_3	R_4	R'_1	R_2	R_3	R_4	R'_1	R_2	R'_3	R_4
$\boxed{\text{b}}$	$\boxed{\text{b}}$	b	b	$\boxed{\text{a}}$	$\boxed{\text{b}}$	$\boxed{\text{b}}$	b	$\boxed{\text{a}}$	b	$\boxed{\text{c}}$	b
a	a	$\boxed{\text{a}}$	$\boxed{\text{a}}$	b	a	a	$\boxed{\text{a}}$	b	$\boxed{\text{a}}$	b	a
c	c	c	c	c	c	c	c	c	c	a	$\boxed{\text{c}}$

project a when it is (strictly) preferred by i to her own assignment at that profile. That is, let $U_i(a) = \{j \in N \mid \text{there exists } R \text{ such that } a\,P_i\,f_i(R) \text{ and } f_j(R) = a\}$. Then limited influence requires that at any profile where agent j is assigned a and agent i weakly prefers a to her own assignment, agent j's assignment does not change for any other reported preferences of agent i. Formally, a rule f satisfies **limited influence** if, for any profile R, any agent $i \in N$, any project $a \in Z$ such that $a\,R_i\,f_i(R)$, if there is a $j \in U_i(a)$ such that $f_j(R) = a$, then for all R'_i, $f_j(R'_i, R_{-i}) = a$.

Limited influence is a powerful axiom. The astute reader would spot that it already contains much of the flavour of the serial priority rule, in that it suggests that agent j gets to pick project a 'before' agent i. Moreover, in combination with neutrality, the limited nature of the influence can be extended to all projects. Indeed, we now show that the combination of limited influence, neutrality and a weaker notion of Pareto-efficiency called non-wastefulness characterise the serial priority rule in the house allocation model. We say a rule f satisfies **non-wastefulness** if, for all R, all $i \in N$, and all $a \in Z$, $aP_if_i(R) \implies$ there exists $j \neq i$ such that $f_j(R) = a$. When $\underline{q} = \bar{q} = 1$, Pareto-efficiency implies non-wastefulness, though this is not true in general (because of minimum capacities). The converse is also not true in general. We first show some useful implications.

Lemma **3.** *Let $\underline{q} = \bar{q} = 1$. A rule that satisfies limited influence and non-wastefulness is strategy-proof.*

Proof: Let f be a rule that satisfies limited influence and non-wastefulness. Let R be a profile, $i \in N$ an agent, and $a \in Z$ such that $a\,P_i\,f_i(R)$. Let R'_i be arbitrary preferences. We show that $f_i(R'_i, R_{-i}) \neq a$. Since $a\ P_i\ f_i(R)$, by non-wastefulness there is some $j \neq i$ such that $f_j(R) = a$. By definition, $j \in U_i(a)$. By limited influence, therefore, $f_j(R'_i, R_{-i}) = a$. In particular, $f_i(R'_i, R_{-i}) \neq a$. Thus f satisfies strategy-proofness. ■

A simple example shows that non-bossiness is independent of these axioms.

Example 3. Let $N = \{1, 2, 3, 4\}$, $Z = \{a, b, c, d, e\}$, $\underline{q} = \bar{q} = 1$. Let f be a rule defined as follows: For any R, if the second-ranked project in R_1 is the same as $top(R_2)$, then the agent ordering is $\sigma = (1, 2, 3, 4)$, and is $\sigma = (1, 2, 4, 3)$ otherwise. It is straightforward to check that f satisfies limited influence, non-wastefulness and neutrality. However, f is bossy, as can be seen from the preferences in Figure 4.

Figure 4: Independence of non-bossiness

R_1	R_2	R_3	R_4	R'_1	R_2	R_3	R_4
[a]	[b]	[c]	c	[a]	[b]	c	[c]
b	a	d	[d]	c	a	[d]	d
c	c	a	a	b	c	a	a
d	d	b	b	d	d	b	b
e	e	e	e	e	e	e	e

When agent 1 changes her preferences from R_1 to R'_1, she changes the assignments of agents 3, 4 without changing her own assignment. ■

We now state and prove our alternate characterisation of the serial priority rule.

Theorem 3. *Let $\underline{q} = \bar{q} = 1$. A rule satisfies non-wastefulness, limited influence and neutrality if and only if it is a serial priority rule.*

Proof: We first show that the serial priority rule satisfies non-wastefulness, neutrality, and limited influence. Let $\sigma \in \Sigma$ be given. We know from Svensson (1999) that the serial priority rule φ^{σ}_{SP} satisfies Pareto-efficiency (and therefore non-wastefulness) and neutrality. To show limited influence, let R be a profile, $i \in N$ be an agent, and $a \in Z$ be a project such that

$a\ R_i\ \varphi^{\sigma}_{SP(i)}(R)$. Let $j \neq i$ be an agent such that $\varphi^{\sigma}_{SP(j)}(R) = a$. Since $\bar{q} = 1$, it follows that $\varphi^{\sigma}_{SP(i)}(R) \neq a$, and thus $a\ P_i\ \varphi^{\sigma}_{SP(i)}(R)$ and $j \in U_i(a)$. By the definition of the serial priority rule, therefore, agent j gets her assignment before agent i. Thus, for any R'_i, since agent i cannot change the order of agents, agent j continues to pick before agent i, and so $\varphi^{\sigma}_{SP(j)}(R'_i, R_{-i}) = a$. Thus φ^{σ}_{SP} satisfies limited influence.

To show the reverse direction, we construct the ordering σ over agents, and show that assignments are made according to the ordering for any preference profile. Let f be a rule satisfying non-wastefulness, limited influence and neutrality. Let R be a profile where $top(R_i) = top(R_j)$ for all $i, j \in N$. Let $a \in Z$ be such that $top(R_i) = a$ for all $i \in N$. By non-wastefulness, there is some $i \in N$ such that $f_i(R) = a$. Note that $a\,P_j\,f_j(R)$ for all $j \neq i$. Let $i \in U_j(a)$ for all $j \neq i$. We now show that agent i is assigned a at any profile where she ranks a top, independent of the preferences of other agents.

Let R'_i be preferences such that $top(R'_i) = a$. Suppose $f_i(R'_i, R_{-i}) \neq a$. Then $a\ P'_i\ f_i(R'_i, R_{-i})$. By non-wastefulness, $f_j(R'_i, R_{-i}) = a$ for some $j \neq i$. But then $j \in U_i(a)$, and so limited influence implies $f_j(R) = a$, which is a contradiction. Thus $f_i(R'_i, R_{-i}) = a$. Let R'_{-i} be an arbitrary sub-profile for agents other than i. Let $j \neq i$, and consider the profile (R'_{ij}, R_{-ij}). By limited influence, $f_i(R'_{ij}, R_{-ij}) = a$. By a recursive argument for all other agents, we get that $f_i(R') = a$. Since R' was arbitrary, $f_i(R') = a$ whenever $top(R'_i) = a$. By neutrality, for any $b \in Z$ and any R, $f_i(R) = b$ whenever $top(R_i) = b$.

Set $\sigma(1) = i$. Define $M^1 = N \setminus \{\sigma(1)\}$, $Z^1 = Z \setminus f_{\sigma(1)}(R)$. By the earlier arguments, there is some agent $j \in M^1$ such that $f_j(R) = top(R_j, Z^1)$ for all R_{M^1}. Set $\sigma(2) = j$. We can repeat the argument for all remaining agents, and show that at each stage t there is an agent k_t who can always get her top-ranked project that is not already the assignment of an earlier agent. Thus we can set $\sigma(t) = k_t$. In this way, we construct σ. At each stage t, $f_{\sigma(t)}(R) = top(R_{\sigma(t)}, Z \setminus \{f_{\sigma(t')}(R)\}_{t'<t})$. This is the definition of a serial priority rule, and thus $f = \varphi^{\sigma}_{SP}$. ■

It should be noted that the rule f in Example 2 does not satisfy limited influence. To see this, let profiles R, R', R'' be given as in Figure 3. Note that $f_1(R) = f_2(R) = b$ and $b P_3 f_3(R)$. Thus $U_3(b) = \{1, 2\}$. In profile $R' = (R'_1, R_2, R_3, R_4)$, we have that $f_2(R) = f_3(R) = b$. Now, consider profile R'', formed from R' by changing only agent 3's preferences. Since $b\, R_3\, f_3(R')$ and $2 \in U_3(b)$, by limited influence we should get $f_2(R'') = a$. But $f_2(R'') = b$. Thus f violates limited influence. We invite the reader to refer to the Appendix for a fuller discussion of limited influence, where we show that the specification of the axiom is 'tight', in that stronger or weaker versions fail.

In order to pin down the $SSPR$ exactly, we also require a stronger version of non-bossiness[14]. To state the property formally, we need a few definitions. We say a two-element partition (N_1, N_2) of N (i.e., such that $N_1 \cup N_2 = N$) is a *bifurcation* of N. Additionally, we say a vector x' is a reshuffling of a vector x if, for every project, the number of agents assigned that project in x' equals the number of agents assigned that project in x. Then x' can be thought of as preserving the aggregate assignment in x and possibly exchanging some individual assignments within it. Formally, for any $M \subseteq N$ with $|M| = k$, and any two vectors $x, x' \in Z^k$, we say that x' is a *reshuffling of* x if, for every $a \in Z$, $|\{i \in M \mid x_i = a\}| = |\{j \in M \mid x'_j = a\}|$. If x' is a reshuffling of x, we denote it by $x' \sim^r x$.

We say a rule is (strong) group-non-bossy if, for any bifurcation of the set of agents, if agents in one element report different preferences and get the same aggregate assignment among themselves, then the assignment of each agent in the other element of the bifurcation is unaffected by the change. Formally, a rule f satisfies **group-non-bossiness** if, for all bifurcations (N_1, N_2) of N, for all R, and for all R'_{N_1}, we have that $[f_{N_1}(R'_{N_1}, R_{-N_1}) \sim^r f_{N_1}(R)] \implies [f_{N_2}(R'_{N_1}, R_{-N_1}) = f_{N_2}(R)]$. It is clear that group-non-bossiness implies non-bossiness (set $|N_1| = 1$ in the definition), but the converse is in general not true. This axiom is also known as separability. See Thomson (2016) for a fuller discussion of non-bossiness and its variants.

We show in Theorem 4 that strategy-proofness, group-non-bossiness, unanimity, neutrality and limited influence are necessary and sufficient conditions for a rule to be an $SSPR$. In Section 7, we show the independence of the axioms.

[14] In Section 7, we give an example of a non-$SSPR$ satisfying the other axioms and non-bossiness.

5 Characterisation

We now state and prove our main result.

Theorem 4. *Let $\underline{q}, \bar{q} \in \{1, \ldots, n\}$. Then a rule satisfies strategy-proofness, group-non-bossiness, limited influence, unanimity and neutrality if and only if it is an SSPR.*[15]

Proof: **Sufficiency:** Let $\sigma \in \Sigma$ be given. We first show that the $SSPR$ φ^σ satisfies the axioms.

Strategy-proofness: Let R be a profile, let $i \in N$, and let $a \in Z$ such that $a \, P_i \, \varphi_i^\sigma(R)$. We have to show that, for any R'_i, $\varphi_i^\sigma(R'_i, R_{-i}) \neq a$. Let $t \in \{1, \ldots, n\}$ be such that $\sigma(t) = i$. By definition of the $SSPR$, we have that $\varphi_i^\sigma(R) = top(R_i, Z_t(s^{t-1}))$. Thus $a \notin Z_t(s^{t-1})$. Since $Z_t(s^{t-1})$ is determined at stage $t-1$, it is independent of preferences of agent i, and we have that, for any R'_i, $a \notin Z_t(s^{t-1})$. Thus $top(R'_i, Z_t(s^{t-1})) \neq a$ and $\varphi_i^\sigma(R'_i, R_{-i}) \neq a$.

Group-non-bossiness: Let R be a profile, $M \subset N$ and let R'_M be preferences such that $\varphi_M^\sigma(R) \sim^r \varphi_M^\sigma(R'_M, R_{-M})$. We have to show that, for all $i \notin M$, $\varphi_i^\sigma(R'_M, R_{-M}) = \varphi_i^\sigma(R)$. Let $i \notin M$ be an arbitrary agent, and suppose $\sigma(t) = i$ for some t. Let $\varphi_i^\sigma(R) = a$. Then $a = top(R_i, Z_t(s^{t-1}))$. Consider (R'_M, R_{-M}). Since $\varphi_M^\sigma(R) \sim^r \varphi_M^\sigma(R'_M, R_{-M})$, it follows that $a \in Z_t(s^{t-1})$ and for any $b \notin \varphi^\sigma(R)$ such that $b \, P_i \, a$, we have that $b \notin Z_t(s^{t-1})$. Since R_i is the same in both profiles, $a = top(R_i, Z_t(s^{t-1}))$, and so $\varphi_i^\sigma(R'_M, R_{-M}) = a$.

Limited influence: Let $i, j \in N$, $a \in Z$ and let R be a profile such that $a \, R_i \, \varphi_i^\sigma(R)$ and $j \in U_i(a)$. Then $\sigma^{-1}(j) < \sigma^{-1}(i)$. Let $\varphi_j^\sigma(R) = a$. Let $t \in \{1, \ldots, n\}$ be such that $\sigma(t) = j$. By the definition of the $SSPR$, for any R'_i, $a = top(R_j, Z_t(s^{t-1}))$, and so $\varphi_j^\sigma(R'_i, R_{-i}) = a$. Since $j \in U_i(a)$ is arbitrary, φ^σ satisfies limited influence.

Unanimity: Let R be such that $top(R) \in A$. Consider an arbitrary round $t \in \{1, \ldots, n\}$, and let $\sigma(t) = i$. It is easy to see that $top(R_i, Z) \in Z_t(s^{t-1})$, and so $top(R_i, Z_t(s^{t-1})) = top(R_i, Z)$. Thus $\varphi_i^\sigma(R) = top(R_i)$. Thus, φ^σ is unanimous.

Neutrality: The names of projects are not referenced at any point in the definition of φ^σ, and so for any permutation π of projects and any R, $\varphi^\sigma(\pi R) = \pi \varphi^\sigma(R)$, and φ^σ is neutral.

[15] As a corollary, by Lemma 1 and Lemma 2, the $SSPR$ is group strategy-proof and Pareto-efficient.

Necessity: Let f satisfy strategy-proofness, group-non-bossiness, limited influence, unanimity and neutrality. By Lemma 1 and Lemma 2, f satisfies group-strategy-proofness and Pareto-efficiency. We show that f is an $SSPR$. That is, we construct an order σ, and show that for any R, $f(R) = \varphi^\sigma(R)$, according to the process outlined by the $SSPR$.

If $\underline{q} = \bar{q} = 1$, we have by Theorem 1 that f is a serial priority rule. The remainder of this proof holds true for this case as well, but note that if we let σ be as defined for the serial priority rule, then f is also an $SSPR$, and we are done. Thus, for what follows, we assume that $\bar{q} > 1$. We start by proving a useful lemma.

Lemma 4. *Let f satisfy strategy-proofness, non-bossiness, limited influence, unanimity and neutrality. There exists a set of agents $M \subseteq N$ with $|M| = \bar{q}$ such that, for any R and any $a \in Z$, $[top(R_M) = (a, \ldots, a)] \implies [f_M(R) = (a, \ldots, a)]$.*

Proof: Let R be a profile where $R_i = R_j$ for all $i, j \in N$. Without loss of generality, let a be the top-ranked project for all agents. Since f satisfies Pareto-efficiency, there exists an $M \subseteq N$ with $|M| = \bar{q}$ such that $f_M(R) = (a, \ldots, a)$. Note that $a \; P_j \; f_j(R)$ for all $j \notin M$. Thus, $i \in U_j(a)$ for all $i \in M, j \notin M$. Let R'_{-M} be an arbitrary sub-profile for agents in $N \setminus M$.

Pick $j \in N \setminus M$ and consider sub-profile (R'_j, R_{-j}). By limited influence, it follows that $f_M(R'_j, R_{-j}) = (a, \ldots, a)$. A repeated argument for all other agents in $N \setminus M$ gives us $f_M(R_M, R'_{-M}) = (a, \ldots, a)$. Then, strategy-proofness and non-bossiness imply that for any R'_M such that $top(R'_M) = (a, \ldots, a)$, we have $f_M(R') = (a, \ldots, a)$. Since f satisfies neutrality, this is true for all $a \in Z$. Since R' is arbitrary, we have the result. ■

We have that $i \in U_j(a)$ for any $i \in M, j \notin M$. By neutrality, $i \in U_j(b)$ for all $b \in Z$. We write this as $i \in U_j$.

If $m = 1$, f satisfies strategy-proofness, and satisfies unanimity for the set M. By Theorem 2, there exists an $i \in M$ such that, for all R, $f_i(R) = top(R_i)$. We can prove the result for this case as well, but for convenience, note that if we set $\sigma(1) = i$, and let σ rank other agents in M next in arbitrary order, followed by agents in $N \setminus M$ in arbitrary order, then for any R, $f_{\sigma(1)}(R) = top(R_{\sigma(1)}, Z)$, $f_{\sigma(t)}(R) = top(R_{\sigma(t)}, Z_t(s^{t-1}))$ for all $\sigma(t) \in M$, and $f_{\sigma(t')}(R) = \emptyset$ for all $\sigma(t') \notin M$. Thus f is an $SSPR$, and we are done. So for what follows, we assume $m > 1$.

Step 1

Let R be the profile of identical preferences as defined above, with project a ranked top by all agents. By Lemma 4, $f_M(R) = (a \ldots, a)$. Set $M^0 = K^0 = \emptyset$. Consider $i \in M$. Let R'_i be such that $top(R'_i) = b$ for some $b \neq a$. We show that agent i is assigned b at (R'_i, R_{-i}).

Claim **1.** $f_i(R'_i, R_{-i}) = b$.

Proof: Suppose not. Then, by strategy-proofness, $f_i(R'_i, R_{-i}) = a$. By non-bossiness, $f(R'_i, R_{-i}) = f(R)$. For all $j \notin M$, let R'_j rank a top and b second. By a similar argument to Lemma 4, the assignment of agents in M does not change if we move to the profile $(R_{M\setminus\{i\}}, R'_{-\{M\setminus\{i\}\}})$. That is, $f_M(R_{M\setminus\{i\}}, R'_{-\{M\setminus\{i\}\}}) = (a, \ldots, a)$. Since $m > 1$, and b is ranked second by all agents outside M, by Pareto-efficiency there is some $M' \subseteq N \setminus M$ such that $f_{M'}(R'_{M\setminus\{i\}}, R_{-\{M\setminus\{i\}\}}) = (b, \ldots, b)$. But this violates Pareto-efficiency, as bP'_ia and aP'_kb for all $k \in M'$. Thus, $f_i(R'_i, R_{-i}) = b$. ■

We show that all other agents in M continue to be assigned a at (R_i, R_{-i}).

Claim **2.** $f_j(R'_i, R_{-i}) = a$ for all $j \in M \setminus \{i\}$.

Proof: Since a is ranked top by all other agents, by Pareto-efficiency, $a \in f(R'_i, R_{-i})$. Let $j \in M \setminus \{i\}$ be an arbitrary agent in M, and suppose for contradiction that $f_j(R'_i, R_{-i}) = x$, where $x \neq a$. Note that $a\ P_j\ x$, and for any $k \in N \setminus M$, we have that $j \in U_k$ and $a\ R_k\ x$. By Pareto-efficiency, there is some $k^1 \in N \setminus M$ such that $f_{k^1}(R'_i, R_{-i}) = a$. Let R'_{k^1} be preferences that rank x top and a second. By strategy-proofness, $f_{k^1}(R'_{ik^1}, R_{-ik^1})\ R_{k^1}\ a$. By Pareto-efficiency, $f_{k^1}(R'_{ik^1}, R_{-ik^1}) = x$. But since $j \in U_{k^1}$, limited influence implies $f_j(R'_{ik^1}, R_{-ik^1}) = x$. Thus, by Pareto-efficiency, there is some $k^2 \in N \setminus M \cup \{k^1\}$ such that $f_{k^2}(R'_{ik^1}, R_{-ik^1}) = a$. In turn, for all $k \in N \setminus M$ we use a repeated argument, constructing R'_k that ranks x top and a second, and so $f_j(R_{M\setminus\{i\}}, R'_{-\{M\setminus\{i\}\}}) = x$. By Pareto-efficiency, $a \in f(R_{M\setminus\{i\}}, R'_{-\{M\setminus\{i\}\}})$, and so there is a $k \in N \setminus M$ such that $f_j(R_{M\setminus\{i\}}, R'_{-\{M\setminus\{i\}\}}) = a$. But this violates Pareto-efficiency, as $a\ P_j\ x$ and xP'_ka. ■

Define $k(i)$ as $k \in N \setminus M$ such that $f_k(R'_i, R_{-i}) = a$. For any $j \in M$ such that $j \neq i$, we can let R'_j be defined the same way as R'_i, and by the same argument as in Claim 1 and

Claim 2, we have that $f_j(R'_j, R_{-j}) = b$, $f_i(R'_j, R_{-j}) = a$ for all $i \in M \setminus \{j\}$, and we can define $k(j)$ as $k \in N \setminus M$ such that $f_k(R'_j, R_{-j}) = a$. We now show that this is the same agent in each case.

Claim **3.** For all $j, l \in M$, $k(j) = k(l)$.

Proof: We have that $f_M(R'_i, R_{-i}) \sim^r f_M(R'_j, R_{-j})$. Thus group-non-bossiness implies $f_k(R'_i, R_{-i}) = f_k(R'_j, R_{-j})$ for all $k \notin M$. In particular, $f_{k(i)}(R'_i, R_{-i}) = f_{k(i)}(R'_j, R_{-j}) = a$. Thus $k(i) = k(j)$. By a repeated argument for all $j, l \in M$, it follows that $k(j) = k(l)$. ■

Let this agent be denoted k^1, and define $K^1 = \{k^1\}$ and $M^1 = M \cup K^1$. Define $\mathcal{R}^\infty$ as the collection of profiles used in the construction above. That is, let $\mathcal{R}^1 = \{R | \exists i \in M^1$ s.t. $top(R_i) = b$ and $top(R_j) = a$ for all $j \neq i\}$. Then, for any $R \in \mathcal{R}^1$, $f_i(R) = top(R_i)$ for all $i \in M^1$. Moreover, this is independent of preferences of agents not in M^1. So set $k^1 \in U_j$ for all $j \notin M^1$.

Step t, $t < m$

We proceed recursively as follows. Let $\mathcal{R}^{t-1}, K^{t-1}, M^{t-1}$ be as derived at Step $t-1$. Let $R \in \mathcal{R}^{t-1}$. We have that $f_i(R) = top(R_i)$ for all $i \in M^{t-1}$. Let $\bar{M} \subseteq M^{t-1}$ be the set of agents who rank a top, i.e., $\bar{M} = \{i \in M^{t-1} \mid top(R_i) = a\}$. Let X denote the set of projects other than a that are ranked top by fewer than $\bar{q}$ agents. That is, $X = \{c \neq a || \{j \in N | top(R_j) = c\}| < \bar{q}\}$.

Consider $i \in \bar{M}$, and let R'_i be preferences such that $top(R'_i) = c$ for some $c \in X$. Since $t < m$, by similar arguments as in Claim 1 and Claim 2, we see that $f_i(R'_i, R_{-i}) = c$, $f_j(R_i, R_{-i}) = f_j(R)$ for all $j \in M^{t-1} \setminus \{i\}$, and there is some $k(i) \notin M^{t-1}$ such that $f_{k(i)}(R'_i, R_{-i}) = a$. We now show that the identity of $k(i)$ does not depend on which project is ranked top by i.

Claim **4.** For any $d \in X$ with $d \neq c$, and any R''_i such that $top(R''_i) = d$, $f_{k(i)}(R''_i, R_{-i}) = a$.

Proof: Suppose not. Let $f_{k(i)}(R''_i, R_{-i}) = x \neq a$. By Pareto-efficiency, there is some agent k such that $f_k(R''_i, R_{-i}) = a$. Note that $k(i) \in U_k(a)$ as $f_{k(i)}(R'_i, R_{-i}) = a$ and $aP_k f_k(R'_i, R_{-i})$. By neutrality, $k(i) \in U_k(b)$ for all $b \in Z$. In particular, $k(i) \in U_k(x)$. Consider R'_k such that $top(R'_k) = x$ and a is ranked second. By Pareto-efficiency, $f_k(R'_k, R''_i, R_{-ik}) = x$. By limited influence, $f_{k(i)}(R'_k, R''_i, R_{-ik}) = x$. Thus, by Pareto-efficiency, there is some $k' \notin \{M^{t-1} \cup k\}$ such that $f_{k'}(R'_k, R''_i, R_{-ik}) = a$. We can repeat the above argument for each such k', noting that at each stage, $k(i) \in U_{k'}(x)$, to derive a contradiction as in Claim 2. ■

Moreover, we can use a similar argument to Claim 3 to show that $k(i) = k(j)$ for all $i, j \in M^{t-1}$. Let this agent be k^t. Let $K^t = K^{t-1} \cup \{k^t\}$ and $M^t = M^{t-1} \cup \{k^t\}$. Define $\mathcal{R}^t = \{R | \exists N_t \subset M^t \text{ s.t } |N_t| = t, \textit{top}(R_i) \neq a \text{ for all } i \in N_t, |\{j \in N_t | \textit{top}(R_i) = b\}| < \bar{q} \text{ for all } b \neq a, \text{ and } \textit{top}(R_j) = a \text{ for all } j \notin N_t\}$. Then, for any $R \in \mathcal{R}^t$, $f_i(R) = \textit{top}(R_i)$ for all $i \in M^t$. Moreover, this is independent of preferences of agents not in M^t. Set $k^t \in U_j$ for all $j \notin M^t$.

Step m

Let $\mathcal{R}^m, K^m, M^m$ be as derived in Step $m-1$. We now show the existence of a set of m agents who can always get their top-ranked projects when they are all distinct. That is, we find a set $L \subseteq N$ with $|L| = m$ such that for all R with $\textit{top}(R_i) \neq \textit{top}(R_j)$ for all $i, j \in L$, we have that $f_i(R) = \textit{top}(R_i)$ for all $i \in L$. There are two cases:

Case 1: $m \geq \bar{q}$. We take a profile in R^m that has different top-ranked projects for each agent in M and the first $m - \bar{q}$ agents added to M^m (i.e., in $K^{m-\bar{q}}$), with one of those projects being a. That is, let $R \in \mathcal{R}^m$ be such that $\textit{top}(R_{i_m}) = a$ for some $i_m \in M$ and for all $i, j \in M \cup K^{m-\bar{q}} \setminus \{i_m\}$, we have $\textit{top}(R_i) \neq \textit{top}(R_j)$ and a is ranked second. Also, $\textit{top}(R_j) = a$ for all other agents j. Let X be the set of projects ranked top by agents in M^m. So $X = \{\textit{top}(R_i) | i \in M^m\}$, and by construction, $|X| = m$. Since $f_i(R) = \textit{top}(R_i)$ for all $i \in M$, we have that m distinct projects are assigned in R. Now we show that no other agent can get a distinct top-ranked project for any other preferences.

Let $\bar{M}$ be the set of agents other than i_m that rank a top, i.e., $\bar{M} = \{j \in N \setminus \{i_m\} \mid \textit{top}(R_j) = a\}$. Let $i \in \bar{M}$, let R'_i be preferences such that $\textit{top}(R'_i) = d$ for some $d \notin X$. Suppose $f_i(R'_i, R_{-i}) = d$. Since m projects are top-ranked by agents in $M^{m-\bar{q}}$, it follows by feasibility that $f_j(R'_i, R_{-i}) \neq \textit{top}(R_j)$ for some $j \in M \cup K^{m-\bar{q}}$. By strategy-proofness, $f_j(R'_i, R_{-i}) = a$. But then, $j \in U_i$ and limited influence imply $f_j(R) = a$, which is a contradiction, as $f_j(R) = \textit{top}(R_j) \neq a$. Thus $f_i(R'_i, R_{-i}) \neq d$. This is true for all $i \in \bar{M}$ and all $d \notin X$. Define $L = M \cup K^{m-\bar{q}}$.

Case 2: $m < \bar{q}$**.** We show by construction that $L \subset M$. Let $\bar{N} \subset M$ be an arbitrary subset such that $|\bar{N}| = m$. Let $R \in R^m$ such that $top(R_{i_m}) = a$ for some $i_m \in \bar{N}$ and $top(R_i) \neq top(R_j) \neq a$ and a is ranked second for all $i \in \bar{N} \setminus \{i_m\}$. In particular, $top(R_i) = a$ for all $i \notin \bar{N}$. Let $X = \{top(R_i) | i \in M^m\}$. Then $|X| = m$. Thus there are m distinct top-ranked projects for agents in $\bar{N}$. Since $f_i(R) = top(R_i)$ for all $i \in M$, we have that m distinct projects are assigned in R. Let $j \in M \setminus \bar{N}$, let R'_j be such that $top(R'_j) = d$ for some $d \notin X$.

If $f_j(R'_j, R_{-j}) = d$, then for some $i \in \bar{N}$, $f_i(R'_j, R_{-j}) \neq top(R_i)$. Set $k \in U_i$ for all $k \in \bar{N} \cup \{j\} \setminus \{i\}$. With some abuse of notation, we redefine $\bar{N}$ by dropping j and adding i. Otherwise, $f_j(R'_j, R_{-j}) = a$, by strategy-proofness. By non-bossiness, $f(R'_j, R_{-j}) = f(R)$. Set $k \in unanimity_j$ for all $k \in \bar{N} \cup \{i\} \setminus \{j\}$.

Repeat for all $k \in M \setminus \bar{N}$ and (R'_k, R_{-k}) where R'_k is such that $top(R'_k) \notin X$. By an iterated argument, we can find $L \subset M$, $|L| = m$, such that $top(R_i) \neq top(R_j) \neq a$ for all $i \in L$, and $f_i(R) = top(R_i)$. Set $i \in U_j$ for all $i \in L, j \notin L$.

In each case above, we have that $top(R_i) \neq top(R_j)$ for all $i, j \in L$, and $f_i(R) = top(R_i)$ for all $i \in L$. By limited influence, and noting that $i \in U_j$ for each $i \in L, j \notin L$, it follows that this is true for all preferences of other agents R'_{-L}. By neutrality, this is independent of the labelling of the projects. Thus, for any R such that $top(R_i) \neq top(R_j)$ for all $i, j \in L$, we have that $f_i(R) = top(R_i)$ for all $i \in L$.

If $m < \bar{q}$, $L \subsetneq M$, and we can extend the ranking to agents in $M \setminus L$. For the remaining agents in $M \setminus L$, we proceed as follows: Let $i, j \in L$, let $a \in Z$, and let R be a profile such that, for all $k, l \in L \setminus \{i, j\}$, $top(R_k) \neq top(R_l) \neq a$, and $top(R_{i'}) = a$ for all $i' \neq k, l$. Let $X = \{top(R_k) | k \in M\}$. Then $f_k(R) = a$ for all $k \in M \setminus L$. Pick $k, l \in M \setminus L$, and let R'_k rank $c \notin X$ top, and a second, and R'_l rank $d \notin X$ top, and a second. Firstly, note that $f_L(R'_{kl}, R_{-kl}) = f_L(R)$. By Pareto-efficiency and feasibility, either $f_k(R'_{kl}, R_{-kl}) = c$ or $f_l(R'_{kl}, R_{-kl}) = d$. If the former, let $k \in U_l$, and let $l \in U_k$ otherwise. We can repeat for all $k, l \in M \setminus L$.

Construction of σ

Define an ordering σ as follows: For any $i, j \in N$, let $[i \in U_j, j \notin U_i] \implies [\sigma^{-1}(i) < \sigma^{-1}(j)]$. To complete the ordering, we have to rank agents in $L \cap M$ and agents in $N \setminus M^m$. Let σ rank agents in $L \cap M$ arbitrarily. Now we rank agents in $N \setminus M^m$.

Let $W = N \setminus M^m$. Let $Z \subset M^m$ with $|Z| = \bar{q} - 1$. Let $top(R_j) \neq a$ for all $j \in M^m \setminus Z$. If $X = \{top(R_j) | j \in M^m \setminus Z\}$, then R is such that, for any $b \in X$, $|\{i \in M^m \setminus Z | top(R_i) = b\}| < \bar{q}$. That is, no project is ranked top by $\bar{q}$ agents in $M^m \setminus Z$. Let $top(R_l) = a$ for all other agents. It follows that $f_i(R) = top(R_i)$ for all $i \in M^m$. In particular, $f_Z(R) = (a, \ldots, a)$. By Pareto-efficiency, there is an agent $k^{m+1} \in W$ such that $f_{k^{m+1}}(R) = a$. By group-non-bossiness and neutrality, and similar arguments to before, this is true for all projects and all permutations of preferences of agents in M^m, thus agent k^{m+1} is uniquely defined. Set $K^{m+1} = K^m \cup \{k^{m+1}\}$, and $M^{m+1} = M^m \cup \{k^{m+1}\}$. Set $\mathcal{R}^{m+1}$ as the collection of all R as constructed above. For any $R \in \mathcal{R}^{m+1}$, $f_i(R) = top(R_i)$ for all $i \in$ By definition, $k^{m+1} \in U_i$ for all $i \in W \setminus \{i\}$, thus set $\sigma^{-1}(k^{m+1}) < \sigma^{-1}(i)$ for all $i \in W \setminus \{k^{m+1}\}$.

Let $R'_{k^{m+1}}$ be such that $top(R'_{k^{m+1}}) = top(R_i)$ for some $i \in M^m \setminus Z$, ensuring that $|\{i \in M^m \cup \{k^{m+1}\} \setminus Z | top(R_i) = top(R'_{k^{m+1}})\}| \leq \bar{q}$. Let $R' = (R'_{k^{m+1}}, R_{-k^{m+1}})$. By a similar argument, there is some $k^{m+1} \in W \setminus \{k^{m+1}\}$ such that $f_{k^{m+2}}(R') = a$. Again, by group-non-bossiness and neutrality, this agent is unique. By definition, $k^{m+2} \in U_i$ for all $i \in W \setminus \{i, j\}$, so set $\sigma^{-1}(k^{m+2}) < \sigma^{-1}(i)$ for all $i \in W \setminus \{k^{m+1}, k^{m+2}\}$. Set $K^{m+2} = K^{m+1} \cup \{k^{m+2}\}$ and $M^{m+2} = M^{m+1} \cup \{k^{m+2}\}$. Set $\mathcal{R}^{m+2}$ as the collection of all profiles R as constructed above. Then, for any $R \in \mathcal{R}^{m+2}$, we have that $f_i(R) = top(R_i)$ for all $i \in M^{m+2}$. Similarly, we can repeat for all other agents, noting that at each stage, the agent determined is unique. This way we fully determine σ.

The Algorithm

It remains to be shown that for any R, assignments are made by f according to σ and the procedure described by the SSPR. Let R be a profile. Let s^0 be the null allocation. Let $|M \cap L| = t'$. Note that σ ranks agents in $M \cap L$ first. For any $i \in M \cap L$, $f_i(R) = top(R_i)$. Thus, for any $t \leq t'$, $f_{\sigma(t)}(R) = top(R_{\sigma(t)}, Z_t(s^{t-1}))$.

Define $Z_{t'+1}(s^{t'}) = \{b \in Z | \exists x \in A$ with $x_i = s_i^{t'}$ for all $i \in \Omega_\sigma(\sigma(t'+1))$, and $x_{\sigma(t'+1)} = b\}$. Then, by the above construction, it follows that $f_{\sigma(t'+1)}(R) = top(R_{\sigma(t'+1)}, Z_{t'+1}(s^{t'}))$. Repeating for $t'+2, \ldots, n$, we have that, for all t, $f_{\sigma(t)}(R) = top(R_{\sigma(t)}, Z_t(s^{t-1}))$. Thus $f(R) = \varphi^\sigma(R)$, and f is an $SSPR$. This completes the proof. ■

6 Discussion and Special Cases

6.1 The Social Choice Model

The social choice model is a special case of capacity constraints in which $\underline{q} = \bar{q} = n$. We show in this case that our axioms are implied by the axioms in Gibbard (1973) and Satterthwaite (1975). In particular, group-non-bossiness and limited influence are satisfied trivially, and so we show that a rule satisfying strategy-proofness and unanimity also satisfies neutrality. We do this by appealing to the properties themselves, and not to the original proofs, and so it is possible to claim their characterisation results as a corollary of ours.

We establish neutrality by first showing that such a rule satisfies pairwise neutrality (to be defined shortly), and then extending this to all cases. We start by proving a useful claim: if f is strategy-proof and unanimous, then for any profile where the set of projects that are top-ranked by some agent is of size 2, the outcome of f at that profile is one of those two projects.

Claim **5.** Let f be strategy-proof and unanimous. For a profile R, let $X_R = \{a \in Z \mid top(R_i) = a$ for some $i \in N\}$. Then, for any R such that $|X_R| = 2$, $f(R) \in X_R$.

Proof: Let R be a profile, and let $X_R = \{a, b\}$. Let $L = \{i \in N \mid top(R_i) = a\}$ and $M = \{i \in N \mid top(R_i) = b\}$. (Then $L \cup M = N$.)

Suppose that $f(R) = c \notin \{a, b\}$. Consider $i \in L$, and let R_i' be preferences such that a and b are ranked first and second in R_i', respectively. By strategy-proofness, $f(R_i', R_{-i}) \neq a$. Repeating for all agents in L, and a sub-profile R_L' such that R_j' ranks a top and b second for all $j \in L$, we have that $f(R_L', R_M) \neq a$. Suppose $f(R_L', R_M) \neq b$. Then $b \, P_i' \, f(R_L', R_M)$ for all $i \in L$. Let R_L'' be a sub-profile such that R_i'' ranks b top for each $i \in L$. By strategy-proofness, we have that $f(R''_L, R_M) \neq b$. But by unanimity, $f(R''_L, R_M) = b$, which is a contradiction. Thus $f(R'_L, R_M) = b$. Let R'_M be a sub-profile such that R'_j ranks b top and a second for each $j \in M$. By strategy-proofness, $f(R') = b$.

Consider $j \in M$, and preferences R'_j as defined above. By a similar argument to the above, $f(R'_j, R_{-j}) \neq b$. Repeating for agents in M, we have that $f(R'_M, R_L) \neq b$ and, in fact, $f(R'_M, R_L) = a$. By strategy-proofness, therefore, $f(R') = a$. But this is a contradiction, as we have two allocations for the same profile. Thus, $f(R) \in \{a, b\}$. ■

We now define pairwise neutrality, which requires that if a profile is formed from another by taking two projects and exchanging only their positions in each agent's preferences, then either the assignment is exchanged as well (if it is one of the switched projects) or is unaffected (if not). To state the axiom formally, we need a few definitions. Let R, R' be profiles, and let $a, b \in Z$ be projects. We say that R' is a flip of R for a, b if R' is formed from R by only exchanging the positions of a, b in every agent's preferences, and preserving the relative ranking of all other projects. Formally, we say that a profile R' is a *flip of R for a, b* if: (1) $aP_ib \implies bP'_ia$ for all $i \in N$; (2) $bP_ia \implies aP'_ib$ for all $i \in N$; (3) For any $c \neq a, b$, $aP_ic \Leftrightarrow aP'_ic$ and $bP_ic \Leftrightarrow bP'_ic$ for all $i \in N$; (4) For any $c, d \notin \{a, b\}$, $cP_id \Leftrightarrow cP'_id$ for all $i \in N$. A rule f satisfies **pairwise neutrality** if, for all projects $a, b \in Z$, and all profiles R, R' such that R' is a flip of R for a, b: $f(R) = a \Leftrightarrow f(R') = b$ and $f(R) = c \Leftrightarrow f(R') = c$ for all $c \notin \{a, b\}$.

Proposition **2.** *Let $\underline{q} = \bar{q} = n$. Let f be a rule that satisfies strategy-proofness and unanimity. Then f satisfies pairwise neutrality.*

Proof: Let R be a profile, $a, b \in Z$, and let R' be a flip of R for a, b. Without loss of generality, we can let a, b be the two top-ranked projects in R_i for each $i \in N$.

Case 1: $f(R) \in \{a, b\}$: Let $f(R) = a$. We have to show that $f(R') = b$. Define $M = \{i \in N \mid a\ P_i\ b\}$, and let $L = \{i \in N \mid b\ P_i\ a\}$. Let R''_L be a sub-profile such that, for each $j \in L$, R''_j ranks b top and some $c \notin \{a, b\}$ second. Consider $i \in L$ and R''_i. By Claim 5, $f(R''_i, R_{-i}) \in \{a, b\}$. Since, by strategy-proofness, $f(R''_i, R_{-i}) \neq b$, it follows that $f(R''_i, R_{-i}) = a$. Repeating for other agents in L, we have that $f(R''_L, R_M) = a$. Let $\hat{R}_L$ be a sub-profile such that $\hat{R}_i$ ranks c top and a second for each $i \in L$. By strategy-proofness,

$f(\hat{R}_L, R_M) = a$. Let $\hat{R}_M$ be a sub-profile such that $\hat{R}_j$ ranks a top and c second for each $j \in M$. By strategy-proofness, $f(\hat{R}) = a$.

Now, consider the profile (R'_M, R_L). Since $top(R'_i) = b$ for every $i \in M$ and $top(R_j) = b$ for every $j \in L$, by unanimity it follows that $f(R'_M, R_L) = b$. Suppose that $f(R') = a$. Consider $i \in L$ and $\hat{R}_i$ as defined above. By strategy-proofness, $f(\hat{R}_i, R'_{-i}) \in \{a, c\}$. Repeating for other agents in L, we have that $f(\hat{R}_L, R'_M) \in \{a, c\}$. By Claim 5, and since no agent ranks a top in $(\hat{R}_L, R'_M)$, it follows that $f(\hat{R}_L, R'_M) = c$. For $i \in M$, by strategy-proofness we have that $f(\hat{R}_L, \hat{R}_i, R'_{M\setminus\{i\}}) = c$. Repeating for all agents in M, we have that $f(\hat{R}) = c$. But this is a contradiction, as $f(\hat{R}) = a$ as established above. Thus $f'(R) \neq a$, and so by Claim 5, $f(R') = b$ as required.

Case 2: $f(R) \notin \{a, b\}$: Let $f(R) = c \notin \{a, b\}$. We have to show that $f(R') = c$. Let R^1 be a profile such that R^1 is a flip of R for a, c. Since $f(R) = c$, by the arguments of Case 1 above, it follows that $f(R^1) = a$. Let R^2 be a flip of R^1 for a, b. We have that $f(R^2) = b$. Finally, let R^3 be a flip of R^2 for b, c. We have that $f(R^3) = c$. It can be easily seen that $R^3 = R'$. Thus $f(R') = c$ as required. ■

Proposition 3. *Let $\underline{q} = \bar{q} = n$. Let f be a rule that satisfies strategy-proofness and unanimity. Then f satisfies group-non-bossiness, neutrality, and limited influence.*

Proof: Let f satisfy strategy-proofness and unanimity. group-non-bossiness is satisfied vacuously. Let R be a profile, $a \in Z$ a project, $i \in N$ an agent, and suppose that $a \, P_i \, f(R)$. Let R'_i be arbitrary preferences. By strategy-proofness, $f(R'_i, R_{-i}) \neq a$. Therefore, $f_j(R) \neq a$ for all $j \in N$. Thus $f_j(R) = a \implies f_j(R'_i, R_{-i}) = a$ is satisfied vacuously. Since j is arbitrary, limited influence is satisfied.

To show neutrality, let R be a profile and let $R' = \pi R$, for some permutation π. We have to show that $f(R') = \pi f(R)$. Since Z is finite, we can see π as a finite sequence of pairwise permutations. So let $\pi = \pi_1 \circ \pi_2 \circ \ldots \circ \pi_{n-1}$, and let $R^1, \ldots, R^n$ be a sequence of profiles such that $R^1 = R$, $R^n = R'$, and for each $i \in 2, \ldots, n$, $R^i = \pi_{i-1} R^{i-1}$. Then, since f is pairwise neutral by Proposition 2, we have that $f(R^n) = \pi_{n-1} f(R^{n-1}) = \pi_{n-2} \circ \pi_{n-1} f(R^{n-2})$ $= \ldots = \pi_1 \circ \ldots \circ \pi_{n-1} f(R^1)$. Since $R^n = R' = \pi R$ and $R^1 = R$, and since $\pi_1 \circ \ldots \circ \pi_{n-1} = \pi$, we have that $f(\pi R) = \pi f(R)$, and f is neutral. ■

6.2 House Allocation

The house allocation model is a special case of capacity constraints in which $\underline{q} = \bar{q} = 1$. We show in this case that our axioms are implied by the (fewer) axioms used by Svensson (1999). Unanimity follows naturally from the fact that strategy-proofness, non-bossiness and neutrality imply Pareto-efficiency. However, showing limited influence and group-non-bossiness requires a little more work. The approach to proving these implications is quite similar to the proof technique used by Svensson (1999) to characterise the serial priority rule, and so that we cannot claim that result as a directly corollary to ours.

Proposition 4. *Let $\underline{q} = \bar{q} = 1$. Let f be a rule that satisfies strategy-proofness, non-bossiness, and neutrality. Then f satisfies unanimity, limited influence, and group-non-bossiness.*

Proof: Let f be a rule that satisfies strategy-proofness, non-bossiness and neutrality. By Svensson (1999), we know that f satisfies Pareto-efficiency. Since Pareto-efficiency implies unanimity, we have that f satisfies unanimity.

For what follows, let R be a profile of identical preferences, i.e., $R_i = R_j$ for all $i, j \in N$. By Pareto-efficiency, there is some agent i_1 such that $f_{i_1}(R) = top(R_{i_1}, Z)$. Similarly, by Pareto-efficiency, for every $k \in \{1, \ldots, n\}$, there is an agent i_k such that $f_{i_k}(R) = top(R_{i_k}, Z \setminus \{f_{i_j}(R)\}_{j<k})$. By definition, for every $i_j, i_k \in N$ such that $j < k$, we have that $i_j \in U_{i_k}(f_{i_j}(R))$. By neutrality, this is true for all projects. Thus $i_j \in U_{i_k}$. For an arbitrary profile R, we say a profile of identical preferences R' *reflects* R if R' is constructed as follows: For any $i \in N$ and any $j, k \in \{1, \ldots, n\}$ with $j < k$, R'_i is such that $f_{i_j}(R)\ P'_i\ f_{i_k}(R)$. That is, R' is constructed by sequentially ranking the assignments of agents in R according to the sequence of agents $(i_1, \ldots, i_n)$ constructed above. By strategy-proofness and non-bossiness, it is easy to see that $f(R') = f(R)$.

To show limited influence, consider a profile R, agents $i, j \in N$, and a project $a \in Z$, such that $a\ R_i\ f_i(R)$ and $f_j(R) = a$. Since $\underline{q} = \bar{q} = 1$, we have that $f_i(R) \neq a$ and so $a\ P_i\ f_i(R)$. Let $\hat{R}$ be a profile of identical preferences that reflects R. Then $f(\hat{R}) = f(R)$. In particular, $f_i(\hat{R}) = f_i(R)$ and $f_j(\hat{R}) = a$. Let R'_i be arbitrary preferences, and suppose $f_i(R'_i, R_{-i}) = b$. By strategy-proofness, $b \neq a$. If $b = f_i(R)$, then by non-bossiness, $f_j(R'_i, R_{-i}) = a$, and limited influence is established. So let $b \neq f_i(R)$. Let $k \in N$ be an agent such that $f_k(R) = b$. Let $\tilde{R}$ be a profile of identical preferences constructed as follows: Let $\tilde{R}$ reflect R, except projects b and $f_i(R)$ are swapped. By neutrality, $f_i(\tilde{R}) = b$ and, in particular, $f_j(\tilde{R}) = a$. By strategy-proofness and non-bossiness, $f(\tilde{R}) = f(R'_i, R_{-i})$, and thus $f_j(R'_i, R_{-i}) = a$, establishing limited influence.

To show group-non-bossiness, let R be a profile, $M \subset N$ a set of agents, and R'_M preferences such that $f_M(R) \sim^r f_M(R'_M, R_{-M})$. Let $\hat{R}$ be a profile of identical preferences that reflects R, and let $\tilde{R}$ be a profile of identical preferences that reflects (R'_M, R_{-M}). By construction, $\hat{R}$ is a permutation of $\tilde{R}$ for assignments of agents in M, and thus by neutrality, $f(\hat{R})$ is a permutation of $f(\tilde{R})$ for assignments of agents in M. In particular, $f_{N\setminus M}(\hat{R}) = f_{N\setminus M}(\tilde{R})$. But since $f(\hat{R}) = f(R)$ and $f(\tilde{R}) = f(R'_M, R_{-M})$, we have that $f_{N\setminus M}(R'_M, R_{-M}) = f_{N\setminus M}(R)$, and f satisfies group-non-bossiness. ■

7 Independence of Axioms

To show that these axioms are independent, we now provide examples of rules satisfying all but one of the axioms in turn.

Example 4. (Strategy-proofness)

Let $\mathcal{N} = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{Z} = \{a, b, c, d\}$, $\underline{q} = \bar{q} = 2$. Let f be a rule similar to the SSPR, but defined as follows. For any profile R, if the second-ranked project in R_2 is the same as $top(R_1)$ and $top(R_2) \neq top(R_3)$, then $\sigma_R = (1, 3, 4, 2, 5, 6)$, otherwise $\sigma_R = (1, 2, 3, 4, 5, 6)$.

Then, for the two preference profiles R, R' in Figure 5, the assignments are marked in boxes. R and R' differ only in agent 2's preferences. We have that $\sigma_R = (1, 3, 4, 2, 5, 6)$, while $\sigma_{R'} = (1, 2, 3, 4, 5, 6)$. It is immediate that this rule is not strategy-proof, as agent 2 can manipulate at R via R'_2.

We show that f satisfies limited influence. For any $a \in Z$, and any profile R'' such that $top(R''_i) = a$ for all $i \in N$, we have that $f_{12}(R'') = a$, and $2 \in U_3(a)$ and $2 \in U_4(a)$. Note that the only profiles where agent 2 does not get her top-ranked project are of the form $\hat{R}$ such that the second-ranked project in $\hat{R}_2$ is the same as $top(\hat{R}_1)$, and $top(\hat{R}_1) \neq top(\hat{R}_2) \neq$

$top(\hat{R}_3) \neq top(\hat{R}_4)$. Thus $U_2(a) = \emptyset$. Let R be a profile, and let R_3' be preferences such that $f_2(R_3', R_{-3}) \neq f_2(R) = a$. It follows that $top(R_1) \neq top(R_2)$. Thus, by construction, $f_3(R) = top(R_3)$. And so, in particular, we have that $f_3(R)\ P_3\ a$. Since we do not have $a\ R_3\ f_3(R)$, limited influence is trivially satisfied. The same is true for agent 4, and so f satisfies limited influence. It is left to the reader to verify that f satisfies the other axioms.

Figure 5: Independence of strategy-proofness

R						R'					
R_1	R_2	R_3	R_4	R_5	R_6	R_1	R_2'	R_3	R_4	R_5	R_6
$\boxed{a}$	b	$\boxed{d}$	$\boxed{c}$	$\boxed{d}$	$\boxed{c}$	$\boxed{a}$	$\boxed{b}$	$\boxed{d}$	c	d	c
d	$\boxed{a}$	a	d	c	d	d	c	a	$\boxed{d}$	c	d
c	c	c	a	a	a	c	a	c	a	$\boxed{a}$	a
b	d	b	b	b	b	b	d	b	b	b	$\boxed{b}$

Example **5.** (Group-Non-Bossiness)

Let $\mathcal{N} = \{1, 2, 3, 4, 5, 6\}, \mathcal{Z} = \{a, b, c, d\}$, $\underline{q} = \bar{q} = 2$. Let f be a rule similar to the SSPR, but defined as follows. For any profile R, if $top(R_1) = top(R_2)$, then $\sigma = (1, 2, 3, 4, 5, 6)$, otherwise $\sigma = (1, 2, 3, 6, 5, 4)$.

Then, for preferences R, R' given in Figure 6, the assignments are marked in boxes. To see that f does not satisfy group-non-bossiness, note that, for $M = \{2, 3\}$, $f_M(R') \sim^r f_M(R)$, but $f_{N \setminus M}(R') \neq f_{N \setminus M}(R)$. It is straightforward to verify that f satisfies the other axioms. Moreover, it can be seen that f is non-bossy, but f is not an $SSPR$. Thus weakening group-non-bossiness to non-bossiness no longer implies the rule is an $SSPR$.

Example **6.** (Limited Influence)

The rule f in Example 2 satisfies strategy-proofness, group-non-bossiness, unanimity and neutrality, but fails to satisfy limited influence.

Example **7.** (Unanimity)

It is easy to see that the rule in Proposition 1 satisfies strategy-proofness, group-non-bossiness, limited influence and neutrality, but violates unanimity.

Figure 6: Independence of group-non-bossiness

R						R'					
R_1	R_2	R_3	R_4	R_5	R_6	R_1	R'_2	R'_3	R_4	R_5	R_6
$\boxed{\text{a}}$	$\boxed{\text{b}}$	$\boxed{\text{a}}$	c	$\boxed{\text{c}}$	$\boxed{\text{c}}$	$\boxed{\text{a}}$	$\boxed{\text{a}}$	$\boxed{\text{b}}$	$\boxed{\text{c}}$	$\boxed{\text{c}}$	c
b	b	d	d	d	d	b	c	d	d	d	d
c	c	a	a	a	a	c	b	a	a	a	a
d	d	b	$\boxed{\text{b}}$	b	b	d	d	b	b	b	$\boxed{\text{b}}$

Example **8.** (Neutrality)

Let $\mathcal{N} = \{1, 2, 3, 4\}$, $\mathcal{Z} = \{a, b, c\}$, $\underline{q} = \bar{q} = 2$. Let f be a rule that operates like the SSPR, but with the following difference: For any profile R, if $top(R_1) = a$, then $\sigma = (1, 2, 3, 4)$, and otherwise $\sigma = (1, 3, 2, 4)$. Since the $SSPR$ satisfies strategy-proofness, group-non-bossiness, limited influence and neutrality, so does this rule. However, f is not neutral, as the specification explicitly depends on the project that is ranked top by agent 1.

References

Abdulkadiroğlu, A. and T. Sönmez (1998): "Random serial dictatorship and the core from random endowments in house allocation problems," *Econometrica*, 66, 689–702.

Bade, S. (2015): "Serial dictatorship: The unique optimal allocation rule when information is endogenous," *Theoretical Economics*, 10, 385–410.

Cechlárová, K. and T. Fleiner (2017): "Pareto optimal matchings with lower quotas," *Mathematical Social Sciences*, 88, 3–10.

Ehlers, L. and B. Klaus (2003): "Coalitional strategy-proof and resource-monotonic solutions for multiple assignment problems," *Social Choice and Welfare*, 21, 265–280.

Fragiadakis, D., A. Iwasaki, P. Troyan, S. Ueda, and M. Yokoo (2015): "Strategyproof matching with minimum quotas," *ACM Transactions on Economics and Computation*, 4.

Gibbard, A. (1973): "Manipulation of voting schemes: A general result," *Econometrica*, 41, 587–601.

Hatfield, J. W. (2009): "Strategy-proof, efficient, and non-bossy quota allocations," *Social Choice and Welfare*, 33, 505–515.

Kamiyama, N. (2013): "A note on the serial dictatorship with project closures," *Operations Research Letters*, 41, 559–561.

Klijn, F. (2017): "Constrained allocation of projects to heterogeneous workers with preferences over peers," Barcelona GSE Working Paper Series, no. 960.

Monte, D. and N. Tumennasan (2013): "Matching with quorums," *Economics Letters*, 120, 14–17.

Pápai, S. (2000a): "Strategyproof assignment by hierarchical exchange," *Econometrica*, 68, 1403–1433.

——— (2000b): "Strategyproof multiple assignment using quotas," *Review of Economic Design*, 5, 91–105.

——— (2001): "Strategyproof and non-bossy multiple assignments," *Journal of Public Economic Theory*, 3, 257–271.

Rhee, S. (2011): "Strategy-proof allocation of individual goods among couples," *Japanese Economic Review*, 62, 289–303.

Satterthwaite, M. (1975): "Strategy-proofness and Arrow's Conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, 10, 187–216.

Sönmez, T. and M. U. Ünver (2010): "House allocation with existing tenants: a characterisation," *Games and Economic Behavior*, 69, 425–445.

Svensson, L.-G. (1999): "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 16, 557–567.

Thomson, W. (2016): "Non-bossiness," *Social Choice and Welfare*, 47, 665–696.

Appendix

We show that the specification of the axiom of limited influence is 'tight', in the following sense:

Claim 1: Weaker versions of limited influence no longer give the $SSPR$ characterisation result.

Claim 2: A stronger version is incompatible with the other axioms.

Consider Claim 1 first. Suppose we were to weaken limited influence as follows:

Definition **2.** A rule f satisfies *weak limited influence 1* if, for any profile R, any agent $i \in N$, and any project $a \in Z$ such that $a \; P_i \; f_i(R)$, for all R'_i and all $j \in unanimity_i(a)$: $[f_j(R) = a] \implies [f_j(R'_i, R_{-i}) = a]$.

The weakening comes from requiring that $a \; P_i \; f_i(R)$, rather than $a \; R_i \; f_i(R)$. We could also consider another version:

Definition **3.** A rule f satisfies *weak limited influence 2* if, for any profile R, any project $a \in Z$, any agents $i, j \in N$, if $a \, R_i \, f_i(R)$ and $f_j(R) = a$, then for all R'_i, $[a \in f(R'_i, R_{-i})] \implies [f_j(R'_i, R_{-i}) = a]$.

The weakening comes from requiring j to be assigned a at the new profile only if a is assigned in the resulting profile. In particular, if $a \notin f(R'_i, R_{-i})$, then it says nothing about agent j's assignment, even if she was assigned a in the original profile.

However, in each of the above cases, the weaker version is not enough to guarantee the $SSPR$. This can be easily seen from the example in Section 7 showing the independence of limited influence. The rule described there does not satisfy limited influence, but it can be verified that it satisfies both versions of weak limited influence. However, as demonstrated, this rule is no longer an $SSPR$.

Consider Claim 2. We look at the natural strengthening of limited influence, where we require that the limit on influence be extended to all cases:

Definition 4. A rule f satisfies *strong limited influence* if, for any profile R, any agent $i \in N$, and any project $a \in Z$ such that $a \, R_i \, f_i(R)$, for all R'_i and all $j \neq i$: $[f_j(R) = a] \implies [f_j(R'_i, R_{-i}) = a]$.

However, this specification is too strong.

Proposition 5. *The SSPR does not satisfy strong limited influence.*

Proof: We give a proof by example. Let $N = \{1, 2, 3, 4\}$, let $Z = \{a, b, c\}$, and let the $SSPR$ φ be given by $\sigma \in \Sigma$, where $\sigma = (1, 2, 3, 4)$. Let R be a profile as given in the left side of Figure 7. Then the allocation prescribed by φ is given in boxes.

Figure 7: The $SSPR$ fails to satisfy strong limited influence

R_1	R_2	R_3	R_4	R'_1	R_2	R_3	R_4
[a]	[b]	[a]	a	[c]	[b]	a	a
b	a	b	[b]	a	a	[b]	b
c	c	c	c	b	c	c	[c]

Let R'_1 be preferences for agent 1, and let the profile be (R'_1, R_{-1}) (given on the right side of Figure 7). Then the allocation prescribed by φ for this profile is given in boxes. However, since $aR_1\varphi_1(R)$ and $\varphi_3(R) = a$, strong limited influence implies that $\varphi_3(R'_1, R_{-1}) = a$, which is not true. ■

In fact, in the presence of non-trivial minimum capacities, strong limited influence is incompatible with strategy-proofness, non-bossiness, unanimity and neutrality.

Theorem 5. *When $\underline{q} > 1$ and $m > 1$, there is no rule f that satisfies strategy-proofness, non-bossiness, unanimity, neutrality and strong limited influence.*

Proof: Let f satisfy strategy-proofness, non-bossiness, unanimity, neutrality and strong limited influence. Let $N = \{1, 2, 3, 4\}$, $Z = \{a, b, c\}$, and let $\underline{q} = \bar{q} = 2$. Then $m = 2$. Let $R = (R_1, R_2, R_3, R_4)$ be a profile given as in Figure 8 below.

Since f satisfies Pareto-efficiency, there are two agents that are assigned a in R. Without loss of generality, let the pair be $\{1, 2\}$, and let the assignment be given in boxes in Figure 8.

Figure 8: Preferences

R_1	R_2	R_3	R_4	R'_1	R''_1	R'_2	R'_3
[a]	[a]	a	a	b	c	b	c
b	b	[b]	[b]	a	a	a	a
c	c	c	c	c	b	c	b

By strong limited influence, for any $\tilde{R}_{34}$, we have that $f_{12}(R_{12}, \tilde{R}_{34}) = (a, a)$. Let R'_2 be preferences for 2 as given in Figure 8. By Pareto-efficiency, $f_2(R'_2, R_{-2}) = b$ (otherwise, by strategy-proofness, $f_2(R'_2, R_{-2}) = a$, and by non-bossiness, $f(R'_2, R_{-2}) = f(R)$, which violates Pareto-efficiency as, for instance, $a\, P_3\, f_3(R'_2, R_{-2})$). By strong limited influence, we have that $f_1(R'_2, R_{-2}) = a$. By feasibility, some agent in $\{3, 4\}$ is assigned a. Without loss of generality, let $f_3(R'_2, R_{-2}) = a$. The allocation is shown in Figure 9. Moreover, for any $\tilde{R}_4$, by strong limited influence we have that $f_{13}(\tilde{R}_4, R'_2, R_{13}) = (a, a)$ and $f_2(\tilde{R}_4, R'_2, R_{13}) = b$. By strategy-proofness and non-bossiness, we have that $f_{13}(\tilde{R}) = (a, a)$ and $f_2(\tilde{R}) = b$ whenever $top(\tilde{R}_{13}) = (a, a)$ and $top(\tilde{R}_2) = b$.

Figure 9: Preferences

R_1	R'_2	R_3	R_4
[a]	[b]	[a]	a
b	a	b	[b]
c	c	c	c

Now, let R'_3 be preferences as in Figure 8. Consider the profile (R'_2, R'_3, R_{14}).

Case 1: $f_3(R'_2, R'_3, R_{14}) = a$

By non-bossiness, $f(R'_2, R'_3, R_{14}) = f(R'_2, R_{-2})$. Let R''_1 be preferences for agent 1 as given in Figure 8, and let $\hat{R} = (R''_1, R'_2, R'_3, R_4)$. For ease of reference, the profile is shown in Figure 10. By Pareto-efficiency, $f_1(\hat{R}) = c$ (otherwise, by strategy-proofness and non-bossiness, $f_{13}(\hat{R}) = (a, a)$, and $f(\hat{R})$ is not an efficient allocation). By strong limited influence, $f_3(\hat{R}) = a$. If $f_2(\hat{R}) = a$, then $f_4(\hat{R}) = c$, violating Pareto-efficiency as agents $3, 4$ can trade. If $f_2(\hat{R}) = c$, this violates Pareto-efficiency, as agents $2, 3$ can trade. So $f_2(\hat{R}) = b$. But this violates feasibility, as more than two projects are assigned for this profile.

Figure 10: Preferences for Case 1

R_1''	R_2'	R_3'	R_4
c	b	c	a
a	a	a	b
b	c	b	c

Case 2: $f_3(R_2', R_3', R_{14}) = c$.

By strong limited influence, $f_1(R_2', R_3', R_{14}) = a$. By feasibility, since only two projects can be assigned, $f_2(R_2', R_3', R_{14}) \neq b$. It follows that $f_2(R_2', R_3', R_{14}) = a$, as otherwise we have $f_2(R_2', R_3', R_{14}) = c$, but agent 2 can move project a to the top of her preferences and get a, violating strategy-proofness. This profile and allocation is shown in Figure 11.

Figure 11: Preferences

R_1	R_2'	R_3'	R_4
[a]	b	[c]	a
b	[a]	a	b
c	c	b	[c]

Claim **6.** For any R such that $top(R_1) \neq top(R_3)$, $f_1(R) = top(R_1)$ and $f_3(R) = top(R_3)$.

Proof: Let $R = (R_2', R_3', R_{14})$ as above. We have that $f_{12}(R) = (a, a)$ and $f_3(R) = (c, c)$. Let $\tilde{R}_4$ be arbitrary preferences. Since $aP_4f_4(R)$ and $cR_4f_4(R)$, by strong limited influence we have that $f(\tilde{R}_4, R_{-4}) = f(R)$. Let $\tilde{R}_2$ be arbitrary preferences that rank b top. By strategy-proofness, $f_2(\tilde{R}_{24}, R_{13}) \neq b$. Suppose $f_2(\tilde{R}_{24}, R_{13}) = c$. By strong limited influence, $f_1(\tilde{R}_{24}, R_{13}) = a$. If $f_3(\tilde{R}_{24}, R_{13}) \neq c$, then by feasibility, $f_3(\tilde{R}_{24}, R_{13}) = a$. But then, for

$\tilde{R}_3$ that ranks a top, by the earlier result we have that $f_2(\tilde{R}_{124}, R_1) = b$. But by strong limited influence, $f_2(\tilde{R}_{124}, R_1) = c$. This is a contradiction. Thus $f_3(\tilde{R}_{24}, R_{13}) = c$. Then, by strategy-proofness and non-bossiness, for any arbitrary preferences $\tilde{R}_2$, we have that $f_1(\tilde{R}_{24}, R_{13}) = a$ and $f_1(\tilde{R}_{24}, R_{13}) = c$. It follows from strategy-proofness and non-bossiness that for all R such that $top(R_1) = a$ and $top(R_3) = c$, we have $f_1(R) = a$ and $f_3(R) = c$. By neutrality, this is true for all $a, c \in Z$. Thus $f_1(R) = top(R_1)$ and $f_3(R) = top(R_3)$ whenever $top(R_1) \neq top(R_3)$. ■

Let R'_1 be preferences as in Figure 8, and consider the profile (R'_1, R_{-1}). By a similar argument to the one above, it follows that $f_1(R'_1, R_{-1}) = b, f_2(R'_1, R_{-1}) = a$, and since $top(R'_1) \neq top(R_3)$, by Claim 6, we have that $f_3(R'_1, R_{-1}) = a$. Finally, consider the profile (R'_{13}, R_{24}), given in Figure 12. Since $top(R'_3) \neq top(R'_1)$, by Claim 6, we have that $f_1(R'_{13}, R_{24}) = b$ and $f_3(R'_{13}, R_{24}) = c$. But by strong limited influence, we have that $f_2(R'_{13}, R_{24}) = a$, which violates feasibility.

Figure 12: Preferences

R'_1	R_2	R'_3	R_4
b	a	c	a
a	b	a	b
c	c	b	c

Thus there is no rule which simultaneously satisfies strategy-proofness, non-bossiness, strong limited influence, unanimity and neutrality. ■