Influence in Private-Goods Allocation

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Abstract

Many private-good allocation rules allow for certain agents - when they change the preferences they report to the rule - to affect the welfare of other agents without affecting their own welfare. Classic examples are the agent-proposing Gale-Shapley rule for matching (Gale and Shapley, 1962) and the Vickrey rule for single-object auctions (Vickrey, 1961). We propose a new methodology, based on the study of this influence that agents might have on each other's welfare, that furthers our understanding of such rules. We use structural conditions on influence to explain why some rules satisfy normatively desirable properties such as Pareto-efficiency, stability, weak Maskin monotonicity, non-wastefulness, and weak group-strategy-proofness, while others do not. Illustrative applications include the efficiency-adjusted deferred acceptance mechanism for matching (Kesten, 2010), and generalised absorbing top-trading-cycles rules in object-allocation models (Aziz and Keijzer, 2011).

Keywords: Private-goods allocation, welfare, strategy-proofness, influence, matching, single-object auctions, object-allocation, non-bossiness, Pareto-efficiency, stability.

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1 Introduction

Public institutions that provide private goods such as housing, education, health, or licenses, often use systematic procedures to distribute these goods among agents in a manner that reflects their preferences. For example, auction rules for licenses use bids to determine winners and their corresponding payments, centralised admissions processes use students' reported preferences and schools' priorities to determine matches, and so on. The design of suitable 'allocation rules' holds a distinguished place in the mechanism design literature. Insights from the theory have successfully transformed many markets, including the design of tenders for public projects, admissions processes to match medical residents to hospitals or students to public schools, allocation schemes for public housing projects, rationing schemes for social endowments, and cost-sharing mechanisms for the provision of public goods.

The design of allocation rules for private goods is qualitatively different from that for public goods. The excludable nature of private goods often necessitates different assignments for different agents, as opposed to determining the common (level of) public goods to be consumed by all agents. An immediate consequence is that designers can escape the dictatorial implications for voting rules of the Gibbard-Satterthwaite theorem (Gibbard, 1973; Satterthwaite, 1975). This has led to a proliferation of non-dictatorial incentive-compatible allocation rules in a variety of applications.

Another consequence is that allocation rules can be more responsive to *changes* in reported preferences. Consider the second-price auction (also sometimes known as the Vickrey rule (Vickrey, 1961)). The bidder with the second-highest bid does not win the auction, but her bid sets the winning price. If this 'price-setting' agent were instead to report a different bid, this might affect the winning price. Elsewhere, when the student-proposing Gale-Shapley rule (Gale and Shapley, 1962) is used to determine which students attend which public schools, a student unilaterally changing her reported preferences could change the matched schools for other students. Crucially, a change in others' welfare could occur *without* a change in the welfare of the agent who reports different preferences: the price-setting agent might continue to lose and pay nothing, and the student in question might continue to be matched to the same school.

We are concerned with allocation rules under which an agent changes her preferences and 'influences' the welfare of some other agent, i.e., changes the welfare of the other agent with no change in her own welfare. In particular, we are interested in when this happens for a specific type of changed preferences: when the agent reports that her originally prescribed assignment is worth relatively more to her than before.[1] As illustration, in the second-price auction, an increase in the relative worth of the price-setting agent's prescribed assignment (not getting the object and paying nothing) is equivalent to a lower valuation for the object, and therefore a lower bid. For the student-proposing Gale-Shapley rule, these are preferences in which a student reports that her prescribed match is relatively higher-ranked with respect to other schools than it was originally.

It should be pointed out that these changed preferences are *hypothetical*; few applications allow agents the opportunity to report preferences twice. Nevertheless, this issue is relevant when preferences are private information, and agents can strategise over which preferences to report. Even if the agent herself might not gain by reporting preferences that are not her true preferences, there might still be individual or social incentives to do so in order to help or hurt others, particularly if such a change is 'risk-free', in that her own welfare is unaffected as a result. For instance, a lowered bid by the price-setting agent improves the welfare of the winning bidder, since it reduces the winning price, leading to possibilities of inducements or collusion (see, e.g., Graham and Marshall (1987)). These incentives could also be social. The ability of some students to report preferences in a way that gives them the same matched school but positively affects the welfare of others is critical to improving the efficiency of the student-proposing Gale-Shapley rule (see, e.g., Kesten (2010)).

Influence of this sort bears a close relation to the 'bossiness' of an allocation rule. Bossiness has received much attention in the mechanism design literature, mostly towards elimi-

[1] Technically, these changed preferences are 'monotonic transformations' of her original preferences at her prescribed assignment, i.e., preferences in which the relative position of her prescribed assignment weakly improves with respect to other alternatives.

nating its effects. The search for 'non-bossy'[2] allocation rules has proved fruitful in object-allocation (Pápai, 2000), division problems (Sprumont, 1991), models of exchange (Barberà and Jackson, 1995), and others. In each of these models, it is possible to find desirable allocation rules under which no agent can influence another's welfare. Non-bossiness has also been shown to have a close connection with normatively desirable properties for allocation rules, such as 'group-strategy-proofness' (Pápai, 2000), 'Maskin invariance' (Svensson, 1999), and even 'Pareto-efficiency' (Pápai, 2001).

Yet non-bossiness sometimes conflicts with other requirements. In matching models, non-bossiness is incompatible with 'stability' (Kojima, 2010). In models of object reallocation, where agents might be indifferent between objects, non-bossiness is incompatible with Pareto-efficiency, 'individual rationality' and 'strategy-proofness' (Bogomolnaia et al., 2005; Jaramillo and Manjunath, 2012). Requiring influence-freeness in these models is tantamount to eliminating plausible rules from consideration, regardless of the other desirable properties in their favour.

The allocation rules we are interested in are essentially bossy.[3] Our approach is to draw the connection between the *bossiness of a rule* and the *influence between agents* under that rule, and is based on the following observation: For a (possibly bossy) allocation rule and a 'profile' of agents' preferences, an agent can either influence the welfare of another agent at that profile or she cannot. By checking this for each ordered pair of agents, we can find the binary 'influence relation' induced by the rule on the set of agents at that profile. The influence relation is a new tool with which to study allocation rules in general, and formalising this idea is our principal methodological contribution.

The induced influence relation is always empty for a non-bossy allocation rule, since by definition no agent can ever influence the welfare of any other agent in this way. However, the influence relation reveals a lot about some bossy allocation rules. For example, take the so-called efficiency-adjusted deferred-acceptance mechanism (EADAM) in models of school

[2]The term is due to Satterthwaite and Sonnenschein (1981).

[3]Technically, bossy in welfare terms under monotonic transformations. See Thomson (2016) for more on bossiness.

choice (Kesten, 2010). Based on obtaining consent from certain students to waive potential priority violations, the EADAM generates efficiency improvements to the allocation prescribed by the student-proposing Gale-Shapley rule, without hurting consenting students. In particular, the EADAM involves following a *specific order* over consenting students. We show that exercising this obtained consent is equivalent to performing a preference change for the consenting student of the sort we are interested in, and this order over agents is closely related to the influence relation induced by the Gale-Shapley rule (see Section 6). In another example, generalised absorbing 'top-trading-cycles' (GATTC) rules for object reallocation, when agents might be indifferent between objects, are bossy (Aziz and Keijzer, 2011; Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012). We characterise all instances of influence under strategy-proof GATTC rules. Our results might also help in characterising all strategy-proof, individually rational and Pareto-efficient rules in such environments, which is as yet an open question.

It is also observed that bossiness is often no impediment to an allocation rule satisfying Pareto-efficiency (Alcalde-Unzu and Molis, 2011; Jaramillo and Manjunath, 2012), 'weak group-strategy-proofness' (Barberà et al., 2016), or even stability, 'weak Maskin monotonicity', or 'non-wastefulness' (Kojima and Manea, 2010). These properties are often invoked directly as normative justifications for using specific rules. Yet their connection with bossiness has never been adequately explained. For instance, why are some bossy rules Pareto-efficient while others are not? Studying the structure of influence under different rules allows us to provide such an explanation. This is our second contribution.

We do this via the following two questions. Firstly, for a given allocation rule, which are the agents with influence? Secondly, what are the welfare consequences of such influence? We propose some simple structural conditions related to these two questions. The conditions are that: (1) the induced influence relation among agents is always 'acyclic'; (2) the welfare effects on influenced agents are always positive ('positivity'); (3) whenever there are influenced agents, there is always at least one who is better off and at least one who is worse off in welfare terms ('oppositeness'); (4) in object-allocation environments, positively-influenced agents always receive objects that were previously assigned to other agents ('displacement'); and (5) the independence of agents from other agents' influence is preserved across certain preference changes ('preservation'). These conditions provide an analytical framework that is useful for classifying different allocation rules. As an illustration of the potential of this approach, we show that the Vickrey rule and the student-proposing Gale-Shapley rule satisfy positivity but not oppositeness, whereas GATTC rules satisfy oppositeness but not positivity.

Using these conditions, we show how various properties of allocation rules arise naturally as a consequence of the structure of influence itself. For all our results, strategy-proofness and acyclicity are essential. In Proposition 1, we show that a strategy-proof allocation rule is 'minimally Maskin welfare-invariant' if it is acyclic. This allows us to show, in Theorem 1, that an allocation rule that is strategy-proof and acyclic is weakly Maskin monotonic if and only if it is positive. This explains why the student-proposing Gale-Shapley rule and the Vickrey rule are weakly Maskin monotonic. In Theorem 3, we show that a rule defined on a 'top-rich' domain that is strategy-proof, acyclic and 'unanimous' is Pareto-efficient if and only if it is opposite. This explains why the GATTC rules are Pareto-efficient. Moreover, the incompatibility between positivity and oppositeness explains precisely why some allocation rules may be either weakly Maskin monotonic or Pareto-efficient, but not both.

In Theorem 2, we show that any allocation rule defined on a 'rich' domain that satisfies acyclicity and either positivity or preservation is weakly group-strategy-proof if and only if it is strategy-proof. This result is related to the equivalence result in Barberà, Berga, and Moreno (2016), but provides new insights as it is based solely on the structure of influence under allocation rules. In pure object-allocation settings, we show in Theorem 4 that a rule defined on a top-rich domain that is strategy-proof, acyclic and unanimous is non-wasteful if it is displacing. And finally, we focus on stability in matching. We show in Theorem 5 that for many-to-one matching models without contracts, a rule defined on a top-rich domain that is 'top-respecting', strategy-proof, unanimous, acyclic, displacing and positive is stable. We believe this result can be extended to more general matching environments as well. The aforementioned incompatibility between positivity and oppositeness also helps illuminate the Pareto-efficiency/stability tradeoff.

The paper is organised as follows. In Section 2, we provide a general model of private-goods allocation that also encompasses the above applications. In Section 3, we present the notion of influence, and as motivating examples show the nature of influence for some widely-studied bossy allocation rules in models of matching, object-allocation and reallocation, and single-object auctions. In Section 4, we define a number of commonly-used properties of an allocation rule and show some basic relations between them. In Section 5, we define the conditions relating to influence. This section also contains our main results. In Section 6, we return to our motivating examples, armed with our results and new analytical tools, and uncover more details about their working. Finally, in Section 7, we discuss limitations, applications, and possible extensions.

2 The Model

The private-goods allocation model we consider involves a finite set of *agents* $N = \{i, j, \ldots\}$. For each agent $i \in N$, there is an associated set of *resources* Z^i which, depending on the application, might represent indivisible objects or quantities of a good. There is also an associated set of *terms* T^i, which might include contracts, payments or transfers to be made in exchange for obtaining resources. Putting them together, a set of *alternatives* for agent $i \in N$ is a set of resource-term pairs, and is denoted $X^i \subseteq Z^i \times T^i$. In general, Z^i and T^i are not restricted to be finite. A special case of this model is when $Z^i = Z^j \equiv Z$ for all agents $i, j \in N$. In this case, Z is a *common* set of resources. Another special case that recurs frequently in applications is when Z is common and finite, and $T^i = \emptyset$ for all $i \in N$. Then $X^i \equiv X = Z$ for all $i \in N$, and we call such an X a set of *objects*, and call this a *pure object-allocation* setting. This covers models of object-allocation without transfers, and matching without contracts, for example. For a subset of agents $M \subseteq N$, X^M denotes the Cartesian product $\times_{i \in M} X^i$. An *allocation* $a \in X^N$ specifies an alternative $a_i \in X^i$ for each agent $i \in N$. The set of *feasible allocations* is determined by the economy in question, and is denoted $\mathcal{A} \subseteq X^N$. For any $a \in \mathcal{A}$ and $M \subseteq N$, $a_M \in X^M$ refers to the *assignments* of M in a. For $i \in N$, the notation $a_i = (z_i, t_i)$ is sometimes used to refer to the specific resource-term pair associated with the assignment a_i.

For each agent $i \in N$, *preferences* are given by a reflexive, complete and transitive binary relation R_i over X^i, where, for two alternatives $x, y \in X^i$, $x \; R_i \; y$ is interpreted as meaning 'x is at least as good as y under preferences R_i'. An additionally antisymmetric R_i is called *strict*. As is the common convention, the asymmetric component of R_i is denoted P_i and the symmetric component I_i. The *strict upper contour set of an alternative* $x \in X^i$ *at* R_i contains all alternatives preferred to x at R_i, i.e., $\bar{U}(R_i, x) \equiv \{y \in X^i \mid y \; P_i \; x\}$ and the *strict lower contour set of* x *at* R_i contains all alternatives less preferred to x at R_i, i.e., $\bar{L}(R_i, x) \equiv \{y \in X^i \mid x \; P_i \; y\}$.

We assume that if an agent is indifferent between two alternatives, her welfare is the same if she receives either of them. Conversely, an agent who prefers an alternative to another is better off in welfare terms if she receives the preferred alternative. Agents' preferences extend from alternatives to allocations in a straightforward manner when agents are selfish. Selfishness assumes that agents care only about the assignment they receive in an allocation, and does not depend for instance on what other agents might receive. A selfish agent's preferences over two allocations can directly be imputed by her preferences over the respective assignment she receives in each allocation. Selfishness is assumed throughout what follows.

For an agent $i \in N$, the *complete* domain of all possible preferences over X^i is written $\tilde{\mathcal{R}}^i$. For $M \subseteq N$, we write $\tilde{\mathcal{R}}^M$ for $\times_{i \in M} \tilde{\mathcal{R}}^i$.[4] An arbitrary domain for $M \subseteq N$ is denoted $\mathcal{R}^M \subseteq \tilde{\mathcal{R}}^M$. For $M \subseteq N$, a *sub-profile* $R_M \in \mathcal{R}^M$ is a specification of preferences for each agent in M. Sub-profiles for all agents (called *profiles*) are denoted $R, R' \in \mathcal{R}^N$. For $M \subseteq N$, a profile $R \in \mathcal{R}^N$ may also be written as (R_M, R_{-M}) to emphasise the role of M, with $R_M \in \mathcal{R}^M$ and $R_{-M} \in \mathcal{R}^{N \setminus M}$.

For an agent $i \in N$, a set of alternatives $Y^i \subseteq X^i$, and preferences $R_i \in \mathcal{R}^i$, denote by *top*(R_i, Y^i) the set of *maximal alternatives* in Y^i according to R_i, i.e. *top*$(R_i, Y^i) = \{x \in$

[4] This Cartesian assumption is not without loss of generality. See Barberà, Berga, and Moreno (2016).

$Y^i \mid x \; R_i \; y$ for all $y \in Y^i\}$. Similarly, for $M \subseteq N$, and $Y^M \subseteq \times_{i \in M} Y^i$, let $top(R_M, Y^M) = \{a_M \in Y^M \mid a_i \in top(R_i, Y^i)$ for all $i \in M\}$.

We now present two structural conditions on domains. For domains in which maximal alternatives exist (e.g., when the set of alternatives is finite), a condition called *top-richness* requires that, for any preferences in the domain and any alternative, there exist preferences in the domain that top-rank that alternative without altering the relative position of other alternatives.

Definition 1. For $i \in N$, a domain $\mathcal{R}^i$ is top-rich if, for any $R_i \in \mathcal{R}^i$, and any $x \in X^i$, there exists $R'_i \in \mathcal{R}^i$ such that $top(R'_i, X^i) = x$, and for any $y, z \in X \setminus \{x\}$, we have that $y \; R_i \; z \iff y \; R'_i \; z$. A domain $\mathcal{R}^N$ is top-rich if $\mathcal{R}^i$ is top-rich for each $i \in N$.

The complete domain of preferences on a finite set of alternatives X is top-rich, and so, in particular, is the domain of strict preferences on X. However, a maximal alternative is not guaranteed to exist for arbitrary sets of resources and preferences (e.g. in models of cost sharing of a public good (Serizawa, 1999; Moulin, 1994)). Nevertheless, such a domain may be called *rich* if, for two preferences in the domain for an agent, preferences can be found in the domain that combine their characteristics in a specific manner. In particular, it requires that, for two preferences R_i and R'_i in the domain and two alternatives x and y with $y \; P_i \; x$, there exists preferences R''_i in the domain that 'raises' the position of y in a manner consistent with both R_i and R'_i, without disturbing the relative position of x in R_i.

Definition 2 (Barberà, Berga, and Moreno (2016)). For $i \in N$, a domain $\mathcal{R}^i$ is rich if, for any $R_i, R'_i \in \mathcal{R}^i$, and any $x, y \in X^i$ such that $y \; P_i \; x$, there exists $R''_i \in \mathcal{R}^i$ such that $\bar{U}(R''_i, y) \subseteq (\bar{U}(R_i, y) \cap \bar{U}(R'_i, y))$, $\bar{L}(R'_i, y) \subseteq \bar{L}(R''_i, y)$, $\bar{U}(R''_i, x) = \bar{U}(R_i, x)$, and $\bar{L}(R''_i, x) = \bar{L}(R_i, x)$. A domain $\mathcal{R}^N$ is rich if $\mathcal{R}^i$ is rich for each $i \in N$.

Richness extends the idea of a top-rich domain to domains without maximal alternatives. In particular, every top-rich domain is also rich (for any $R_i, R'_i \in \mathcal{R}^i$ and $x, y \in X^i$ such that $y \; P_i \; x$, top-richness guarantees the existence of preferences R''_i that top-ranks y without altering the relative position of other alternatives in R_i, and this R''_i satisfies the requirements of richness) but the converse is not true in general.

Given an arbitrary domain $\mathcal{R}^N$ and a set of feasible allocations $\mathcal{A}$ associated with a particular application, an *allocation rule* (or simply a *rule*) is a function $f : \mathcal{R}^N \to \mathcal{A}$ that prescribes a feasible allocation for every profile in the domain. For $R \in \mathcal{R}^N$, $f_i(R)$ and $f_M(R)$ denote the assignment(s) of agent i and agents in $M \subseteq N$, respectively.

A certain sort of preference change for agents is central to our analysis and we define it here. 'Monotonic transformations' of an agent's preferences for a particular alternative are preferences in which that alternative weakly improves its position relative to other alternatives. This can be stated in terms of the relation between the strict upper and lower contour sets of that alternative at the two preferences. Formally, for $i \in N$, $x \in X^i$ and $R_i \in \mathcal{R}^i$, preferences $R'_i \in \mathcal{R}^i$ are a *monotonic transformation of* R_i *for* x if $\bar{U}(R'_i, x) \subseteq \bar{U}(R_i, x)$ and $\bar{L}(R_i, x) \subseteq \bar{L}(R'_i, x)$. The set of monotonic transformations of R_i for x in $\mathcal{R}^i$ is denoted $\mathcal{M}(\mathcal{R}^i, R_i, x) \subseteq \mathcal{R}^i$. Extending the definition to sets of agents, for $M \subseteq N$, $R_M \in \mathcal{R}^M$, and $a_M \in X^M$, a sub-profile $R'_M \in \mathcal{R}^M$ is a monotonic transformation of R_M for a_M if $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, a_i)$ for all $i \in M$. $\mathcal{M}(\mathcal{R}^M, R_M, a_M) \subseteq \mathcal{R}^M$ denotes the set of all monotonic transformations of R_M for a_M in $\mathcal{R}^M$.

3 Influence

Since our notion of influence between agents is based on the bossiness under monotonic transformations of an allocation rule, it is useful to first formally define this concept. An allocation rule is called 'bossy' in welfare terms under monotonic transformations if there is a profile and a pair of agents such that the first agent can change the welfare of the second by reporting preferences that are a monotonic transformation of her assignment in the original profile, in a welfare-neutral way for herself according to her original preferences. A rule is 'non-bossy' if it is not bossy.

Definition **3.** An allocation rule f defined on $\mathcal{R}^N$ is bossy (in welfare terms) under monotonic transformations if there is a profile $R \in \mathcal{R}^N$, agents $i, j \in N$, and preferences $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ such that $f_i(R'_i, R_{-i})\ I_i\ f_i(R)$ and either $f_j(R'_i, R_{-i})\ P_j\ f_j(R)$ or $f_j(R)\ P_j\ f_j(R'_i, R_{-i})$. An allocation rule is non-bossy under monotonic transformations if it is not bossy under monotonic transformations.

It is easy to see that if we remove the additional requirement that R'_i be a monotonic transformation of R_i at $f_i(R)$, we recover the original definition of non-bossiness (Satterthwaite and Sonnenschein, 1981). We now fix an allocation rule, and reformulate this idea directly in terms of influence between agents under the rule. For a given allocation rule, an agent is said to have *influence* (in welfare terms) over another agent at a profile if there is a monotonic transformation of her assignment at that profile such that she is welfare-neutral across the two allocations (according to her original preferences), but the other's welfare changes. Conversely, an agent is *independent* of the influence of another agent at that profile if no such monotonic transformation for the second agent exists. Formally:

Definition 4. Let f be an allocation rule defined on $\mathcal{R}^N$. For $i, j \in N$, agent i has influence (in welfare terms) over j (for short, $\underline{i \text{ has influence with } j}$) at $R \in \mathcal{R}^N$ if there exists $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ such that $f_i(R'_i, R_{-i})\ I_i\ f_i(R)$ and either $f_j(R'_i, R_{-i})\ P_j\ f_j(R)$ or $f_j(R)\ P_j\ f_j(R'_i, R_{-i})$. Conversely, we say agent j is independent of the influence of i at R ($\underline{j \text{ is independent of } i}$) if, for any $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$, $f_i(R'_i, R_{-i})\ I_i\ f_i(R) \implies f_j(R'_i, R_{-i})\ I_j\ f_j(R)$.

The definition of influence is consistent with the definition of bossiness under monotonic transformations. However, the influence relationship is defined between agents under a given rule, rather than on the rule itself. The reasoning behind using monotonic transformations is straightforward: if an agent's welfare is unchanged when she drops her claims on desirable objects that she does not obtain, it might be easier for her to change her reported preferences in this way, rather than some non-monotonic transformation that might increase the risk of being obtaining an undesirable outcome.[5] Moreover, many existing results that use non-bossiness in fact use non-bossiness under monotonic transformations (see Section 4), thus

[5]This argument also forms the central thesis of the efficiency-adjusted deferred acceptance mechanism (Kesten, 2010; Tang and Yu, 2014), where the incentives and welfare of an agent are unharmed by giving consent.

relaxing this requirement is natural. We discuss other possible definitions of influence in Section 7.

It is also worth mentioning at this point that, for an agent, having influence with other agents is not the same as having a say over her own assignment. For instance, the well known *serial priority rule* (or 'serial dictatorship') in object-allocation models gives earlier agents in the priority a greater chance of getting their preferred objects, and thus a greater say over their assignments. But since this rule is non-bossy (Svensson, 1999), no agent has any influence with any other. Conversely, as we will see, agents with influence under the Gale-Shapley rule for matching models may not have any increased say in their assignments.

Observe that our definition of influence pertains to pairs of agents: one has influence with another at a profile, or does not. We use this idea to generate a binary 'influence relation' for subsets of agents. For an allocation rule f defined on $\mathcal{R}^N$ and a profile $R \in \mathcal{R}^N$, an *influence relation* b_R^f is the set of all ordered pairs in N such that the first agent in the pair has influence with the second at R. Formally, $b_R^f \equiv \{(i,j) \in N \times N \mid \text{there is } R_i' \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R)) \text{ such that } i \text{ has influence with } j \text{ at } R \text{ via } R_i'\}$. We can also identify independent agents at a profile: those that are not influenced by any other agent at that profile. For an allocation rule f, a profile $R \in \mathcal{R}^N$, and a set of agents $C \subseteq N$, the set of *independent agents in C at R* is $I_R^f(C) \equiv \{i \in C \mid (j,i) \notin b_R^f \text{ for all } j \in C\}$. We write $I_R^f(N)$ simply as I_R^f.

The influence relation is a widely applicable concept and could be quite arbitrary. The relation could be different for different rules at the same profile. The relation could also be different for the same rule at different preference profiles. The set of independent agents might be empty at some profiles, and not at others. It is clear, however, that for a non-bossy rule, the influence relation is always empty, and thus the set of independent agents is the set of agents itself. Moreover, the relation when it exists is always irreflexive: $(i,i) \notin b_R^f$. It should also be noted that, for a given rule f, b_R^f is *uniquely* determined for every R: for every $i,j \in N$, either $(i,j) \in b_R^f$ or not.

3.1 Illustrative Examples

An examination of influence under some well-known bossy allocation rules helps illustrate the concept.

3.1.1 Matching

For the sake of exposition, first consider the basic one-sided many-to-one matching model without contracts, also known as 'school choice' (Abdulkadiroğlu and Sönmez, 2003; Gale and Shapley, 1962). For simplicity, we assume a pure object-allocation setting. Preferences for an agent $i \in N$ are represented by a strict relation R_i over $X \cup \{\emptyset\}$, where $\emptyset$ denotes the 'outside option' for the agent. Given preferences R_i, every object $x \in X$ such that $x \; P_i \; \emptyset$ is called 'acceptable'. Each object $x \in X$ has a 'capacity', which is a non-negative integer q_x that denotes the maximum number of agents that it can accommodate. Moreover, each object $x \in X$ has a 'priority' over N, denoted by a strict relation $\succ_x$. In a one-sided matching model such as this, it is assumed that priorities are commonly known, and the focus is instead on the welfare of agents. For what follows, we fix a particular 'capacity profile' q and a particular 'priority profile' $\succ$ for X.

For any profile of (strict) preferences $R \in \mathcal{R}^N$ for agents, the *Gale-Shapley algorithm* proceeds as follows. N is denoted the 'proposing' side of the market. In the first round, each agent $i \in N$ applies to his or her most-preferred object in X according to her preferences R_i. Each object $x \in X$ tentatively accepts up to q_x of its highest-priority applicants (according to $\succ_i$), rejecting all other applications (if there are fewer than q_x applications, all are tentatively accepted). In each subsequent round, each rejected agent in the previous round applies to her most-preferred object from among those she considers acceptable and that have not rejected her already, if any such objects remain. Each object x considers its revised set of applications, including the applications held from the previous round, if any, and tentatively accepts up to q_x highest-priority of these, rejecting all others. If at any point a rejected agent finds no remaining object acceptable, he or she leaves the process unmatched. The procedure terminates when no more applications are made.

Given q and $\succ$, the algorithm prescribes a unique allocation (or 'matching') for any $R \in \mathcal{R}^N$. Given q and $\succ$, the agent-proposing Gale-Shapley rule σ^N identifies for every R the allocation $\sigma^N(R)$ that results from running the Gale-Shapley algorithm at R with agents proposing. The following example illustrates influence under the Gale-Shapley rule:

Example 1. Let $N = \{i_1, i_2, i_3\}$ and $X = \{x_1, x_2, x_3\}$. Let each object have a capacity of 1. Given priorities for objects and two profiles of preferences for agents as in Figure 1, the allocations made by σ^N are given in boxes. For simplicity, we do not list unacceptable objects or agents.

Figure 1: Example 1

R						(R'_{i_2}, R_{-i_2})					
R_{i_1}	R_{i_2}	R_{i_3}	$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$	R_{i_1}	R'_{i_2}	R_{i_3}	$\succ_{x_1}$	$\succ_{x_2}$	$\succ_{x_3}$
x_1	x_1	x_2	i_3	i_1	i_2	$\boxed{x_1}$	$\boxed{x_3}$	$\boxed{x_2}$	i_3	i_1	i_2
$\boxed{x_2}$	$\boxed{x_3}$	$\boxed{x_1}$	i_2	i_3	$\emptyset$	x_2	$\emptyset$	x_1	i_2	i_3	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	i_1	i_2		$\emptyset$		$\emptyset$	i_1	i_2	

The situation on the right differs from the one on the left only in the preferences of i_2. In particular, $R'_{i_2} \in \mathcal{M}(\mathcal{R}^{i_2}, R_{i_2}, \sigma^N_{i_2}(R))$. Since the assignments (and thus the welfare) of agents i_1 and i_3 change with no change in the assignment of i_2 (and thus her welfare under her original preferences), we have by definition that i_2 has influence with i_1 and i_3 at R via R'_{i_2}.

We have the following well-known facts about the agent-proposing Gale-Shapley rule:

Fact 1. For any instance of influence under the agent-proposing Gale-Shapley rule, the change in welfare for influenced agents is positive.

This fact depends crucially on influence being defined with respect to monotonic transformations. The Gale-Shapley rule exhibits bossiness more generally as well (e.g., consider the preference change R'_{i_2} to R_{i_2} for i_2). For an arbitrary case of bossiness (i.e., under preference changes that are non-monotonic transformations), Fact 1 may not hold.

For a profile R and the corresponding allocation $\sigma^N(R)$, we say a feasible allocation $a \in \mathcal{A}$ is a *feasible Pareto-improvement to* $\sigma^N(R)$ if all agents are at least as well off in a as at $\sigma^N(R)$, and some agent is better off, i.e., $a_i \; R_i \; \sigma_i^N(R)$ for all $i \in N$ and $a_j \; P_j \; \sigma_j^N(R)$ for some $j \in N$. Then:

FACT **2.** Any feasible Pareto-improvement to $\sigma^N(R)$ involves agents exchanging assignments along cycles.

For a detailed exposition of these facts, the reader is invited to refer to Kesten (2010) and Tang and Yu (2014). The critical observation for our purposes is that the only agents in N who have influence with other agents in N at a profile R are those who are 'interrupters' at R. An interrupter is a agent $i \in N$ who initiates a 'rejection cycle': i is tentatively matched to some object $x \in X$ at some step t in the algorithm, causing some other agent $i' \in N$ to be rejected by x in favour of i at step t, and this sets off a sequence of rejections in subsequent steps that eventually lead to some agent $j \in N$ applying to object x at a step t' ($t' > t$), at which point i is herself rejected by x in favour of j. Importantly, Kesten (2010) demonstrates that an interrupter i may be associated with an 'unrealised cycle', where if i were not to initiate the above rejection cycle between steps t and t' (via a monotonic transformation of her assignment in R, say), her own assignment would not change, but some non-empty set of agents $B \subset N$ would all be better off as a result, by effectively trading their assignments along the previously unrealised cycle. In our analysis, this is precisely the influence of i with respect to agents in the corresponding unrealised cycle B.

This also yields some insight into the working of the so-called efficiency-adjusted deferred-acceptance mechanism (EADAM). As mentioned in the introduction, Kesten (2010) showed that the sometimes large welfare losses that result from the Gale-Shapley rule in school choice settings can be mitigated by running the EADAM procedure. Based on obtaining consent from potential interrupters to waive potential priority violations, the EADAM generates efficiency improvements to the Gale-Shapley allocation by 'dropping' the resource in question from the consenting student's preferences, thereby allowing previously unrealised cycles to realise without hurting the consenting student. The idea of dropping a resource from preferences is equivalent to performing a specific monotonic transformation of the original assignment (e.g., raising it to the position 'just above' the resource), and thus the resulting efficiency improvements are tantamount to instances of influence. Moreover, the EADAM handles interrupters in a specified order that bears a close relation to the influence structure at the original profile (see Section 6).

While we have focused here on school choice, the Gale-Shapley algorithm is the cornerstone of rules that have been used in many extended models. A general model of matching with contracts (Hatfield and Milgrom (2005)) subsumes models of college admissions (Gale and Shapley, 1962), school choice, and the labour market auction model (Kelso and Crawford, 1982), among others. In this model, there is a set of workers N, a set of firms Z, and a set of contract terms T. Alternatives for a worker $w \in N$ can be represented as a (common) set of contracts $X = Z \times T$. Preferences for worker $w \in N$ are given by a total order R_w over X (which implies they are strict). Each firm $f \in Z$ has a capacity q_f. Preferences for firm $f \in Z$ is assumed to be represented by a choice function over $N \times T$ satisfying substitutability, 'strictness', and the law of aggregate demand. Since firms and terms are finite, the complete domain of strict preferences $\mathcal{R}^N \subset \tilde{\mathcal{R}}^N$ is top-rich. An allocation in this model is a set of contracts $X' \subset X$ satisfying some feasibility requirements. The generalised worker-proposing Gale-Shapley rule makes feasible allocations for any specification of preferences for firms and workers. The nature of influence for agents as described above also exists in the extended rules.

3.1.2 Single-good auctions

In a simple single-good auctions model, there is a single seller, who owns m identical copies of an indivisible good, and wishes to exchange these with a larger (but finite) set of buyers N,[6] in return for monetary transfers (payments). For simplicity, we assume that the seller has a

[6]The agents in this model are assumed to be buyers alone; the seller is ignored.

reservation price of zero. For buyers in N, alternatives take the common form $X = \delta \times \mathbb{R}_+$, the first component $\delta = \{0, 1\}$ a binary variable that takes the value 1 if the good is received by the buyer and 0 otherwise, and the second component denotes the corresponding payment. A feasible allocation in this setting is a vector $(x, p) \in \delta^N \times \mathbb{R}_+^N$, with $\sum_{b \in N} x_b = m$ and $p_b \geq 0$ for all $b \in N$ (non-negative payments). The set of all feasible allocations is denoted $\mathcal{A}$.

Each buyer $b \in N$ is assumed to have a valuation $v_b \in \mathbb{R}_+$ representing the maximum he or she is willing to pay to obtain a copy of the good. Additional assumptions are that obtaining the good is preferable to not for the same level of payment, and preferences are quasi-linear in payment. Under these assumptions, a valuation v_b can be equivalently represented by preferences R_i over X in the following way: The alternative $(1, 0)$ is maximal, which represents obtaining the good at zero payment. Paying less is better, i.e., for any $x_b \in \delta$ and $p_b, q_b \in \mathbb{R}_+$, $(x_b, p_b) \; R_b \; (x_b, q_b) \iff p_b \leq q_b$. Moreover, the buyer is indifferent between receiving the object and paying her valuation and not receiving the object and paying nothing, i.e., $(1, v_b) \; I_b \; (0, 0)$. The set of all preferences for buyer b is given by $\mathcal{R}^b$, and each $R_b \in \mathcal{R}^b$ is associated with a valuation $v_b \in \mathbb{R}_+$.

The seller conducts an auction. Each buyer $b \in N$ submits a bid $c_b \in \mathbb{R}_+$. A bid profile is denoted $c \in \mathbb{R}_+^N$. A particular auction rule is the Vickrey rule $V : \mathbb{R}_+^N \to \mathcal{A}$ (Vickrey, 1961), which produces a feasible allocation for every bid profile $c \in \mathbb{R}_+^N$ as follows: Assume an exogenous and fixed strict ordering $\succ$ over N. Arrange the bids in c in descending order, breaking any ties according to $\succ$. That is, for a given c, write N as $(\tau_1(c), \tau_2(c), \ldots, \tau_N(c))$ with $\tau_1(c) \in N$ the buyer with the highest bid in c, and subsequent buyers ranked by reducing bids, such that when multiple buyers have the same bid, the order for them is determined by the tie-breaker $\succ$ with the highest-ranked buyer first. Define $W(c) = \{\tau_1(c), \ldots, \tau_m(c)\}$ and $L(c) = N \setminus W(c)$. Then the allocation prescribed by the Vickrey rule $V(c) = (x(c), p(c))$, where $x_b(c) = 1 \iff b \in W(c)$, and winning agents pay an amount equal to the $(m+1)^{th}$ bid, i.e., $p_b(c) = c_{\tau_{m+1}(c)}$ for all $b \in W(c)$ and $p_b(c) = 0$ for all $b \in L(c)$. We call $\tau_{m+1}(c)$ the *price-setting buyer* at the bid profile c.

Assume that a profile of bids c for buyers is given. Note that, for a non-winner $b \in L(c)$, a monotonic transformation of her assignment $(0,0)$ is equivalent to a reduction in her valuation of the object, i.e., for $b \in L(c)$, $R'_b \in \mathcal{M}(\mathcal{R}^b, R_b, V_b(c)) \iff v'_b < v_b$. For a winning buyer $b \in W(c)$, a monotonic transformation of her assignment $(1, p_b(c))$ involves an increase in her valuation of the object, i.e., for $b \in W(c)$, $R'_b \in \mathcal{M}(\mathcal{R}^b, R_b, V_b(c)) \iff v'_b > v_b$.

It is well known that it is a weakly dominant strategy for buyers to bid their true valuations under the Vickrey rule (Vickrey, 1961). Using this observation, for a given bid profile c, we note that buyers in $W(c)$ do not have influence with any other buyer in N at c. This is because, for any monotonic transformation (i.e., increase in their valuation and therefore bid), the set of winning agents as well as the winning price remains the same. Similarly, any monotonic transformation for a losing buyer (i.e., a decrease in the valuation and therefore bid) does not affect the set of winning agents or the winning price if she is not the price-setting buyer. In fact, the only buyer who has influence is the price-setting buyer, since her bid determines the price of the good. To see this, note that a monotonic transformation of her preferences is a lowered valuation and therefore bid, or $v'_{\tau_{m+1}(c)} < v_{\tau_{m+1}(c)}$. But then the set of winners remains the same, and only the winning payment reduces. Thus this agent has influence with the winning buyers, and the losing buyers are all independent of any agent's influence. Moreover, the following fact becomes readily apparent:

Fact **3.** For any instance of influence under the Vickrey rule, the change in welfare for influenced agents is positive.

Fact 3 is also sensitive to the notion of influence being defined in terms of monotonic transformations. The price-setting agent $\tau_{m+1}(c)$ can also affect the welfare of winning buyers by raising her bid. However, as argued above, this is not a monotonic transformation of her assignment, and so it falls outside our purview of influence. Another remark is that the influence structure could change if the price-setting agent reduces her new bid sufficiently: she may no longer be the price-setting agent for the new profile of bids, and a new price-setting agent could emerge in the new bid profile.

3.1.3 Object Reallocation with Indifferences

In models of object reallocation, there is a common set of objects Z, and each object in Z is initially owned by some agent in N. For simplicity, we assume that each agent initially owns at most one object, and restrict our attention to the pure object-allocation setting. Agents' preferences over objects in X are initially assumed to be strict (we relax this later). An allocation consists of a specification of at most one object for each agent.

The Top Trading Cycles (TTC) algorithm (Shapley and Scarf, 1974) works as follows: Given a profile of preferences R, each agent initially 'points' to her most-preferred object, and each object points to its owner. Since the set of agents is finite, there will always be at least one cycle formed (for instance, an agent pointing to the object she owns is a cycle.) Agents in a cycle are assigned their most-preferred object and removed from the market. The same is done for all cycles that might arise. The procedure repeats with remaining agents and objects: agents point to their most-preferred object from the remaining ones, remaining objects point to their owners, and agents in a cycle are assigned their most-preferred remaining object and removed from the market with their assignments. This continues until there are no more agents remaining. It has been shown that this procedure determines a unique allocation (the 'core') for every profile of preferences. The TTC rule associates with each profile the corresponding core allocation found by the TTC algorithm.

Extended models allow for the possibility that agents initially own multiple objects (Pápai, 2000), allow for partial ownership rights (e.g., Sönmez and Ünver (2010)), allow for monetary transfers (Miyagawa, 2001), allow for agents to be assigned multiple objects (Pápai, 2001; Hatfield, 2009) and allow for objects to be assigned to multiple agents (Rhee, 2011; Raghavan, 2017). In the domain of strict preferences, such TTC-based rules are non-bossy, individually rational, strategy-proof and Pareto-efficient (Pápai, 2000; Ma, 1994). Thus no agent has influence.

However, once we introduce indifferences in agents' preferences, no rule is strategy-proof, Pareto-efficient, individually rational and non-bossy (Bogomolnaia, Deb, and Ehlers, 2005; Jaramillo and Manjunath, 2012). In such environments, top trading absorbing sets (TTAS) rules (Alcalde-Unzu and Molis, 2011) and top cycles rules with priority (TCP) (Jaramillo and Manjunath, 2012) are strategy-proof, Pareto-efficient and individually rational. A general formulation that contains these rules is the class of generalised absorbing top-trading-cycles (GATTC) rules (Aziz and Keijzer, 2011). Not all GATTC rules are strategy-proof. But they are all Pareto-efficient and individually rational. Thus, given the impossibility results just stated, strategy-proof GATTC rules are bossy. Henceforth, we restrict our attention to strategy-proof GATTC rules. We illustrate influence under GATTC rules with an example.

Example **2.** [Jaramillo and Manjunath (2012)] Let $N = \{1, 2, 3\}$, $Z = \{a, b, c\}$. Let the endowment be as follows: $\omega = (a, b, c)$, i.e., agent 1 initially owns a, agent 2 initially owns b and agent 3 initially owns c. Let the exogenous ordering be $1 \prec 2 \prec 3$. Then, for profiles given in Figure 2, the allocations prescribed by the TCP rule are given in boxes.

Figure 2: Example 2

R_1	R_2	R_3	R'_1	R_2	R_3
$a\ b\ \boxed{c}$	$\boxed{a}$	a	$\boxed{c}$	a	$\boxed{a}$
	b	$\boxed{b}$	$a\ b$	$\boxed{b}$	b
	c	c		c	c

It is evident from the example that agent 1 has influence with agents 2, 3 at R via R'_1. However, as we argue in Section 6, this influence is quite restricted, and occurs only in special circumstances. Moreover, we note the following facts.

Fact **4.** For any instance of influence under GATTC rules, the changes in welfare for influenced agents are in opposite directions, in that the welfare effect is positive for some agent and negative for at least one other.

Fact **5.** For any instance where an agent is positively-influenced under GATTC rules, the new object she receives was previously the assignment of some other agent.

4 Properties of Allocation Rules

In this section, we present several normatively attractive and widely-studied properties of allocation rules. We start with notions of strategy. Since preferences are private information, an allocation rule can take as input only those preferences that are reported to it by agents. This could be thought of as inducing a 'revelation game' associated with the rule, in which agents might strategise over which preferences to report. In particular, an agent who finds that reporting false preferences leads to an assignment for her that she prefers to the one she would have received by telling the truth, might be encouraged to 'manipulate' the rule in this way. This leads to the requirement of 'strategy-proofness', or non-manipulability, of an allocation rule: truth-telling must always be a weakly dominant strategy for each agent in the preference revelation game associated with the rule.

Definition **5.** A rule f defined on $\mathcal{R}^N$ is *manipulable at* $R \in \mathcal{R}^N$ *by* $i \in N$ if there exists $R_i' \in \mathcal{R}^i$ such that $f_i(R_i', R_{-i})\ P_i\ f_i(R)$. A rule f is strategy-proof if it is not manipulable at any $R \in \mathcal{R}^N$ by any $i \in N$.

Strategy-proofness is limited to precluding manipulations on the part of individual agents. However, we might still worry about the possibility of joint manipulations by several agents. This is especially a concern in small economies, where agents have more opportunities to jointly plan their strategies. Thus we might wish to extend the non-manipulability of a rule to groups of agents as well. Several extensions to groups have been proposed (e.g., transfer-proofness, bribe-proofness, etc. See Thomson (2013) for a survey.) For our present purposes, we limit our attention to two of them. 'Strong' group-strategy-proofness precludes cases where a group of agents that attempt to manipulate the rule remain as well off, with at least one of them becoming better off.

Definition **6.** A rule f defined on a domain $\mathcal{R}^N$ is *weakly manipulable at* $R \in \mathcal{R}^N$ *by* $C \subseteq N$ if there exists $R_C' \in \mathcal{R}^C$ such that $f_i(R_C', R_{-C})\ R_i\ f_i(R)$ for all $i \in C$ with $f_j(R_C', R_{-C})\ P_j\ f_j(R)$ for some $j \in C$. A rule f is strongly group-strategy-proof if it is not weakly manipulable by any $C \subseteq N$ at any $R \in \mathcal{R}^N$.

A less restrictive version of group-strategy-proofness only precludes cases where agents in a group reporting false preferences are *all* better off as a result.

DEFINITION **7.** A rule f defined on $\mathcal{R}^N$ is *strongly manipulable at* $R \in \mathcal{R}^N$ *by* $C \subseteq N$ if there exists $R'_C \in \mathcal{R}^C$ such that $f_i(R'_C, R_{-C})\ P_i\ f_i(R)$ for all $i \in C$. A rule f is weakly group-strategy-proof if it is not strongly manipulable by any $C \subseteq N$ at any $R \in \mathcal{R}^N$.

The following equivalence is well-known.

LEMMA **1** (Pápai (2000))**.** *An allocation rule f defined on a top-rich*[7] *domain $\mathcal{R}^N$ is strongly group-strategy-proof if and only if it is strategy-proof and non-bossy under monotonic transformations.*

Though originally stated for non-bossiness in general, a closer examination of the proof reveals that non-bossiness under monotonic transformations is the critical factor.

Another commonly sought-after attribute of an allocation rule is 'welfare monotonicity', which in general represents the notion that when some agents reduce their claims on desirable alternatives, any change in their welfare should be non-negative. To illustrate this idea, imagine that agents report preferences and the allocation rule specifies an allocation. Now, some agents instead consider reporting preferences that drop their claims on some alternatives that they considered at least as good as their assignments, but did not obtain: more precisely, they would report monotonic transformations of their current assignments. These preferences, along with the original preferences for other agents, would form a new profile, for which the rule would specify a new allocation. Welfare monotonicity refers to restrictions on the changes in welfare that might occur across these two allocations: these changes must be non-negative. Different versions of the condition are used in different contexts, depending on whether the change in welfare is required to be exactly zero ('welfare-invariance'), or merely non-negative, and whether the condition pertains to some or all agents. For instance, a strong requirement would be to insist on welfare-invariance for all agents who change their preferences across the two profiles:

[7] While the original proof used a top-rich domain, the argument can easily be extended to a rich domain.

Definition 8. A rule f defined on $\mathcal{R}^N$ satisfies Maskin welfare-invariance if, for any $C \subseteq N$ and any $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$: $f_i(R'_C, R_{-C})\ I'_i\ f_i(R)$ for all $i \in C$.

Observe that the definition above differs from the classical definition (Dasgupta, Hammond, and Maskin, 1979; Maskin, 1999). For one, it relaxes the original requirement of *assignment-invariance* to a more permissive *welfare-invariance*: the specific assignments could change, provided they change in a welfare-neutral way.[8] Secondly, the original definition pertains only to monotonic transformations for single agents. However, it is easy to see that a repeated application of the property can extend to sets of agents effecting monotonic transformations as well. Moreover, the original definition requires welfare-invariance for *all* agents; our choice of formulation requires it only for agents who change their preferences. Finally, the welfare-invariance is with respect to the changed preferences, rather than the original. It is well-known that the non-bossiness of a strategy-proof rule is sufficient for it to be Maskin welfare-invariant. A version of this implication for strict preferences (and for all agents) is proved in Svensson (1999), but the argument can easily be extended to subsets of agents and indifferences. We state this below without proof.

Lemma 2. *A strategy-proof allocation rule that is non-bossy under monotonic transformations is Maskin welfare-invariant.*

A weaker notion of welfare monotonicity requires instead that the welfare change be non-negative for all agents changing their preferences, though not necessarily invariant.

Definition 9. A rule f defined on $\mathcal{R}^N$ satisfies weak Maskin monotonicity if, for any $C \subseteq N$ and any $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$: $f_i(R'_C, R_{-C})\ R'_i\ f_i(R)$ for all $i \in C$.

This more permissive property is satisfied by a larger class of rules, including the Vickrey rule for buyers, and a version for all agents is one of the properties that characterises the Gale-Shapley rule (Kojima and Manea, 2010). However, it is still too restrictive for some other desirable rules, such as the extended top-trading cycles rules for models of object-allocation with indifferences. This is discussed in more detail in Section 6. In our quest to

[8] Of course, this is only relevant if there are indifferences in preferences.

find a weaker version of monotonicity that is satisfied by as many rules as possible while retaining the spirit of the original property, we propose a version requiring welfare-invariance for *at least one* of the agents changing their preferences.

DEFINITION **10.** A rule f defined on $\mathcal{R}^N$ satisfies minimal Maskin welfare-invariance if, for any $C \subseteq N$, and any $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$, there exists an $i \in C$ such that $f_i(R'_C, R_{-C})\ I'_i\ f_i(R)$.

Minimal Maskin welfare-invariance is silent on the change in welfare of agents other than i in the definition: it could be positive or negative. Arguably, it is the minimal formulation of monotonicity that is consistent with Maskin welfare-invariance. It should be pointed out that, formulated in this way, weak Maskin monotonicity and minimal Maskin welfare-invariance are independent properties. Weak Maskin monotonicity requires non-negative welfare changes for all agents changing their preferences, but does not require any of the changes to be zero. On the other hand, minimal Maskin welfare-invariance requires some changing agent's welfare change to be zero, but does not insist on the welfare changes to be non-negative for the others. However, for a weakly group-strategy-proof rule, the latter condition is implied by the former.

LEMMA **3.** *For a weakly group-strategy-proof rule, weak Maskin monotonicity implies minimal Maskin welfare-invariance.*

Proof: Let f be a rule defined on $\mathcal{R}^N$ that is weakly group-strategy-proof and weakly Maskin monotonic. Take $R \in \mathcal{R}^N$ and for $M \subseteq N$, let $R'_M \in \mathcal{M}(\mathcal{R}^M, R_M, f_M(R))$. Weak Maskin monotonicity implies $f_i(R'_M, R_{-M})\ R'_i\ f_i(R)$ for all $i \in M$. It cannot be that $f_i(R'_M, R_{-M})\ P'_i\ f_i(R)$ for all $i \in M$, because then $f_i(R'_M, R_{-M})\ P_i\ f_i(R)$ for all $i \in M$, violating weak group-strategy-proofness. Thus $f_i(R'_M, R_{-M})\ I'_i\ f_i(R)$ for some $i \in M$, and f is minimally Maskin welfare-invariant. ■

The notion of minimal Maskin welfare-invariance proves useful in two ways. We show how it is a common denominator for a wide set of allocation rules, including those that do not satisfy weak Maskin monotonicity as defined above. We also relate this property in a close way to weak group-strategy-proofness (see Section 5).

We now turn to the notion of efficiency of an allocation rule. There are many ways to formulate this property. For instance, we could consider utilitarian-efficiency, in which an allocation maximises the sum of agents' utilities. However, such a definition would not cover applications with only ordinal preferences. Since we seek a general understanding of a wide class of allocation rules, we limit ourselves to the weaker notion of 'Pareto-efficiency'.

Definition **11.** A rule f defined on $\mathcal{R}^N$ is Pareto-efficient if, for all $R \in \mathcal{R}^N$, there is no feasible allocation $a \in \mathcal{A}$ such that $a_i \ R_i \ f_i(R)$ for all $i \in N$ and $a_j \ P_j \ f_j(R)$ for some $j \in N$.

The Gale-Shapley rule is not Pareto-efficient for the proposing side of the market. In fact, defined this way, the Vickrey rule is not Pareto-efficient for buyers.[9] In top-rich domains, we might want Pareto-efficiency at least in situations of minimal conflict. That is, we might require the rule to be *unanimous*, in that it assigns each agent one of her maximal alternatives when it is collectively feasible to do so. For $R \in \mathcal{R}^N$, define $\mathcal{A}(R) = \{a \in A \mid a_i \in \mathit{top}(R_i, X)$ for all $i \in N\}$.

Definition **12.** A rule f defined on a top-rich domain $\mathcal{R}^N$ is unanimous if, for all $R \in \mathcal{R}^N$: $\mathcal{A}(R) \neq \emptyset \implies f(R) \in \mathcal{A}(R)$.

When preferences are strict, this definition becomes the standard version of unanimity (e.g. Raghavan (2017)). Another weak version of Pareto-efficiency is 'non-wastefulness' of resources. We propose this definition for an object-allocation setting (i.e., Z is common and finite, and $T^i = \emptyset$ for all $i \in N$; thus $X = Z$.) An allocation rule is said to be *non-wasteful* if there is no agent and a preferred object in that agent's object set, which can be assigned to that agent in a feasible manner, keeping the rest of the allocation fixed.

Definition **13.** Let X be a set of objects, and let f be an allocation rule defined on $\mathcal{R}^N$. Let $R \in \mathcal{R}^N$, with $f(R)$ the corresponding allocation. The rule f is *wasteful* at R if there

[9] A reduction in the winning price benefits all winners without affecting any of the losing buyers. Of course, this does not take into account the seller.

exists $i \in N$ and $x \in X \setminus \{f_j(R)\}_{j\in N}$ such that $x \; P_i \; f_i(R)$ and $(x, f_{-i}(R)) \in \mathcal{A}$. A rule is non-wasteful if it is not wasteful at R for any $R \in \mathcal{R}^N$.

Non-wastefulness defined this way is appropriate in situations where resources might be physical objects, or where monetary transfers are not allowed, such as some models of object-allocation or matching. Yet we could have also defined it so it has meaning in auction settings: the interpretation would be that none of the objects to be auctioned should remain unsold if there is a buyer who is willing to purchase it, and it is feasible for her to do so, given the rest of the allocation. However, such a definition involves separating resources and terms in the definition, and for simplicity we do not use a more general version here.

The final type of property we consider here is 'stability', usually formulated for matching models but applicable more generally as well. For exposition, we present a highly simplified version of this property. Consider the school-choice problem without contracts described in Section 3. Let f be an allocation rule for this problem. Given a capacity profile q and a priority profile $\succ$ for X, a profile of preferences $R \in \mathcal{R}^N$ for agents, and the corresponding allocation $f(R)$, a 'blocking pair' for this allocation consists of an unmatched agent-object pair that rank each other higher in their preferences and priorities than their current match. That is, a pair (i, x) with $i \in N$ and $x \in X$ is called an *blocking pair in* $f(R)$ if $x \; P_i \; f_i(R)$ and there exists $j \in N$ with $f_j(R) = x$ such that $i \succ_x j$. A rule is said to be *stable* if for every profile it is non-wasteful and there are no blocking pairs.

Definition **14.** Let capacities q and priorities $\succeq$ for X be given. A rule f defined on $\mathcal{R}^N$ is stable if, for any $R \in \mathcal{R}^N$, $f(R)$ is non-wasteful and contains no blocking pair.

We have defined stability as applicable to school-choice models without contracts. However, it is possible to define stability more generally.

5 Conditions on Influence

As illustrated by the examples above, the influence structure varies across different allocation rules. Even for the same rule, it is possible that some agents have influence with others at some profiles, but be influenced by those selfsame agents at other profiles. For instance, it is easy to construct bid profiles for the Vickrey rule where some agent i is a winner and some agent j is the price-setting agent, and another where agent j is a winner and agent i is the price-setting agent. In what follows, we attempt to organise the structure of influence for allocation rules under a general framework. As we will see, many of these rules share some commonality in how influence manifests under them. Moreover, some simple characteristics of the nature of influence go a long way towards explaining why different rules satisfy (or fail to satisfy) the different properties presented in Section 4.

Our first condition is a structural requirement on the influence relation: that of acyclicity. A binary relation is called acyclic if it contains no cycles. Formally, given a rule f and a profile $R \in \mathcal{R}^N$, we say the influence relation b^f_R is *acyclic* if, for any integer $2 \leq k \leq |N|$, and $i_1, \ldots, i_k \in N$ such that $(i_j, i_{j+1}) \in b^f_R$ for all $j \in \{1, \ldots, k-1\}$, we have that $(i_k, i_1) \notin b^f_R$. The notion of acyclicity of an influence relation extends naturally to an allocation rule:

Definition 15. An allocation rule f defined on $\mathcal{R}^N$ satisfies *acyclicity of the induced influence relation* (acyclicity) if, for all $R \in \mathcal{R}^N$, the influence relation b^f_R is acyclic.

The idea of acyclicity of a relation has been used in many contexts. Satterthwaite and Sonnenschein (1981) define an 'affects relation' based on whether an agent can affect the assignment of another agent or not (with or without assignment-invariance on her part), and show that non-bossiness and a weak richness condition imply that the affects relation is acyclic. Further afield, in a Bayesian model of information revelation (Hagenbach, Koessler, and Perez-Richet, 2014), the acyclicity of the so-called 'masquerade relation' plays an essential role. It is arguably a minimal 'rationality' condition on a binary relation.

An acyclic influence relation induces a partial order on N. That is, we can find an ordered partition of any subset of N into 'influenced equivalence classes', where all agents in the same cell of the partition are independent of each other's influence, and where agents are in different cells of the partition only if there is an 'influence chain' of agents that leads from one to the other. This is similar to the partition of a set of resources into 'indifference classes' by a preference relation defined on the set, where resources are in the same indifference class if they do not dominate each other in preference terms, and resources are in different indifference classes only if there is a preference 'dominance-chain' of resources in the classes that separate them. We formalise this idea in the following lemma:

LEMMA **4.** *Let f be a rule defined on $\mathcal{R}^N$, let $R \in \mathcal{R}^N$. For any $C \subseteq N$, an acyclic influence relation b^f_R induces a unique partition $\{C_1(R), \ldots, C_K(R)\}$ of C such that elements of the partition can be ordered by influence, i.e.:*

1. *$C_1(R) = \{i \in C \mid (j, i) \notin b^f_R$ for all $j \in C\}$.*
2. *For any $k \geq 2$, if we recursively define $C^{k-1} = C \setminus \bigcup_{j<k} C_j(R)$, then, $C_k(R) = \{i \in C^{k-1} \mid (j, i) \notin b^f_R$ for all $j \in C^{k-1}\}$.*
3. *$K \equiv min(k \mid C^k = \emptyset)$.*

The proof involves a simple constructive exercise, and is relegated to the appendix. A consequence of Lemma 4 is that if the induced influence relation at a profile R is acyclic, we can identify for any subset $C \subseteq N$ at least one agent in C who is independent of the influence of other agents in C at R, and at least one agent in C who cannot influence any other agent in C at R. This allows us to formulate a useful way of moving across profiles related by monotonic transformations. We define an *influence-respecting transform*, which is an algorithm that will be used often in what follows. Formally:

DEFINITION **16.** Let f be an acyclic rule defined on $\mathcal{R}^N$, let $R \in \mathcal{R}^N$ be a profile, let $C \subseteq N$, and let $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$. An influence-respecting transform $R \to (R'_C, R_{-C})$ is defined as:

Step 1: Let $C^1 = C$ and $R^1 = R$. Let the ordered partition of C^1 be $\{C_1(R^1), \ldots, C_{K^1}(R^1)\}$ (Lemma 4). Take any $i \in C_{K^1}(R^1)$ and define $R^2 = (R'_i, R^1_{-i})$. Define $C^2 = C^1 \setminus \{i\}$. If $C^2 = \emptyset$, stop. Otherwise, go to Step 2.

Step $k \geq 2$: Let R^k and C^k be as defined in Step $k-1$. By Lemma 4, we determine the ordered partition $\{C_1(R^k), \ldots, C_{K^k}(R^k)\}$ of C^k. Take any $i_k \in C_{K^k}(R^k)$, and define

$R^{k+1} = (R'_{i_k}, R^k_{-i_k})$, and $C^{k+1} = C^k \setminus \{i_k\}$. If $C^{k+1} = \emptyset$, we stop. Otherwise, we go to Step $k+1$.

This procedure is well-defined and terminates in exactly $|C|$ steps. At each step, an agent from the last cell of the induced partition on remaining agents is selected, and her preferences are changed to the required monotonic transformation. At each step, acyclicity ensures that the last cell is non-empty. We repeat this process one by one until we have done so for all agents in C. It is easy to see that the final profile $R^{|C|+1} = (R'_C, R_{-C})$. The procedure is not unique: it could differ in the selection of the agent in the last cell of the partition at any step. However, we show that, for any version of this procedure, we have welfare-invariance for some agent in C. In particular:

Proposition 1. *If a rule f defined on an arbitrary domain is strategy-proof and acyclic, then f satisfies minimal Maskin welfare-invariance.*

The proof is in the appendix, but we briefly describe the intuition here. For any $C \subseteq N$ and any $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$, acyclicity allows us to specify an influence-respecting transform $R \to (R'_C, R_{-C})$. At each step, the agent selected is from the last cell of the ordered partition of remaining agents. Strategy-proofness ensures this agent's welfare is unchanged when she effects her corresponding preference change, and since this is a monotonic transformation of her assignment, the properties of the last cell of the partition ensure that no remaining agent's welfare changes. In particular, after the penultimate step of the influence-respecting transform, the sole remaining agent's welfare is unchanged from the original profile R, and strategy-proofness ensures that her welfare is unchanged when her preferences are changed in the last step to give us (R'_C, R_{-C}). This is enough to satisfy minimal Maskin welfare-invariance.

At each step of the influence-respecting transform, the selected agent does not influence any of the remaining agents. However, this says nothing about any influence this agent might have on agents that have been selected in previous steps. These agents could be influenced by the selected agent, and moreover, any welfare changes for the influenced agents could be positive or negative. However, as noted in Fact 1 and Fact 3, the Gale-Shapley rule and the

Vickrey rule always produce positive welfare effects as a result of influence under monotonic transformations. We formalise this notion in the following definition:

Definition **17.** A rule f defined on $\mathcal{R}^N$ satisfies *positive welfare effects of influence* (positivity) if, for any $R \in \mathcal{R}^N$, and any $i, j \in N$, if i has influence with j at R via some R'_i, then $f_j(R'_i, R_{-i})\ P_j\ f_j(R)$.

There is a close connection between positivity and weak Maskin monotonicity.

Theorem **1.** *A rule f defined on $\mathcal{R}^N$ that is strategy-proof and acyclic is weakly Maskin monotonic if and only if it is positive.*

The proof is in the appendix. Essentially, we show that the arguments of Proposition 1 and positivity together imply that all agents are at least as well off at the end of the influence-respecting transform, which is weak Maskin monotonicity. For the converse, if we can find an instance where positivity is violated, we can define a suitable monotonic transformation under which some agent is worse off, violating weak Maskin monotonicity.

As discussed previously, different agents could have influence and different agents could be independent at different profiles. However, in some cases, an independent agent in a set might remain independent in that set even after certain preferences changes. In particular, we propose a condition we call 'preservation', which requires that if each agent in some set C is at least as well off in welfare terms when they all report different preferences, then any agent in C that is independent of the influence of any other agent in C remains independent in the new profile as well. Formally:

Definition **18.** For an allocation rule f defined on $\mathcal{R}^N$, f satisfies *preservation of independent agents under welfare improvements* (preservation) if, for any $C \subseteq N$, any $R \in \mathcal{R}^N$, and any $R'_C \in \mathcal{R}^C$ such that $f_i(R'_C, R_{-C})\ R_i\ f_i(R)$ for all $i \in C$, defining $R' \equiv (R'_C, R_{-C})$, we have that $j \in I^f_R(C) \implies j \in I^f_{R'}(C)$ for all $j \in C$.

Put differently, preservation says that the independence of agents in C cannot vanish under welfare improvements, even under non-monotonic transformations. In particular, if there is an agent in C such that no other agent in C had influence with her at the original profile R, then no agent in C can have influence with her in the new profile R' if all agents in C are all at least as well off in R' as in R. We show in Section 6 that preservation is not satisfied by the Vickrey rule. That is, we show that there are instances where an independent agent loses her independence even though all agents are at least as well off after some preference change. Preservation also has implications for weak group-strategy-proofness. Our next result is that for an allocation rule that is defined on a rich domain and that satisfies acyclicity and either preservation or positivity, strategy-proofness is equivalent to weak group-strategy-proofness.

THEOREM **2.** *A rule defined on a rich domain that satisfies acyclicity, and either preservation or positivity, is weakly group-strategy-proof if and only if it is strategy-proof.*

The proof is in the appendix. To give a flavour, note that if f is not weakly group-strategy-proof on $\mathcal{R}^N$, there is an instance of a profile R, a set of agents $C \subseteq N$, and preferences $R'_C \in \mathcal{R}^C$ such that each agent in C is better off in (R'_C, R_{-C}) than in R according to preferences in R. Setting $R' = (R'_C, R_{-C})$, we suitably construct a profile $R'' = (R''_C, R_{-C})$ that is a monotonic transformation for C of $f_C(R)$ at R_C as well as of $f_C(R')$ at R'_C, in a manner that is allowed by richness of the domain. Acyclicity and strategy-proofness ensure the minimal Maskin welfare-invariance of f (Proposition 1), thus there is some set of agents who are welfare-neutral for each of the influence-respecting transforms $R \to R''$ and $R' \to R''$. If f satisfies either preservation or positivity, we show that the respective sets of welfare-neutral agents for each transform have a non-empty intersection, and thus there is some agent in C whose welfare is the same in R and R'. But since we had assumed that all agents in C are better off in R' than in R, this yields the desired contradiction. This also establishes the close connection between minimal Maskin welfare-invariance and weak group-strategy-proofness.

Theorem 2 is similar to the equivalence result in Barberà, Berga, and Moreno (2016). Though the richness condition in both results is the same, the properties of allocation rules used to derive the same equivalence in that exercise (called joint-monotonicity and respectfulness) are different. Their properties are in fact independent of acyclicity and preservation (demonstrated by Example 8 and Example 9 in Appendix B), though positivity is closely linked to joint-monotonicity (Theorem 1). However, though the two approaches are independent, it is worth pointing out two improvements resulting from our approach. Firstly, our conditions allow us to cover not just all the rules considered in Barberà, Berga, and Moreno (2016), but also the GATTC rules discussed in Section 3, which do not satisfy joint-monotonicity (see Example 10 in Appendix B). Secondly, our result is more foundational, in that we express the equivalence result solely in terms of the influence characteristics of an allocation rule. We also demonstrate via examples that the conditions in Theorem 2 are independent (see Example 3 to Example 7 in Appendix B).

The two properties of preservation and positivity in Theorem 2 explain the equivalence of strategy-proofness and weak group-strategy-proofness for different rules. We show in Section 6 that positivity is satisfied by the Gale-Shapley rule and the Vickrey rule, but is not satisfied by GATTC rules. On the other hand, preservation is satisfied by all the rules except the Vickrey rule.

We now turn to efficiency. A moment of reflection reveals that a positive rule cannot be Pareto-efficient. This is because, for any instance of influence, if all influenced agents are better off (with independent agents retaining the same assignment), then this new allocation is feasible, but Pareto-dominates the original allocation, which then could not have been Pareto efficient. Among our motivating examples, the Vickrey rule and the Gale-Shapley are positive for buyers and proposers, respectively, and it comes as no surprise that they are not Pareto-efficient. However, GATTC rules are Pareto-efficient for agents. Perhaps this is linked to Fact 4, which notes that the welfare effects of influence under these rules are in opposite directions? We formalise this idea below.

Definition **19.** A rule f defined on $\mathcal{R}^N$ satisfies *opposite welfare effects of influence* (oppositeness) if, for any $R \in \mathcal{R}^N$, any $i \in N$, and any $C \subset N$ such that i has influence with all agents in C at R for some R'_i: if there exists $k \in C$ such that $f_k(R'_i, R_{-i})\ P_k\ f_k(R)$, then there exists $k' \in C$ such that $f_{k'}(R)\ P_{k'}\ f_{k'}(R'_i, R_{-i})$ and vice versa.

It is clear that no allocation rule that admits at least one non-trivial instance of influence can be both positive and opposite at the same time.[10] We cement the usefulness of oppositeness by establishing the close link with Pareto-efficiency.

THEOREM **3.** *A rule defined on a top-rich domain and that is strategy-proof, acyclic and unanimous is Pareto-efficient if and only if it is opposite.*

The proof is in the appendix. For a brief flavour, note that in one direction it easily follows that if the rule is not opposite, then the corresponding instance of influence results in a welfare-improvement for some agent without hurting any other, violating Pareto-efficiency. For the other direction, if a rule is not Pareto-efficient, then there exists some feasible Pareto-improvement to the allocation at some profile. Invoking a suitable profile from top-richness, we use influence-respecting transforms to establish a conflict between oppositeness and unanimity. This result also gives us the following incompatibility.

COROLLARY **1.** *No allocation rule defined on a top-rich domain, that has a non-empty influence relation at some profile, and that is strategy-proof, unanimous and acyclic, can be both weakly Maskin monotonic and Pareto-efficient.*

For everything that follows, we restrict our attention to pure object-allocation settings (i.e. Z is common and finite, and $T^i = \emptyset$ for all $i \in N$). Recall that Fact 2 notes that, under the Gale-Shapley rule, any feasible Pareto-improvement to the prescribed allocation at a profile involves agents trading their assignments along a cycle. In particular, this means that when an agent is positively-influenced, her newly assigned object was assigned to some other agent previously. We formalise this notion. We say that an allocation rule satisfies *displacement of assigned objects under influence* if, at any instance where an agent i has a positive influence with agent j, the changed assignment for j involves an object that was previously assigned to some other agent k at the original profile. Formally:

DEFINITION **20.** Let X be a set of objects. A rule f defined on $\mathcal{R}^N$ satisfies *displacement of assigned objects under positive influence* (displacement) if, for any $R \in \mathcal{R}^N$ and any $i, j \in N$

[10] Non-bossy allocation rules satisfy both properties, though vacuously. A rule might also satisfy neither.

such that i has positive influence with j at R via some R_i', there exists $k \in N$ such that $f_j(R_i', R_{-i}) = f_k(R)$.

The implication of course is that agent i also has influence with k at this profile, since k must be assigned a different object in the new profile. Moreover, if k is also positively-influenced, then her newly assigned object was assigned to some other agent previously, and so on (k could also be negatively-influenced, of course, in which case this implication does not hold). In particular, if all influenced agents are positively-influenced, then there are necessarily cycles of trade that involve them. We first show that displacement is closely associated with non-wastefulness.

Theorem **4.** *Let X be a set of objects. A rule defined on a top-rich domain and that is strategy-proof, acyclic and unanimous, is non-wasteful if it is displacing.*

The proof is in the appendix, and is similar to the proof of Theorem 3. We turn now to stability. For two-sided markets (i.e., where objects' preferences are also taken into account), it is already known that there does not exist any stable rule which is non-bossy on both sides (Kojima, 2010). We first show that for one-sided markets such as the ones we consider, there is no allocation rule that is unanimous, non-bossy, and stable.

Proposition **2.** *No allocation rule is unanimous, non-bossy on N, and stable.*

Proof: See Example 1. Let $q_x = 1$ for all $x \in X$, and let $\succ$ for X be as given. For the profile $R \in \mathcal{R}^N$, there is a unique stable allocation $(i_1 \leftrightarrow x_2, i_2 \leftrightarrow x_3, i_3 \leftrightarrow x_1)$. Unanimity requires the unique allocation $f(R_{i_2}', R_{-i_2}) = (i_1 \leftrightarrow x_1, i_2 \leftrightarrow x_3, i_3 \leftrightarrow x_2)$. However, this means i_2 has influence with i_1 and i_3 at R, and so f is bossy for agents at R. ■

In order to establish the conditions required for stability, we start by proposing a weak condition called top-respectfulness, which requires that, for profiles where multiple agents rank the same object as most-preferred, if the lower-ranked agent according to the object's priority is assigned to that object, then so must the higher-ranked agent. Formally, a rule f is $\succ$-*top-respecting* if, for any $\succ$ for X, any $x \in X$ and $i, j \in N$, and any $R \in \mathcal{R}^N$ such that

$top(R_i, X) = top(R_j, X) = x$, $[i \succ_x j] \implies [f_j(R) = x \implies f_i(R) = x]$. Top-respectfulness only applies to very particular instances. Firstly, the condition only applies to specific profiles, in particular where some agents rank a particular object on top. Secondly, it does not require that any of the agents be assigned that object for that profile; only that if the lower-ranked agent according to the object's priority is assigned to that object, then so must the higher-ranked agent. It is easy to think of examples of rules that are top-respecting but not stable, e.g. the so-called immediate acceptance rule (Doğan and Klaus, 2017). However, under some additional conditions, a $\succ$-top-respecting rule is stable.

Theorem **5.** *Let X be a set of objects. Let capacities q and priorities $\succ$ for X over N be given. A rule f defined on a top-rich domain $\mathcal{R}^N$ that is $\succ$-top-respecting, strategy-proof, unanimous, acyclic, displacing and positive is stable.*

The proof is in the appendix. Essentially, we show that if the rule is not stable, there exists a blocking pair (non-wastefulness is ruled out by Theorem 4). Performing a suitable influence-respecting transform, we establish a contradiction between $\succ$-top-respectfulness and positivity. Moreover, the incompatibility of positivity and oppositeness also illuminates the incompatibility between stability and Pareto-efficiency.

6 Applications

In this section, we return to our motivating examples in Section 3 and use our results in Section 5 to shine fresh light on some well-known results in the literature.

6.1 The Gale-Shapley rule

Proposition **3.** *The domain of strict preferences $\mathcal{R}^N$ is top-rich. The agent-proposing Gale-Shapley rule σ^N defined on $\mathcal{R}^N$ satisfies acyclicity, preservation, positivity, and displacement, but not oppositeness.*

The proof is in the appendix. Top-richness, positivity, displacement and preservation are easy to show. Showing acyclicity requires a little more work, but is also straightforward.

Essentially, we rely on showing that only certain interrupters have influence with other agents at a given profile, and showing that there exists a 'last-step' interrupter with whom no other agent has influence at that profile. Iterating this argument, we can derive acyclicity by moving backwards along the order of interrupters. Thus the arguments of Proposition 3 also shed some light on the working of the EADAM, as discussed in Section 3, since the order in which the EADAM handles interrupters is precisely in the reverse order of the steps in which they become interrupters, which is consistent with the acyclic partial order over agents. We also derive the following well-known properties of the agent-proposing Gale-Shapley rule as corollaries.

PROPOSITION **4.** *The agent-proposing Gale-Shapley rule is weakly group-strategy-proof (Dubins and Freedman, 1981). It is weakly Maskin monotonic and non-wasteful (Kojima and Manea, 2010). It is stable (Gale and Shapley, 1962).*

These results are summarised in Figure 3.

Figure 3: The agent-proposing Gale-Shapley rule

Conditions	**Domain**	**Properties**
(Proposition 3)	(Proposition 3)	
Acyclicity	Top-richness	Weak Maskin monotonicity
✓	✓	✓ (Theorem 1)
Preservation	Richness	Weak group-strategy-proofness
✓	✓	✓ (Theorem 2)
Oppositeness		Pareto-efficiency
✗		✗ (Theorem 3)
Displacement		Non-wastefuless
✓		✓ (Theorem 4)
Positivity		Stability
✓		✓ (Theorem 5)

6.2 The Vickrey rule

We now show which of our conditions the Vickrey rule satisfies. Note that displacement, as currently defined, does not apply because the auction environment includes possible transfers.

Proposition **5.** *The domain of preferences $\mathcal{R}^B$ is rich but not top-rich. The Vickrey rule satisfies acyclicity and positivity, but does not satisfy preservation or oppositeness. It also satisfies unanimity.*

The proof is in the appendix, and is very straightforward. Proposition 5 gives us the following well-known properties of the Vickrey rule as corollaries, and these results are summarised in Figure 4.

Proposition **6.** *The Vickrey rule satisfies weak group strategy-proofness and weak Maskin monotonicity, but not Pareto-efficiency.*

Figure 4: The Vickrey rule

Conditions	**Domain**	**Properties**
(Proposition 5)	(Proposition 5)	
Acyclicity	Top-richness	Weak Maskin monotonicity
✓	✗	✓ (Theorem 1)
Preservation	Richness	Weak group-strategy-proofness
✗	✓	✓ (Theorem 2)
Oppositeness		Pareto-efficiency
✗		✗ (Theorem 3)
Positivity		
✓		

6.3 GATTC rules

We start by showing which of our conditions are satisfied by strategy-proof GATTC rules.

Proposition **7.** *The complete domain of preferences $\mathcal{R}^N$ is top-rich. Strategy-proof GATTC rules satisfy acyclicity, preservation, oppositeness and also displacement, but not positivity. They are also unanimous.*

The proof is in the appendix and is straightforward. As a by-product of the proof, we characterise all instances of influence under strategy-proof GATTC rules. Essentially, an agent can have influence with other agents under a strategy-proof GATTC rule only under specific conditions. These conditions are intimately tied in to the tie-breaker used by the rule to determine which of possibly multiple potential cycles is selected for trade at any given step. In particular, an agent can have influence over other agents only if she belongs to two potential cycles, with one favoured over the other by the tie-breaker, and such that she participates in two different cycles along the way. After she effects a monotonic transformation, the originally favoured cycle is no longer available, but the disfavoured cycle is, and consequently realises, giving this agent the same assignment as before. This affects the assignments of some of the agents who belong to one or other of these cycles, and results in influence.

We believe this insight might be useful in potential characterisations of all Pareto-efficient, strategy-proof, and individually rational rules in environments with indifferences, which is as yet an open question. Moreover, it allows us to derive the following properties of strategy-proof GATTC rules as corollaries. The results are summarised in Figure 5.

Proposition **8.** *Strategy-proof GATTC rules are Pareto-efficient (Aziz and Keijzer, 2011), weakly group-strategy-proof, and non-wasteful. They are not weakly Maskin monotonic.*

7 Discussion

In this paper, we have proposed an alternative analytical toolkit to further our understanding of private-good allocation rules. This toolkit is based on the influence that agents might have over each other's welfare under an allocation rule. As we have seen, this allows us to classify and illuminate different bossy rules based on which conditions they satisfy.

Figure 5: Strategy-proof GATTC rules

Conditions	**Domain**	**Properties**
(Proposition 7)	(Proposition 7)	
Acyclicity	Top-richness	Weak Maskin monotonicity
✓	✓	✗ (Theorem 1)
Preservation	Richness	Weak group-strategy-proofness
✓	✓	✓ (Theorem 2)
Oppositeness		Pareto-efficiency
✓		✓ (Theorem 3)
Displacement		Non-wastefuless
✓		✓ (Theorem 4)
Positivity		
✗		

Our definition of influence involves preferences changes that are monotonic transformations. It might be possible to find allocation rules that are bossy but not bossy under monotonic transformations. Such rules are outside the scope of our present analysis. However, many of our conditions on influence, such as positivity, are meaningful only under monotonic transformations. Moreover, acyclicity of the influence relation, which is central to all our results, is guaranteed for some rules only for monotonic transformations.

Consider for example the Vickrey rule. As discussed, at a given bid profile, a monotonic transformation for a winning agent is an increase in her valuation and thus bid, and for a losing agent is a reduction in her valuation and thus bid. Take an example where there are only two bidders, and they have the same valuation and thus bid, and so the winner is determined purely by the tie-breaking rule. Under monotonic transformations, only the losing bidder has influence with the winning bidder, and thus the influence relation is acyclic under monotonic transformations. But note that the winning bidder has a welfare of zero, since she pays her valuation. A unilateral preference change in which she has a lower valuation and thus bid is not a monotonic transformation, but by doing so she now loses (with the same welfare of zero) but the other agent now wins and has a positive welfare change. Thus the winning bidder can also affect the welfare of the losing bidder at the original bid profile without affecting her own welfare, via a non-monotonic preference change. Thus a more general acyclicity does not hold.

As we have shown, the nature of influence explains several properties of allocation rules. We have focused on a few in this paper, but the literature abounds with desirable properties. Further work might be required to illuminate the connection between influence and these properties. For instance, an open question is the relation between influence and utilitarian efficiency. We have focused in the present paper on Pareto-efficiency, which only requires that the prescribed allocation be undominated in welfare terms. However, in models of cardinal utility, such as single-object auctions, this requirement is too weak. The Vickrey rule is not Pareto-efficient, but it is utilitarian-efficient, in that it allocates the m copies of the good to the agents with the m highest valuations, i.e., those who want it the most. It might be possible to explain utilitarian-efficiency in terms of influence.

We might also formulate influence itself differently. For instance, take the following scenario: Agents report a profile of preferences, and an allocation rule determines a feasible allocation for that profile. Some agent leaves the market with her assignment, and the allocation rule is re-run for remaining agents and alternatives. If the assignments of some agents change when the rule is re-run, then we could say that the leaving agent has influence with the agents whose assignments change. This is a 'consistency'-related interpretation of influence.[11] This is possibly easier to motivate, since it does not require definitions such as monotonic transformations, and instead deals with agents leaving the market with their assignments. However, such an interpretation requires defining various concepts with respect to variable populations, since rules and allocations have to be defined with respect to all combinations of 'remaining' agents and alternatives. Also, in other settings, particularly

[11] The author thanks Battal Doğan for suggesting this interpretation.

with private endowments of objects, we would need to define consistency-influence with a bit more care, as we might not wish for an object to leave the market without the agent who initially owns it leaving as well. Nevertheless, this exercise might be fruitful in many applications, especially in school-choice settings (see, e.g., Doğan and Yenmez (2017)).

Another approach, often used in school-choice settings, is to interpret an agent whose changed preferences rank the 'outside option' on top as having left the market. If this agent were originally assigned the outside option, this translates to a monotonic transformation of her assignment. Then properties such as 'weak population monotonicity' (see, e.g., Afacan and Dur (2018)) can be interpreted as weak Maskin monotonicity, and our results in Theorem 1 apply. We might suitably generalise this interpretation for all possible assignments, and this might allow us to define a fixed-population version of consistency-influence.

A line of research that our work might help with is the refinement of classes of allocation rules. For instance, there is a line of literature on comparing allocation rules based on their vulnerability to manipulation (see, e.g., Pathak and Sönmez (2013)). This is particularly useful when no strategy-proof rules are available, and designers might wish to select a rule that minimises the vulnerability to manipulation. There are also economies where no 'good' non-bossy allocation rule is available. If we have to live with a certain amount of bossiness, our analytical tools might help in refining the class of 'good' bossy rules, and possibly finding specific rules that are 'best' in terms of minimising influence. This might have particular application in auction settings, where several potential auction rules are available, and all of them are bossy.

Most of all, this approach should provide a normative justification for using bossy allocation rules, instead of always seeking to eliminate them. Since rules with influence might nevertheless satisfy normatively desirable properties (and we have shown how), we could just as easily make use of them, especially if other requirements are at stake.

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A Omitted Proofs

A.1 Proof of Lemma 4

We prove this by construction. Let $R \in \mathcal{R}^N$ and let b^f_R be acyclic. Let $C \subseteq N$. We first construct $C_1(R)$. Pick an arbitrary agent $i_1 \in C$. If $(j, i_1) \notin b^f_R$ for all $j \in C$, set $i_1 \in C_1(R)$. Otherwise, there is some $i_2 \in C$ such that $(i_2, i_1) \in b^f_R$. Since b^f_R is acyclic, $(i_1, i_2) \notin b^f_R$. If $(j, i_2) \notin b^f_R$ for all $j \in C$, set $i_2 \in C_1(R)$. Otherwise, there is some $i_3 \in C$ such that $(i_3, i_2) \in b^f_R$. By acyclicity of b^f_R, $(i_k, i_3) \notin b^f_R$ for $k = 1, 2$. We repeat this argument for all agents. Eventually, since C is finite, we either find $i_k \in C$ such that $(j, i_k) \notin b^f_R$ for all $j \in C$

(in which case set $i_k \in C_1(R)$), or we reach the last agent i_C. It follows that $(j, i_C) \notin b_R^f$ for all $j \in C$ (otherwise we can find a cycle in b_R^f). Thus $i_C \in C_1(R)$, and $C_1(R) \neq \emptyset$.

Let $C_1(R) = \{i \in C \mid (j,i) \notin b_R^f \text{ for all } j \in C\}$. Define $C^1 = C \setminus C_1(R)$. If $C^1 \neq \emptyset$, by the above argument, we can find $C_2(R) = \{i \in C^1 \mid (j,i) \notin b_R^f \text{ for all } j \in C^1\}$. And similarly, we define all $C_k(R)$ until $C^k = \emptyset$. We set the first such k with $C^k = \emptyset$ as K. For a given b_R^f, it is easy to see that the ordered partition $(C_1(R), \ldots, C_K(R))$ as constructed above is unique.

A.2 Proof of Proposition 1

Let f defined on $\mathcal{R}^N$ be strategy-proof and acyclic. Let $R \in \mathcal{R}^N$. Let $C \subseteq N$, and let $R'_C \in \mathcal{M}(R_C, f_C(R))$. We have to show that there exists $i \in C$ such that $f_i(R'_C, R_{-C})\ I'_i\ f_i(R)$. Let $R \to (R'_C, R_{-C})$ be an influence-respecting transform. Let $k \geq 1$, and consider Step k. The remaining agents are C^k, and the corresponding profile is R^k. For any agent $i \in C_{K^k}(R^k)$ selected from the last cell of the partition, strategy-proofness implies $f_i(R'_i, R^k_{-i})\ I_i\ f_i(R^k)$, and Lemma 4 implies that $f_j(R'_i, R^k_{-i})\ I_j\ f_j(R^k)$ for all $j \in C^k$. In particular, after Step $|C| - 1$, for the (sole) remaining agent $i \in C^{|C|}$, we have that $f_i(R_i, R'_{C\setminus i}, R_{-C})\ I_i\ f_i(R)$. Since $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$, by strategy-proofness we get that $f_i(R'_C, R_{-C})\ I_i\ f_i(R)$, and, in particular, $f_i(R'_C, R_{-C})\ I'_i\ f_i(R)$ proving minimal Maskin welfare-invariance.

A.3 Proof of Theorem 1

Let f defined on $\mathcal{R}^N$ be strategy-proof and acyclic. Let $C \subseteq N$ be a set of agents, and let $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$.

In one direction, let $R \to (R'_C, R_{-C})$ be an influence-respecting transform. Let $k \geq 1$, and consider Step k. For any agent $i \in C_{K^k}(R^k)$ selected from the last cell of the partition, as in Proposition 1, any agent j influenced by i at R^k via R'_i has already changed preferences to R'_j at an earlier step. Positivity then implies $f_j(R'_i, R^k_{-i})\ P'_j\ f_j(R^k)$. For each independent agent j' at that step, $f_{j'}(R'_i, R^k_{-i})\ R_{j'}\ f_{j'}(R^k)$ if $j' \in C^k$, and $f_{j'}(R'_i, R^k_{-i})\ R'_{j'}\ f_{j'}(R^k)$ if $j' \in C \setminus C^k$. Repeating for all steps, we get that $f_j(R'_C, R_{-C})\ R'_j\ f_j(R)$ for all $j \in C$, which is weak Maskin monotonicity.

For the other direction, if f is not positive, then there is a profile $R \in \mathcal{R}^N$, agents $i, j \in N$, and preferences $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ such that $f_i(R'_i, R_{-i})\ I_i\ f_i(R)$ and $f_j(R)\ P_j\ f_j(R'_i, R_{-i})$. If we set $R'_j = R_j$, then $R'_j \in \mathcal{M}(\mathcal{R}^j, R_j, f_j(R))$. By strategy-proofness, $f_i(R'_i, R_{-i})\ I'_i\ f_i(R)$. However, $f_j(R)\ P'_j\ f_j(R'_{ij}, R_{-ij})$, violating weak Maskin monotonicity.

A.4 Proof of Theorem 2

It is clear that weak group-strategy-proofness implies strategy-proofness (set $|C| = 1$ in the definition). For the converse, we first show a useful lemma, which states that if f is strategy-proof, acyclic and additionally satisfies preservation, then at least one agent who was independent at the original profile is welfare-invariant after a monotonic transformation for a set of agents that includes her. That is:

Lemma **5.** *If a rule f defined on $\mathcal{R}^N$ is strategy-proof, acyclic, and preserving, then for any $C \subseteq N$, any $R \in \mathcal{R}^N$, and any $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$, we have that $f_i(R'_C, R_{-C})\ I'_i\ f_i(R)$ for all $i \in \mathcal{I}^f_R(C)$.*

Proof: Let f defined on $\mathcal{R}^N$ be strategy-proof, acyclic and preserving. Let $R \in \mathcal{R}^N$, let $C \subseteq N$, and let $R'_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$. Define $R' \equiv (R'_C, R_{-C})$. Let $R \to R'$ be an influence-respecting transform. Let $k \geq 1$, and consider Step k. The remaining agents are C^k, and the corresponding profile is R^k. For any agent $i \in C_{K^k}(R^k)$ selected from the last cell of the partition, setting $R^{k+1} = (R'_i, R^k_{-i})$, Proposition 1 and preservation imply $\mathcal{I}^f_{R^k}(C) \subseteq \mathcal{I}^f_{R^{k+1}}(C)$. The result easily follows. ■

Let f defined on a rich domain $\mathcal{R}^N$ be strategy-proof and acyclic. Suppose for contradiction that f is not weakly group-strategy-proof. Then there exists a profile $R \in \mathcal{R}^N$, agents $C \subseteq N$, and a sub-profile $R'_C \in \mathcal{R}^C$ such that $f_i(R'_C, R_{-C})\ P_i\ f_i(R)$ for all $i \in C$. For notational convenience, define $R' = (R'_C, R_{-C})$. For each $i \in C$, set $y_i = f_i(R')$ and $x_i = f_i(R)$. Then, note that $y_i\ P_i\ x_i$ for all $i \in C$.

For each $i \in C$, pick $R''_i \in \mathcal{R}$ such that $\bar{U}(R''_i, y_i) \subseteq (\bar{U}(R_i, y_i) \cap \bar{U}(R'_i, y_i))$, $\bar{L}(R'_i, y_i) \subseteq$

$\bar{L}(R''_i, y_i)$, $\bar{U}(R''_i, x_i) = \bar{U}(R_i, x_i)$, and $\bar{L}(R''_i, x_i) = \bar{L}(R_i, x_i)$. By richness, such an R''_i exists for each $i \in C$. By construction, we have that $R''_C \in \mathcal{M}(\mathcal{R}^C, R_C, f_C(R))$ and $R''_C \in \mathcal{M}(\mathcal{R}^C, R'_C, f_C(R'))$. Define $R'' = (R''_C, R_{-C})$.

f satisfies preservation

By Lemma 5:

(1) $f_i(R'')\ I''_i\ x_i$ for all $i \in I^f_R(C)$.

(2) $f_j(R'')\ I''_j\ y_j$ for all $j \in I^f_{R'}(C)$. By transitivity of R'', it follows that $f_j(R'')\ P''_j\ x_j$ for all $j \in I^f_{R'}(C)$.

Note that preservation and $f_i(R')\ P_i\ x_i$ for all $i \in C$ imply that $\mathcal{I}^f_R(C) \subseteq \mathcal{I}^f_{R'}(C)$. Thus, for any agent $j \in \mathcal{I}^f_R(C)$, we have $f_j(R'')\ I''_j\ x_j$ and $f_j(R'')\ P''_j\ x_j$, which is a contradiction.

f satisfies positivity

By Proposition 1, there exists $k \in C$ such that $f_k(R'')\ I''_k\ x_k$. By Theorem 1, f satisfies weak Maskin monotonicity, and so $f_j(R'')\ R''_j\ y_j$ for all $j \in C$. In particular, $f_k(R'')\ R''_k\ y_k$. Transitivity of R''_k and $y_k\ P''_k\ x_k$ implies $f_k(R'')\ P''_k\ x_k$. However, it cannot be that $f_k(R'')\ I''_k\ x_k$ and $f_k(R'')\ P''_k\ x_k$, yielding a contradiction.

A.5 Proof of Theorem 3

Let f defined on a top-rich domain $\mathcal{R}^N$ be strategy-proof, acyclic and unanimous.

In one direction, suppose f does not satisfy oppositeness. Then there exists $R \in \mathcal{R}^N$, agent $i \in N$ and preferences $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ such that all influenced agents are better off, i.e., $M_1 \equiv \{j \in N \mid f_j(R'_i, R_{-i})\ P_j\ f_j(R)\}$ with $M_1 \neq \emptyset$, and $\{j \in N \mid f_j(R)\ P_j\ f_j(R'_i, R_{-i})\} = \emptyset$.[12] By definition, $f_k(R)\ I_k\ f_k(R'_i, R_{-i})$ for all $k \in N \setminus M_1$. Thus $f(R'_i, R_{-i})$ is a feasible Pareto-improvement on $f(R)$, which violates Pareto-efficiency.

For the other direction, suppose for contradiction that f is not Pareto-efficient. Then there is a profile $R \in \mathcal{R}^N$ and a feasible allocation $a \in \mathcal{A}$ such that $a_i\ R_i\ f_i(R)$ for all

[12]The case where all influenced agents are worse off can be treated symmetrically.

$i \in M$ and $a_j \; P_j \; f_j(R)$ for some $j \in N$. Construct a profile R' such that, for any $i \in N$, $top(R'_i, X^i) = a_i$ and other alternatives are ranked exactly the same way as in R_i. Such a profile is guaranteed to exist by top-richness. Consider the influence-respecting transform $R \to R'$. Since f satisfies oppositeness, it is straightforward to work out that $a_i \; P'_i \; f_i(R')$ for some $i \in N$. But by unanimity, $f_i(R') \; I'_i \; a_i$ for all $i \in N$, yielding the desired contradiction.

A.6 Proof of Theorem 4

Let X be a set of objects. Let f be defined on an top-rich domain $\mathcal{R}^N$, and let f be strategy-proof, acyclic, unanimous and displacing. Suppose f is wasteful. Then there is a profile $R \in \mathcal{R}^N$, an agent $i \in N$, and an object $x \in X \setminus \{f_j(R)\}_{j \in N}$ such that $x \; P_i \; f_i(R)$ and $(x, f_{-i}(R)) \in \mathcal{A}$. By an argument similar to that in the proof of Theorem 3, construct a profile R' with $top(R'_i, X) = x$ with all other alternatives in X ranked the same in R'_i as in R_i, and for all $j \neq i$, R'_j is such that $top(R'_j, X) = f_j(R)$ and other alternatives are ranked the same in R'_j as in R_j. Such a profile is guaranteed to exist by top-richness. Consider the influence-respecting transform $R \to R'$. Since $x \notin \{f_j(R)\}_{j \in N}$, displacement implies $f_i(R') \neq x$, and so $x \; P'_i \; f_i(R')$. But by unanimity, $f_i(R') \; I'_i \; x$, yielding the desired contradiction.

A.7 Proof of Theorem 5

Let q and $\succ$ for X be given. Let f be a rule defined on an top-rich domain $\mathcal{R}^N$ that is $\succ$-top-respecting, strategy-proof, unanimous, acyclic, displacing and positive. Let $R \in \mathcal{R}^N$. Suppose, for contradiction, that f is not stable. By Theorem 4, f is non-wasteful, and so there must exist a blocking pair (i, x) with $i \in N$, $x \in X$, and some $j \in N$ such that $f_j(R) = x$, $x \; P_i \; f_i(R)$, and $i \succ_x j$. Let $C \subset N$ denote all agents assigned x in R, i.e., $C = \{k \in N \mid \sigma_k^N(R) = x\}$. Then $|C| = q_x$ by non-wastefulness.

Let $R'_i \in \mathcal{R}^i$ be such that $top(R'_i, X) = x$ and all other alternatives are ranked the same as in R_i. This is permitted by top-richness. Similarly, for $k \in C$, let R'_k be such that $top(R'_k, X) = x$ and all other alternatives are ranked the same as in R_k. Define $R' =$

$(R'_C, R'_i, , R_{N\setminus\{C\cup\{i\}\}})$. Since f is acyclic, we can perform the influence-respecting transform $R \to R'$, and we have by positivity that $x \; R'_k \; f_k(R')$ for all $k \in C$. Thus $f_k(R') = x$ for all $k \in C$, and in particular $f_j(R') = x$. Then, $\succ$-top-respectfulness implies $f_i(R') = x$. However, this means $q_x + 1$ agents are assigned x in R', violating the capacity of x, and thus feasibility.

A.8 Proof of Proposition 3

Positivity and displacement are easy to show. Thus σ^N does not satisfy oppositeness. Moreover, it is easy to see that the domain of strict preferences $\mathcal{R}^N$ is top-rich. We also have the following fact.

Fact **6** (Gale and Shapley (1962))**.** The agent-proposing Gale-Shapley rule is agent-optimal in the class of stable rules. Agent-optimality means that, given $\succ$ and any profile $R \in \mathcal{R}^N$, there is no feasible and stable allocation a such that $a_i \; R_i \; \sigma^N_i(R)$ for all $i \in N$ and $a_j \; P_j \; \sigma^N_j(R)$ for some $j \in N$.

For what follows, fix the priority profile $\succ$ for objects in X. For a profile $R \in \mathcal{R}^N$ for agents, we say that a set of agents $B \subset N$ is an 'unrealised cycle' for $\sigma^N(R)$ if B comprises a cycle of agents such that each agent in B prefers the assignment of the next agent in the cycle to her own under preferences in R. Formally, we say that $B \subset N$ with $B = \{i_1, \ldots, i_k, i_{k+1} \equiv i_1\}$ is an *unrealised cycle* for $\sigma^N(R)$ if $\sigma^N_{i_{j+1}}(R) \; P_{i_j} \; \sigma^N_{i_j}(R)$ for all $j \in \{1, \ldots, k\}$. For a given unrealised cycle B, we can define the allocation that would result if the cycle would realise, i.e., if agents in the cycle traded assignments along the cycle until they received their preferred assignments, holding all other assignments fixed. Formally, given an unrealised cycle B for $\sigma^N(R)$ in R, the *realised allocation for* B, $a^B \in \mathcal{A}$, is defined as follows: For every $i_j \in B$, $a^B_{i_j} = \sigma^N_{i_{j+1}}(R)$, and for every $k \in N \setminus B$, $a^B_k = \sigma^N_k(R)$. We have the following fact:

Fact **7.** (Kesten, 2010) For every unrealised cycle $B \subset N$ for $\sigma^N(R)$, there exists a pair (i, x) with $i \in N \setminus B$ and $x \in X$ such that $i \succ_x j$ and $a^B_j = x$ for some $j \in B$. We call such a pair (i, x) a *blocking pair for* B *in* R.

Let $R \in \mathcal{R}^N$ be a profile, and let $\mathcal{B}(R)$ be the set of all unrealised cycles for $\sigma^N(R)$. We define the set of agents in N that form a part of a blocking pair for some $B \in \mathcal{B}(R)$. For any $B \in \mathcal{B}(R)$, define $\mathcal{C}(B) = \{i \in N \mid (i, x) \text{ is a blocking pair for } B \text{ for some } x \in X\}$. Then we can define $\mathcal{C}(\mathcal{B}(R)) = \{i \in N \mid (i, x) \text{ is a blocking pair for some } B \in \mathcal{B}(R) \text{ for some } x \in X\}$. We show that no agent who is not in $\mathcal{C}(\mathcal{B}(R))$ can have influence with any other agent $j \in N$ at R.

LEMMA **6.** *Let $i \in N$ be such that $i \notin \mathcal{C}(\mathcal{B}(R))$. Then i does not have influence with any $j \in N$ at R.*

Proof: Let $i \in N$ be an agent such that $i \notin \mathcal{C}(\mathcal{B}(R))$, and let $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, \sigma_i^N(R))$. By strategy-proofness, $\sigma_i^N(R'_i, R_{-i}) = \sigma_i^N(R)$. Suppose for contradiction that i has influence with some $B \subset N$ at R. By displacement and positivity, B is a cycle, and $\sigma_k^N(R'_i, R_{-i})\ P_k\ \sigma_k^N(R)$ for all $k \in B$. Thus $B \in \mathcal{B}(R)$, and by Fact 7, $\mathcal{C}(B) \neq \emptyset$. Since $i \notin \mathcal{C}(\mathcal{B}(R))$, this implies $i \notin \mathcal{C}(B)$.

Since σ^N is stable and B is realised in $\sigma^N(R'_i, R_{-i})$, we have that $\sigma_j^N(R'_i, R_{-i})\ P_j\ \sigma_j^N(R)$ for all $j \in \mathcal{C}(B)$. By displacement, each $j \in \mathcal{C}(B)$ is part of a cycle B^j. Since B^j was unrealised in R, $B^j \in \mathcal{C}(\mathcal{B}(R))$, and by Fact 7, $\mathcal{C}(B^j) \neq \emptyset$ for all $j \in \mathcal{C}(B)$. But stability implies $\sigma_k^N(R'_i, R_{-i})\ P_k\ \sigma_k^N(R)$ for all $k \in \mathcal{C}(B^j)$, for each $j \in \mathcal{C}(B)$. Iterating, at each stage it follows that each agent in the corresponding cycle is better off, and so each blocking agent for that cycle is better off. Since N and X are finite, this process eventually terminates. By definition, the realised allocation $\sigma^N(R'_i, R_{-i})$ is stable. Moreover, $\sigma_j^N(R'_i, R_{-i})\ P_j\ \sigma_j^N(R)$ for all $j \in \mathcal{C}(\mathcal{B}(R))$ and $\sigma_k^N(R'_i, R_{-i})\ R_k\ \sigma_k^N(R)$ for all $k \in N \setminus \mathcal{C}(\mathcal{B}(R))$. But since $i \notin \mathcal{C}(\mathcal{B}(R))$, the allocation $\sigma^N(R'_i, R_{-i})$ is stable at R, and is Pareto-improving on $\sigma^N(R)$, violating agent-optimality of σ^N. Since $i \notin \mathcal{C}(\mathcal{B}(R))$ was arbitrary, this is true for all such agents, completing the proof. ■

For any $R \in \mathcal{R}^N$, $B \in \mathcal{B}(R)$, and a blocking pair (i, x) for B in R, let t^i denote the step in the Gale-Shapley algorithm in which i is rejected by z. To show that the agent-proposing Gale-Shapley rule σ^N satisfies acyclicity, we rely on an additional fact. This requires the notion of an *essentially-underdemanded* object. We use the concept as defined in Tang and Yu (2014). At a profile $R \in \mathcal{R}^W$ and the corresponding allocation $\sigma^N(R)$, a object z is *underdemanded* at $\sigma^N(R)$ if no agent prefers object z to her assignment in $\sigma^N(R)$, i.e., $\sigma_i^N(R)\ R_i\ z$ for all $i \in W$. A object is *tier-0 underdemanded* at $\sigma^N(R)$ if it is underdemanded at $\sigma^N(R)$. Similarly, for any $k > 0$, a object is *tier-k underdemanded* at $\sigma^N(R)$ if only agents assigned to lower-tier underdemanded objects prefer it to their assignments in $\sigma^N(R)$ and, additionally, at least one agent assigned to a tier-(k-1) object prefers it to her assignment in $\sigma^N(R)$. Thus an *essentially underdemanded* object at $\sigma^N(R)$ if it is tier-k underdemanded at $\sigma^N(R)$ for some integer $k \geq 0$. Then we have the following fact and result:

Fact **8** (Tang and Yu (2014, Lemma 3))**.** There exists $i_1 \in \mathcal{C}(\mathcal{B}(R))$ such that $t^{i_1} \geq t^j$ for all $j \in \mathcal{C}(\mathcal{B}(R))$, and i_1 is assigned an essentially underdemanded object.

We now show acyclicity. Given Lemma 6, it suffices to show acyclicity on $\mathcal{C}(\mathcal{B}(R))$. Let $\succ$ be given, and let $R \in \mathcal{R}^N$. We rely on the following claim:

Claim **1.** No agent $j \in N$ has influence with i_1 at R.

Proof: We know from Lemma 6 that no agent $j \notin \mathcal{C}(\mathcal{B}(R))$ can have influence with i_1 at R. Suppose some $j \in \mathcal{C}(\mathcal{B}(R))$ has influence with i_1 at R via some $R_j' \in \mathcal{M}(\mathcal{R}^j, R_j, \sigma_j^N(R))$. Then $i_1 \in B$ for some $B \in \mathcal{B}(R)$, and (j, x) is a blocking pair for B at R for some $x \in X$. However, by Fact 8, $\sigma_{i_1}^N(R)$ is essentially underdemanded at R. Thus agent $k \in B$ who prefers $\sigma_{i_1}^N(R)$ to $\sigma_k^N(R)$ is assigned a lower tier underdemanded object in R. Similarly, agent $k' \in B$ who prefers $\sigma_k^N(R)$ to $\sigma_{k'}^N(R)$ is assigned an even lower tier underdemanded object. Extending, eventually we find an agent $k'' \in B$ who is part of the cycle but is assigned a tier-0 underdemanded object. By definition, however, no other agent prefers the assignment of k'' to her own, and thus B cannot be a cycle. This gives the desired contradiction. ■

Define $\mathcal{C}^1 = \mathcal{C}(\mathcal{B}(R)) \setminus \{i_1\}$. An application of Fact 8 and Claim 1 gives us a agent $i_2 \in \mathcal{C}^1$ such that $t^{i_2} \geq t^j$ for all $j \in \mathcal{C}^1$, and no agent in $N \setminus \{i_1\}$ has influence with i_2 at R. An iterated application of the above argument establishes acyclicity. Preservation easily follows from the observation that when some agents report preferences such that they are all as well off as before under σ^N, the number of unrealised cycles weakly reduces. Thus, in particular, the set of independent agents weakly expands in the new profile.

A.9 Proof of Proposition 5

It is easy to see that the domain $\mathcal{R}^B$ is rich but not top-rich. Unanimity is also straightforward. Acyclicity follows from the fact that, at any profile of bids, only the price-setting bidder has influence with any other bidder, and no other bidder has influence with her. Positivity follows naturally from the fact that only the price-setting bidder has influence, and a monotonic transformation of her preferences is a lowered bid, which improves the welfare of all winning agents by reducing the winning price. Thus the Vickrey rule does not satisfy oppositeness.

To see that the Vickrey rule does not satisfy preservation, consider the following simple example. There are two bidders, i.e., $N = \{i, j\}$. Let $R \in \mathcal{R}^N$ be such that $v_i = v_j = 2$. Let the tie-breaking rule be $1 \succ 2$. Then, for the profile of bids $c = (2, 2)$, $V_i(c) = (1, 2)$ (i wins the object and pays 2) and $V_j(c) = (0, 0)$ (j does not win the object and pays nothing). Note that since $p_i(c) = v_i$, we have that $(1, 2)\ I_i\ (0, 0)$ by assumption. Also, $j \in \mathcal{I}^V_R$.

Take a profile $R' \in \mathcal{R}^N$ such that $v'_i = 1$ and $v'_j = 1.5$. At the profile of bids $c' = (1, 1.5)$, the assignment for i is $(0, 0)$ and for j is $(1, 1.5)$. Note that $V_i(c')\ I_i\ V_i(c)$ and $V_j(c')\ P_j\ V_j(c)$. However, i has influence with j at R', and so $j \notin \mathcal{I}^V_{R'}$, violating preservation.

A.10 Proof of Proposition 7

The top-richness of the complete domain $\mathcal{R}^N$ is obvious. Let f be a strategy-proof GATTC rule. Oppositeness and displacement are easy to show. Thus f does not satisfy positivity. We now show acyclicity.

We need a few definitions. Let a strategy-proof GATTC rule be given by f defined on a top-rich domain $\mathcal{R}^N$. Since there is the possibility of multiple overlapping cycles, strategy-proof GATTC rules depend on a tie-breaking rule to determine which cycle is favoured in any step. The TCP rule uses a tie-breaking rule over agents, while the TTAS rule uses a tie-breaking rule over objects. In general, let $\succ^f$ be the tie-breaking rule associated with f. Each agent initially owns an object. Let $\omega_i \in X$ denote the endowment of agent $i \in N$. Let the *endowment profile* be given by $\omega \in X^N$. For any profile $R \in \mathcal{R}^N$, let $f(R)$ be the corresponding allocation. For a profile R, a *potential cycle* is a set of agents $B = \{i_1, \ldots, i_k, i_{k+1} \equiv i_1\}$ and a set of objects $Y = \{x_1, \ldots, x_k\}$ such that $x_i \in Y$ is 'held' by $i \in B$, and $x_{i+1}\ R_i\ x_i$ for all $i \in B$. Denote by $\bar{\mathcal{B}}(R)$ the set of all potential cycles in R that might occur at some step in the process. Denote by $\bar{\mathcal{B}}_i(R)$ the set of all potential cycles in R for agent $i \in N$. Denote by $\mathcal{B}(R) \subseteq \bar{\mathcal{B}}(R)$ the set of all cycles that realise in R under f. For any two potential cycles $B, B' \in \mathcal{B}(R)$, we write $B \succ^f_R B'$ if B is favoured by the tie-breaker $\succ^f$ over B' at R. Then we have the following facts:

Fact **9.** For any $R \in \mathcal{R}^N$, any $i \in N$, and any $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$, if there is a cycle $B \in \mathcal{B}(R) \cap \mathcal{B}(R'_i, R_{-i})$ such that $i \in B$, then $\mathcal{B}(R'_i, R_{-i}) = \mathcal{B}(R)$.

Fact **10.** Let $R, R' \in \mathcal{R}^N$. If $\mathcal{B}(R) = \mathcal{B}(R')$, then $f(R) = f(R')$.

Fact **11.** Let $R \in \mathcal{R}^N$. Let $i \in N$ and $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$. Define $R' \equiv (R'_i, R_{-i})$. Let $B, B' \in \bar{\mathcal{B}}(R) \cap \bar{\mathcal{B}}(R')$ be two potential cycles. Then $B \succ^f_R B' \iff B \succ^f_{R'} B'$.

The following claim characterises all situations of influence under f.

Lemma **7.** *Let $R \in \mathcal{R}^N$. Let $i \in N$, $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ and $C \subset N$ be such that i has influence with agents in C at R via R'_i. Then, there exist $j, k \in C$ and $B, B', B'' \in \bar{\mathcal{B}}(R)$ such that:*

1. $i, j \in B$, $i, k \in B'$, $i, k \in B''$.

2. $B \in \mathcal{B}(R)$, $B' \notin \mathcal{B}(R)$, $B'' \in \mathcal{B}(R)$.

3. $B' \in \mathcal{B}(R'_i, R_{-i})$, $B \notin \bar{\mathcal{B}}(R'_i, R_{-i})$

Proof: Let $R \in \mathcal{R}^N$. Let $i \in N$, $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$ and $C \subset N$ such that i has influence with agents in C at R via R'_i. Let $\omega_i \in X$ be the endowment of i. Firstly, note that $f_i(R) \neq \omega_i$ and $f_i(R'_i, R_{-i}) \neq \omega_i$. To see this, suppose that $f_i(R) = \omega_i$ and $f_i(R'_i, R_{-i}) = \omega_i$.

Then, by Fact 9 and Fact 10, $f(R) = f(R'_i, R_{-i})$, contradicting the assumption that i has influence with C at R via R'_i. If $f_i(R) = \omega_i$, then it follows that $f_i(R'_i, R_{-i}) = \omega_i$, leading to the same contradiction. If $f_i(R) \neq \omega_i$, then there is a $B \in \mathcal{B}(R)$ such that $i \in B$. But since $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R))$, we have that $B \in \bar{\mathcal{B}}(R'_i, R_{-i})$, and by Fact 11, $B \in \mathcal{B}(R'_i, R_{-i})$. It follows that $f(R) = f(R'_i, R_{-i})$, leading to the same contradiction.

So $f_i(R) = y \neq \omega_i$. If $f_i(R'_i, R_{-i}) \neq y$, then there is some $B \in \mathcal{B}(R'_i, R_{-i})$ such that $i \in B$. But since $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, f_i(R)$, $B \in \bar{\mathcal{B}}(R)$ as well, and by Fact 11, $B \in \mathcal{B}(R)$, and so $f_i(R) \neq y$, which is a contradiction. Thus $f_i(R'_i, R_{-i}) = y$.

Let $B \in \mathcal{B}(R)$ be such that $i \in B$. If $\bar{\mathcal{B}}_i(R) = B$, then $\bar{\mathcal{B}}_i(R'_i, R_{-i}) = B$, and by Fact 10, $f(R) = f(R'_i, R_{-i})$. Thus there is some $B' \neq B$ such that $B' \in \bar{\mathcal{B}}_i(R)$. Since $B \in \mathcal{B}(R)$, we have that $B \succ^f B'$. Let $j \in B$ be such that ω_j is held by i after B realises. If $f_i(R) = \omega_j$, then $f_i(R'_i, R_{-i}) = \omega_j$, and by Fact 10, $f(R) = f(R'_i, R_{-i})$. Thus there exists some $B'' \neq B$ such that $B'' \in \mathcal{B}(R)$ and $i \in B''$. Let $k \in B''$ be such that $f_i(R) = \omega_k$. Moreover, $B \succ^f B''$.

Since $R'_i \in \mathcal{M}(\mathcal{R}^i, R_i, \omega_k)$, by strategy-proofness we have that $f_i(R'_i, R_{-i}) = \omega_k$. Thus $k \in B'$, and $B' \in \mathcal{B}(R'_i, R_{-i})$. Finally, it follows that $B \notin \bar{\mathcal{B}}(R'_i, R_{-i})$, because otherwise, by Fact 9, $B \succ^f_{(R'_i, R_{-i})} B'$, and by Fact 11, $B \in \mathcal{B}(R'_i, R_{-i})$ and $B' \notin \mathcal{B}(R'_i, R_{-i})$, which is a contradiction. This completes the proof. ■

Acyclicity then follows from the fact that an influenced agent i at R cannot herself influence any other agent at R. To see this, note that Lemma 7 implies that if agent i influences some agents $C \subset N$ at R, then she trades with some agent in C. But if she is herself influenced at R, this means that she also trades with the agent j who influences her at R. However, this means that there is a larger cycle $B' \in \bar{\mathcal{B}}(R)$ such that $B' \notin \mathcal{B}(R)$, which violates Pareto-efficiency.

Preservation follows easily from the observation that after a preference change for some agents such that that they are all at least as well off, Pareto efficiency implies that the cycles that involve them remain the same. It is clear that at the new profile, no agent from the set who was independent at the original profile is influenced by some other agent in the new profile.

B Examples

We first present examples to show that the axioms in Theorem 2 are independent, and are sufficient for weak group-strategy-proofness.

Example 3. A rule that satisfies strategy-proofness, acyclicity, preservation, and positivity, but is not defined on a rich domain, and is thus not weakly group-strategy-proof. Let $N = \{1, 2\}$, $X = \{a, b, c, d\}$, and let $\mathcal{R}^i = (R_i, R'_i)$ for each $i \in N$, where the preferences are given in Figure 6. Let the allocation prescribed by f be given in boxes in each case.

Figure 6: Example 3

R_1	R_2	R'_1	R_2	R_1	R'_2	R'_1	R'_2
a	a	d	a	a	d	d	d
b	b	$\boxed{c}$	$\boxed{b}$	$\boxed{b}$	$\boxed{c}$	c	c
$\boxed{c}$	$\boxed{c}$	b	c	c	b	$\boxed{b}$	$\boxed{b}$
d	d	a	d	d	a	a	a

The domain is not rich as neither $\mathcal{R}^1$ nor $\mathcal{R}^2$ is rich. It can be seen that f is strategy-proof. Moreover, the set of monotonic transformations is always empty for all agents. That is, for any $i \in N$, any R, and any $R''_i \in \mathcal{R}^i$, we have that $\mathcal{M}(\mathcal{R}^i, R''_i, f_i(R)) = \emptyset$. Thus f is trivially acyclic, preserving and positive. However, f is not weakly group-strategy-proof, as agents in N can strongly manipulate at R via R', and vice versa.

Example 4. A rule defined on a rich domain that is strategy-proof, preserving, and positive, but is not acyclic, and not weakly group-strategy-proof.

Let $N = \{1, 2\}$, $X = \{a, b\}$, and let $\mathcal{R}^i = (R_i, R'_i)$ for each $i \in N$, where the preferences are given in Figure 7. Let the allocation prescribed by f be given in boxes in each case.

It can be verified that f is strategy-proof. But f is not weakly group-strategy-proof, as N can strongly manipulate at R via R'. It is easy to check that the domain is trivially rich,

Figure 7: Example 4

R_1	R_2	R'_1	R_2	R_1	R'_2	R'_1	R'_2
a	a	$\boxed{\text{b}}$	$\boxed{\text{a}}$	$\boxed{\text{a}}$	$\boxed{\text{b}}$	b	b
$\boxed{\text{b}}$	$\boxed{\text{b}}$	a	b	b	a	$\boxed{\text{a}}$	$\boxed{\text{a}}$

that f satisfies preservation and positivity, but that it violates acyclicity, as agents 1 and 2 have influence with each other at R via R'_1 and R'_2, respectively.

Example **5.** A rule defined a rich domain that satisfies strategy-proofness and acyclicity, but is not preserving or positive, and not weakly group-strategy-proof. Let $N = \{1, 2\}$, let $X = \{a, b, c\}$, and for each $i \in N$, let $\mathcal{R}^i = \{R_i, R'_i\}$ as shown in Figure 8. Let the allocation prescribed by f for each profile be given in boxes.

Figure 8: Example 5

R_1	R_2	R'_1	R_2	R_1	R'_2	R'_1	R'_2
a	a	$a, \boxed{c}$	$\boxed{a}$	$\boxed{a}$	a, c	$\boxed{a}, c$	$\boxed{a}, c$
$\boxed{b}, c$	$\boxed{b}, c$	b	b, c	b, c	$\boxed{b}$	b	b

It can be verified that f is strategy-proof. It is not weakly group-strategy-proof, as N can strongly manipulate at R via R'. Also, it can be seen that the domain is rich, as the required preferences are one of the two permissible preferences in each case. Note that, at R, neither agent has influence with the other (trivially, as $\mathcal{M}(R_1, f_1(R)) = \mathcal{M}(R_2, f_2(R)) = \emptyset$). Thus $I^f_R = N$. Moreover, as $f_2(R_1, R'_2)\ R_2\ f_2(R)$ and $f_1(R_1, R'_2)\ R_1\ f_1(R)$, if f satisfied preservation we would have that $I^f_{(R'_i, R_{-i})} = N$. However, it can be seen that agent 2 has influence with agent 1 at (R_1, R'_2) via $R_2 \in \mathcal{M}(R'_2, f_2(R_1, R'_2))$. Thus $1 \notin I^f_{(R_1, R'_2)}$, which violates preservation. Moreover, it can be seen that f does not satisfy positivity. It can be verified that f satisfies acyclicity. For instance, observe that 1 has influence with 2 at R' via R_1, but 2 does not have influence with 1 at R' as $f_1(R')\ I'_1\ f_1(R_1, R'_2)$.

Example **6.** A rule defined on a rich domain that satisfies acyclicity, preservation, and positivity, but is not strategy-proof, and not weakly group-strategy-proof. The 'immediate acceptance rule' is non-bossy (Doğan and Klaus, 2017), and so satisfies acyclicity, preservation and positivity. Since the entire domain of strict preferences is allowed, the domain also satisfies richness. It is well-known that this rule is not strategy-proof (see, e.g., Abdulkadiroğlu and Sönmez (2003)).

Example **7.** A rule that satisfies strategy-proofness, acyclicity, positivity, but not preservation, is the Vickrey rule (Vickrey, 1961). A rule that satisfies strategy-proofness, acyclicity, preservation, but not positivity, is a strategy-proof GATTC rule (Aziz and Keijzer, 2011).

We give two examples to show that acyclicity and preservation are not necessary for weak group-strategy-proofness. These examples also demonstrate that these properties are independent of joint-monotonicity and respectfulness (Barberà, Berga, and Moreno, 2016).

Example **8.** A rule defined on a rich domain, that is weakly group-strategy-proof and satisfies preservation, but not acyclicity. Let $N = \{1, 2\}$, $X = \{a, b\}$, and let $\mathcal{R}^i = (R_i, R'_i)$ for each $i \in N$, where the preferences are given in Figure 9. Let the allocation prescribed by f be given in boxes in each case.

Figure 9: Example 8

R_1	R_2	R'_1	R_2	R_1	R'_2	R'_1	R'_2
a	a	$\boxed{\text{b}}$	$\boxed{\text{a}}$	$\boxed{\text{a}}$	$\boxed{\text{b}}$	$\boxed{\text{b}}$	$\boxed{\text{b}}$
$\boxed{\text{b}}$	$\boxed{\text{b}}$	a	b	b	a	a	a

It is easy to see that f is strategy-proof and weakly group-strategy-proof. The domain satisfies richness. However, f does not satisfy acyclicity, as agent 1 has influence with agent 2 at (R_1, R_2) via $R'_1 \in \mathcal{M}(\mathcal{R}^1, R_1, f_1(R))$, and agent 2 also has influence with agent 1 at (R_1, R_2) via $R'_2 \in \mathcal{M}(\mathcal{R}^2, R_2, f_2(R))$. It can be seen that f trivially satisfies preservation. It can also be verified that f satisfies N-joint-monotonicity and N-respectfulness.

Example **9.** A rule defined on a rich domain that is weakly group-strategy-proof, and acyclic, but not preserving, is the Vickrey rule (see Proposition 5). However, the Vickrey rule satisfies N-joint-monotonicity and N-respectfulness (Barberà, Berga, and Moreno, 2016).

Example **10.** An example to show that strategy-proof GATTC rules do not satisfy joint-monotonicity. Let $N = \{1, 2, 3\}$, $Z = \{a, b, c\}$. Let the endowment be as follows: $\omega = (a, b, c)$, i.e., agent 1 initially owns a, agent 2 initially owns b and agent 3 initially owns c. Let the exogenous ordering be $1 \prec 2 \prec 3$. Then, for profiles given in Figure 10, the allocations prescribed by the TCP rule are given in boxes.

Figure 10: Example 10

R_1	R_2	R_3	R'_1	R'_2	R_3
$a\ b\ \boxed{c}$	$\boxed{a}$	a	$\boxed{c}$	a	$\boxed{a}$
	b	$\boxed{b}$	$a\ b$	c	b
	c	c		$\boxed{b}$	c

It is clear that the rule does not satisfy joint-monotonicity, as agent 2 is worse off in (R'_1, R'_2, R_3) than in R with respect to R_2.