Solidarity for public goods under single-peaked preferences: characterizing target set correspondences*

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Abstract

We consider the problem of choosing a set of locations of a public good on the real line $\mathbb{R}$ when agents have single-peaked preferences over points. We ordinally extend preferences over compact subsets of $\mathbb{R}$, and extend the results of Ching and Thomson (1996), Vohra (1999), and Klaus (2001) to choice correspondences. We show that *efficiency* and *replacement-dominance* characterize the class of target point functions (Corollary 2) while *efficiency* and *population-monotonicity* characterize the class of target set correspondences (Theorem 1).

JEL Classification Numbers: C71, D63, D78, H41.

Keywords: single-peaked preferences; population-monotonicity; replacement-dominance; target point functions, target set correspondences.

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1 Introduction

We study the social choice problem where a non-empty and compact set (of points) is chosen on the real line $\mathbb{R}$. We consider this (chosen) set to represent a public good such that each point in the set represents an option for the public good together with its location (e.g., a parking spot together with its location, see Appendix A). We assume that agents have single-peaked preferences over points, that is, an agent's welfare is strictly increasing up to a certain point, his "peak", and is strictly decreasing beyond this point. Given a non-empty and compact set (of points) that represents the public good's options and their locations, an agent -although in good knowledge of all options and their respective locations- is unable to compute his chance of obtaining the public good at a particular location, e.g., in the case of parking spaces along a street, an agent knows that he will (eventually) find a parking spot somewhere along the street but he does not know where this will be: we consider situations where decisions are made under *ignorance* (Peterson, 2009, p. 40). We assume that agents, when comparing sets, focus on their best (most favorite) point(s) and their worst (least favorite) point(s) in each set (see Appendix A). Finally, we assume that the set has adequate capacity to accommodate all agents, that is, all agents have access to the public good but possibly at different locations. Another example for the type of social choice problems we are interested in would be a social planner who needs to draft an "if-needed" list of candidate locations to build a public facility, e.g., a hospital. She does so in an effort to narrow down future construction scenarios while at the same time respecting preferences. Then, if at some future time the need to build a hospital materializes, each location in this list is examined and one of them is chosen.

More specifically, we look into the situation where the social planner wishes to make a choice by providing the public good in a way that is *(Pareto) efficient* and that satisfies some notion of solidarity between agents towards changes in circumstances. Solidarity requires that all agents not responsible for the change should be affected in the same direction. The changes in circumstances we study in this paper are changes in some agent's preferences and changes in the population. *Replacement-dominance*, introduced in the context of quasi-linear binary public decision (Moulin, 1987), applies to a model with a fixed population of agents and requires that if the preferences of an agent change, then the other agents, whose preferences remained unchanged, should all be made at least as well off as they were initially, or they should all be made at most as well off. *Population-monotonicity*, introduced in the context of bargaining (Thomson, 1983a,b), applies to a model with a variable population of agents and requires that if additional agents join a population, then the agents who were initially present should all be made at least as well off as they were initially, or they should all be made at most as well off.

Many social choice problems can be phrased as problems of providing a public good by choosing a location on or an interval of the real line $\mathbb{R}$, or more generally, a tree network,[1] when agents have single-peaked preferences. In these types of problems, it is very natural for changes in preferences (e.g., through the influence of public media or social networks) or changes in the population (e.g., through a change in the birth or migration rate) to arise. Hence, the properties of *replacement-dominance* and *population-monotonicity* have been studied, together or individually, in a variety of contexts. For the special case where the tree network is a closed interval, the problem coincides with the problem of providing a public good by choosing its level when agents have single-peaked preferences (Moulin, 1980). Apart from the provision of public parking or the provision of a hospital by choosing an "if-needed" list of locations, further examples of providing a public good in one or more locations include the provision of (one or more) schools, parks, or libraries on a tree network that represents an infrastructure. We give a detailed survey of the literature on solidarity properties for public good allocation under single-peaked preferences in Section 4.

On the domain of single-peaked preferences, we show that the class of choice correspondences satisfying *efficiency* and *population-monotonicity* is the class of "target set correspondences" (Theorem 1). Each target set correspondence is determined by a "target set" $[a, b]$: if this set is *efficient*, it is chosen; if it is not *efficient*, then its largest *efficient* subset is chosen, if such a subset exists; otherwise, the closest *efficient* point to the target set is chosen. We also show that *efficiency* and *replacement-dominance* characterize the sub-class of "target point functions," i.e., $a = b$ (Corollary 2). Hence, we obtain corresponding results with the literature (Ching and Thomson, 1996; Thomson, 1993; Vohra, 1999). Our results parallel the case where the public good is provided via a lottery over locations on an interval, and probabilistic target choice functions are characterized on the basis of *efficiency* and *population-monotonicity* (Ehlers and Klaus, 2001).

The paper proceeds as follows. Section 2 explains the model and introduces choice correspondences and their properties. Section 3 contains the definition of target set correspondences and presents our characterization results. We conclude with a literature review and a discussion of model assumptions (Section 4).

[1] A tree network is a connected graph that contains no cycles.

2 The model

Denote the set of natural numbers by $\mathbb{N}$. There is a set of "potential" agents, indexed by $\mathbb{P} \subseteq \mathbb{N}$, where $\mathbb{P}$ contains at least 3 agents. We denote the class of non-empty and finite subsets of $\mathbb{P}$ by $\mathcal{P}$. A set of agents $N \in \mathcal{P}$ is called a *population*.

Each $i \in \mathbb{P}$ is equipped with *preferences* R_i, defined on the real line $\mathbb{R}$, that are *complete*, *transitive*, and *reflexive*. As usual, $x \, R_i \, y$ is interpreted as "x is at least as desirable as y", $x \, P_i \, y$ as "x is preferred to y", and $x \, I_i \, y$ as "x is indifferent to y". Moreover, for preferences R_i there exists a number $p(R_i) \in \mathbb{R}$, called the *peak (level) of* i, with the following property: for each pair x, $y \in \mathbb{R}$ such that either $y < x \leq p(R_i)$, or $y > x \geq p(R_i)$, we have $x \, P_i \, y$. We call such preferences *single-peaked* and denote the *domain of single-peaked preferences* by $\mathcal{R}$.

For each $N \in \mathcal{P}$, we denote the set of *(preference) profiles* $R = (R_i)_{i \in N}$ where for each $i \in N$, $R_i \in \mathcal{R}$, by $\mathcal{R}^N$. For each pair $N, M \in \mathcal{P}$, with $N \subseteq M$, we denote the restriction $(R_i)_{i \in N} \in \mathcal{R}^N$ of $R \in \mathcal{R}^M$ to N by R_N. Given $R \in \mathcal{R}^N$, for each pair $i, j \in N$ we also use the notation R_{-i} instead of $R_{N \setminus \{i\}}$ and $R_{-i,j}$ instead of $R_{N \setminus \{i,j\}}$.

Given $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$, we denote the (set of) *peaks* in R as $p(R) = \{p(R_i)\}_{i \in N}$. Let the *smallest peak* in R be $\underline{p}(R) \equiv \min \{p(R_i)\}_{i \in N}$ and the *largest peak* in R be $\bar{p}(R) \equiv \max \{p(R_i)\}_{i \in N}$. Let the *convex hull* of the peaks in R be $\text{Conv}(R) \equiv [\underline{p}(R), \bar{p}(R)]$.

Denote the class of non-empty and compact subsets of $\mathbb{R}$ by $\mathcal{C}$.[2] For each $X \in \mathcal{C}$, the *minimum (point)* is denoted by $\underline{X}$ and the *maximum (point)* by $\bar{X}$. For $X \in \mathcal{C}$ and $R_i \in \mathcal{R}$, let the set of most preferred point(s) or *best point(s)* of i in X be $b_X(R_i) \equiv \{x \in X : \text{for each } y \in X, \, x \, R_i \, y\}$. Similarly, let the set of least preferred point(s) or *worst point(s)* of i in X be $w_X(R_i) \equiv \{x \in X : \text{for each } y \in X, \, y \, R_i \, x\}$. By single-peakedness, $b_X(R_i) \subseteq \{\underline{X}, p(R_i), \bar{X}\}$. By single-peakedness, $w_X(R_i) \subseteq \{\underline{X}, \bar{X}\}$ and if $w_X(R_i) = \{\underline{X}, \bar{X}\}$ (only if $p(R_i) \in (\underline{X}, \bar{X})$), then $\underline{X} \neq \bar{X}$ and $\underline{X} \, I_i \, \bar{X}$. With some abuse of notation, we treat sets $b_X(R_i)$ and $w_X(R_i)$ as if they are points and for each $x \in X$, we write $b_X(R_i) \, R_i \, x \, R_i \, w_X(R_i)$.

We will consider choice correspondences that assign outcomes in $\mathcal{C}$ under *complete uncertainty* (or *ignorance*) with the interpretation that any agent "*knows the set of possible outcomes …, but has no information about the probabilities of those outcomes or about their likelihood ranking*" (Bossert et al., 2000, p. 295).[3] We assume that agents when evaluating

[2]As discussed in Section 4.2.2, the requirement for sets in $\mathcal{C}$ to be compact is without loss of generality.

[3]For a survey of criteria and methods for ranking subsets of a set of outcomes under complete uncertainty we refer to Barberà et al. (2004, Section 3).

outcomes focus exclusively on the best and worst points of the outcomes. Various preference extensions with different degrees of optimism or pessimism do so (see Appendix A for a more detailed discussion) but all have in common that given $X, Y \in \mathcal{C}$, an agent prefers X to Y if he prefers his best point(s) in X to his best point(s) in Y and his worst point(s) in X to his worst point(s) in Y. To strike a balance between the opposite assumptions of optimistic versus pessimistic preference extensions, we use the following *best-worst* extension of preferences over sets (we use the same symbols to denote preferences over points and preferences over sets).

Best-worst extension of preferences to sets. For each $i \in \mathbb{P}$ with $R_i \in \mathcal{R}$ and each pair $X, Y \in \mathcal{C}$, we have

$$X \; R_i \; Y \text{ if and only if } b_X(R_i) \; R_i \; b_Y(R_i) \text{ and } w_X(R_i) \; R_i \; w_Y(R_i)$$

and

$$X \; P_i \; Y \text{ if and only if } X \; R_i \; Y \text{ and } [b_X(R_i) \; P_i \; b_Y(R_i) \text{ or } w_X(R_i) \; P_i \; w_Y(R_i)].$$

This extension of preferences is *transitive*; however, it is not *complete* (there exist sets $X, Y \in \mathcal{C}$ such that neither $X \; R_i \; Y$ nor $Y \; R_i \; X$).

In Appendix A we give a normative foundation of our preference extension based on Bossert et al. (2000, Theorem 1) and illustrate it with an example of public parking allocation. Furthermore, we discuss various optimistic and pessimistic preference extensions and why results would differ if focusing on them.

We use the standard notion of Pareto optimality/efficiency as our efficiency notion.

Efficient sets. Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$. Set $X \in \mathcal{C}$ is *(Pareto) efficient* if and only if there is no set $Y \in \mathcal{C}$ such that for each $i \in N$, $Y \; R_i \; X$, and for at least one $j \in N$, $Y \; P_j \; X$. We denote the class containing all efficient sets for $R \in \mathcal{R}^N$ by $\mathrm{E}(R)$.

The next characterization of efficient sets coincides with the well-known characterization of (Pareto) efficient points for choice functions.

Proposition 1. *For each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$, a set $X \in \mathcal{C}$ is efficient if and only if the following two conditions hold.*

(i) *X is a subset of the convex hull of the agents' peaks, i.e.,*

$$X \subseteq \mathrm{Conv}(R).$$

(ii) *All of the agents' peaks that lie in the convex hull of X are included in X, i.e.,*

$$\text{Conv}\,(X) \cap p(R) \subseteq X.$$

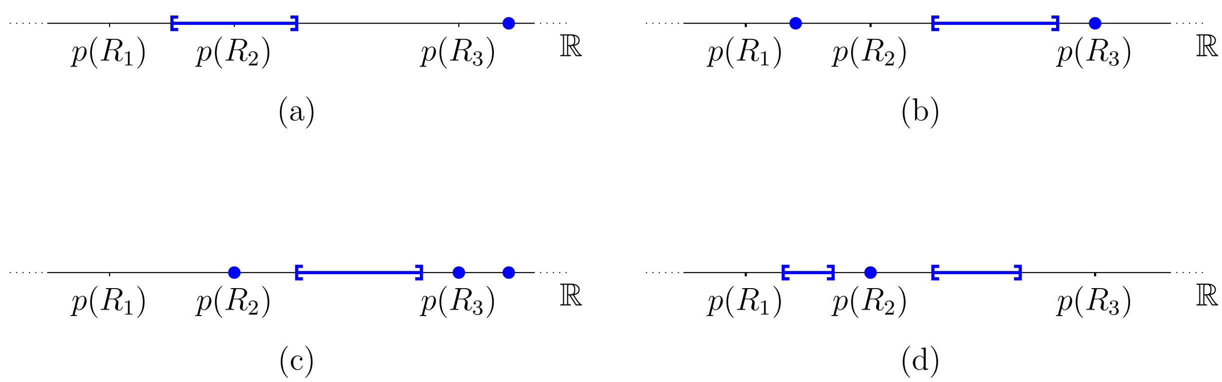

Figure 1: Proposition 1. Let $N = \{1, 2, 3\}$ with $R \in \mathcal{R}^N$ and $p(R) = \{p(R_1), p(R_2), p(R_3)\}$. Sets under consideration are shown in bold. The set in (a) satisfies neither condition (i) nor (ii). The set in (b) satisfies condition (i) but not (ii). The set in (c) satisfies condition (ii) but not (i). The set in (d) satisfies both conditions (i) and (ii), hence it is efficient.

Corollary 1. *Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$.*

(a) *For each convex set $X = \text{Conv}(X) \in \mathcal{C}$, $X \in \text{E}(R)$ if and only if $X \subseteq \text{Conv}(R)$.*

(b) *For each $M \subseteq N$ such that $\text{Conv}(R_M) = \text{Conv}(R)$, if $X \in \text{E}(R)$, then $X \in \text{E}(R_M)$.*

(c) *If $X \in \text{E}(R)$, then for each $i \in N$, $X \; I_i \; \text{Conv}(X)$. Moreover, if $Y \in \mathcal{C}$ is such that for each $i \in N$, $Y \; I_i \; X$, then $\text{Conv}(X) = \text{Conv}(Y)$.*

Since by Corollary 1(c) agents are indifferent between any efficient set and its convex hull, we represent any efficient set by its convex hull.

A *choice correspondence* F assigns to each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$ a set $F(R) \in \mathcal{C}$, i.e., $F \colon \bigcup_{N \in \mathcal{P}} \mathcal{R}^N \to \mathcal{C}$. We denote the family of choice correspondences F by $\mathcal{F}$. If a choice correspondence assigns to each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$ a set consisting of a single point, it is essentially a *choice function*.

We will also consider *fixed population choice correspondences* $F^N \colon \mathcal{R}^N \to \mathcal{C}$ ($N \in \mathcal{P}$). For $N \in \mathcal{P}$, we denote the family of choice correspondences F^N by $\mathcal{F}^N$. Two basic properties of choice correspondences follow.

Efficiency. $F \in \mathcal{F}^N$ ($N \in \mathcal{P}$) is *efficient* if for each $R \in \mathcal{R}^N$, $F(R) \in \text{E}(R)$.

Extreme-peaks-onliness. $F \in \mathcal{F}^N$ $(N \in \mathcal{P})$ satisfies *extreme-peaks-onliness* if the chosen set only depends on the smallest and the largest peaks of the profile, i.e., for each $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(R) = \text{Conv}(\bar{R})$, $F(R) = F(\bar{R})$.

We consider two solidarity properties of choice correspondences. The first solidarity property expresses the solidarity among agents against changes in preferences (Moulin, 1987): if the preferences of an agent change, then the other agents, whose preferences remained unchanged, should all be made at least as well off as they were initially, or they should all be made at most as well off.

Replacement-dominance. $F \in \mathcal{F}^N$ $(N \in \mathcal{P})$ is *replacement-dominant* if for each $j \in N$ and each $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$, the following holds:

$$\text{for each } i \in N \setminus \{j\},\ F(R)\ R_i\ F(\bar{R}) \text{ or for each } i \in N \setminus \{j\},\ F(\bar{R})\ R_i\ F(R).$$

For a population of one or two agents, *replacement-dominance* imposes no restriction on a choice correspondence. Given a population of at least three agents, *efficiency* and *replacement-dominance* have the following implications.

Proposition 2. *Let $F \in \mathcal{F}^N$ $(N \in \mathcal{P},\ |N| \geq 3)$ satisfy efficiency and replacement-dominance. Then,*

(a) *for each $j \in N$, each $R, \bar{R} \in \mathcal{R}^N$ such that $[R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)]$, and each $i \in N \setminus \{j\}$, $F(\bar{R})\ R_i\ F(R)$. In particular, if $\text{Conv}(\bar{R}) = \text{Conv}(R)$, then $F(\bar{R}) = F(R)$;*

(b) *F satisfies extreme-peaks-onliness.*

Proposition 3. *Let $F \in \mathcal{F}^N$ $(N \in \mathcal{P},\ |N| \geq 3)$ satisfy efficiency and replacement-dominance. Then, F is choice function.*

The second solidarity property we consider expresses the solidarity among agents against changes in the population (Thomson, 1983a,b): if additional agents join a population, then the agents who were initially present should all be made at least as well off as they were initially, or they should all be made at most as well off.

Population-monotonicity. $F \in \mathcal{F}$ is *population-monotonic* if for each pair $N, M \in \mathcal{P}$ such that $N \subseteq M$ and each $R \in \mathcal{R}^M$ the following holds:

$$\text{for each } i \in N,\ F(R_N)\ R_i\ F(R) \text{ or for each } i \in N,\ F(R)\ R_i\ F(R_N).$$

A choice correspondence satisfies *peak-monotonicity* (Ching, 1994) if whenever an agent's preferences change such that his peak moves to the left (right), the chosen set moves (weakly) to the left (right).

Peak-monotonicity. $F \in \mathcal{F}^N$ $(N \in \mathcal{P})$ satisfies *peak-monotonicity* if for each $j \in N$ and each $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } p(R_j) \leq p(\bar{R}_j), \text{ then } \underline{F}(R) \leq \underline{F}(\bar{R}) \text{ and } \bar{F}(R) \leq \bar{F}(\bar{R}).$$

A choice correspondence satisfies *uncompromisingness* (Border and Jordan, 1983) if whenever an agent's preferences change such that his peaks, before and after this change, both lie on the same side of the minimum (maximum) point chosen, then the minimum (maximum) point chosen does not change.

Uncompromisingness. $F \in \mathcal{F}^N$ $(N \in \mathcal{P})$ satisfies *uncompromisingness* if for each $j \in N$ and each $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

(i) if $[p(R_j) < \underline{F}(R)$ and $p(\bar{R}_j) \leq \underline{F}(R)]$ or $[p(R_j) > \underline{F}(R)$ and $p(\bar{R}_j) \geq \underline{F}(R)]$, then $\underline{F}(R) = \underline{F}(\bar{R})$ and

(ii) if $[p(R_j) > \bar{F}(R)$ and $p(\bar{R}_j) \geq \bar{F}(R)]$ or $[p(R_j) < \bar{F}(R)$ and $p(\bar{R}_j) \leq \bar{F}(R)]$, then $\bar{F}(R) = \bar{F}(\bar{R})$.

Given a choice correspondence satisfying *efficiency* and *population-monotonicity*, we have the following implications.

Proposition 4. *Let $F \in \mathcal{F}$ satisfy efficiency and population-monotonicity.*

(a) *For each pair $N, M \in \mathcal{P}$ such that $N \subsetneq M$, each $R \in \mathcal{R}^M$, and each $i \in N$, $F(R_N)\ R_i\ F(R)$. In particular, if $\text{Conv}(R_N) = \text{Conv}(R)$, then $F(R_N) = F(R)$.*

Let $N \in \mathcal{P}$, $|N| \geq 3$, and $F \in \mathcal{F}^N$. Then,

(b) *for each $j \in N$, each $R, \bar{R} \in \mathcal{R}^N$ such that $[R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)]$, and each $i \in N \setminus \{j\}$, $F(\bar{R})\ R_i\ F(R)$. In particular, if $\text{Conv}(\bar{R}) = \text{Conv}(R)$, then $F(\bar{R}) = F(R)$;*

(c) *F satisfies extreme-peaks-onliness;*

(d) *F satisfies peak-monotonicity;*

(e) *F satisfies uncompromisingness.*

3 Characterizing target set correspondences

Any *target set correspondence* is determined by its non-empty, closed, and convex target set: if the target set is efficient, it is chosen. If the target set is not efficient, the (unique) maximal efficient subset of the target set is chosen, if one exists; otherwise, the (unique) closest efficient point to the target set is chosen.

Target set correspondence $\boldsymbol{F^{a,b} \in \mathcal{F}^N}$. Let $[a,b] \subseteq \mathbb{R} \cup \{-\infty, \infty\}$, $N \in \mathcal{P}$, and $R \in \mathcal{R}^N$. Then,

$$F^{a,b}(R) = \begin{cases} \{\underline{p}(R)\} & \text{if } b < \underline{p}(R) \\ \{\bar{p}(R)\} & \text{if } a > \bar{p}(R) \\ [a,b] \cap \mathrm{Conv}(R) & \text{otherwise.} \end{cases}$$

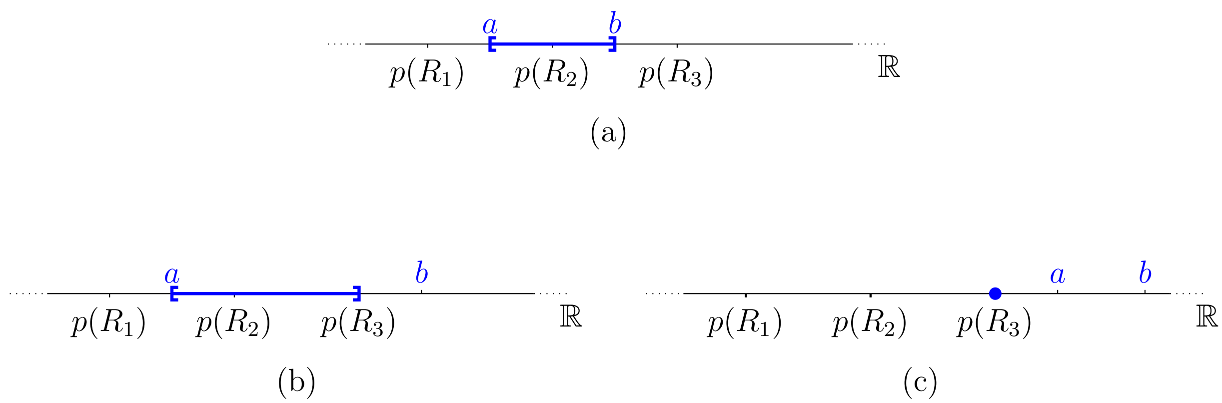

Figure 2: Target set correspondence. Let $N = \{1,2,3\}$, $R \in \mathcal{R}^N$, and $F^{a,b} \in \mathcal{F}^N$. The chosen sets in each case are shown in bold. The target set in (a) is efficient and is chosen, (b) is not efficient but the maximal efficient subset exists and it is chosen, and (c) is not efficient and no maximal efficient subset exists; hence the closest efficient point is chosen.

Target point function $\boldsymbol{f^a \in \mathcal{F}^N}$. Let $a \in \mathbb{R} \cup \{-\infty, \infty\}$, $N \in \mathcal{P}$, and $R \in \mathcal{R}^N$. Then,

$$f^a(R) = F^{a,a}(R).$$

Our first characterization result is a corollary from Proposition 3 and Thomson's (1993) characterization result for choice functions.[4]

[4]In Klaus and Protopapas (2017) we provide a self-contained proof of Corollary 2 as well as a characterization of target set correspondences by *efficiency* and a weaker *replacement-dominance* property called *one-sided replacement-dominance*.

Corollary 2. *A choice correspondence* $F \in \mathcal{F}^N$ $(N \in \mathcal{P},\ |N| \geq 3)$ *satisfies efficiency and replacement-dominance if and only if it is a target point function.*

Our second characterization result states that *efficiency* and *population-monotonicity* characterize target set correspondences.

Theorem 1. *A choice correspondence* $F \in \mathcal{F}$ *satisfies efficiency and population-monotonicity if and only if it is a target set correspondence with the same target set for all populations.*

The properties in our two characterizations are independent. A constant choice correspondence that always chooses a fixed set satisfies *replacement-dominance* and *population-monotonicity* but violates *efficiency*. A choice correspondence that always chooses the peak of the agent with the lowest index satisfies *efficiency*, but it violates *replacement-dominance* and *population-monotonicity*.

Remark 1. Our characterizations in Corollary 2 and Theorem 1 confirm an observed pattern concerning solidarity in public goods models: *replacement-dominance* is at least as strong as *population-monotonicity*. Here and, for example, in Miyagawa (2001), Ehlers (2002, 2003), Ehlers and Klaus (2001), and Gordon (2007a,b), the set of choice functions satisfying *efficiency* and *replacement-dominance* is a subset of the set of choice functions satisfying *efficiency* and *population-monotonicity*. □

4 Conclusion

We considered a problem of choosing a set of locations of a public good on the real line $\mathbb{R}$ when agents have single-peaked preferences over points and evaluate outcome sets under the assumption of complete uncertainty with a best-worst preference extension. We show that *efficiency* and *replacement-dominance* characterize the class of target point functions while *efficiency* and *population-monotonicity* characterize the larger class of target set correspondences. These characterizations resemble previous results in various public goods settings, which we survey in the next subsection.

In the subsection following the literature review, we check the robustness of our results with respect to various model assumptions we made. Our assumption on the preference extension used is separately discussed in Appendix A.

4.1 Literature review: solidarity for public goods models

For choice functions that assign a public good on an interval, or on a tree network, the solidarity properties *replacement-dominance* and *population-monotonicity*, have been considered. Specifically, for the location problem on an interval (on a tree network), it was shown that *efficiency* and *population-monotonicity* characterize the class of "target point functions" on the domain of single-peaked preferences (Ching and Thomson, 1996; Thomson, 1993).[5] and for constant sets of agents *efficiency* and *replacement-dominance* characterize the class of "target point functions" on the domains of single-peaked preferences and symmetric single-peaked preferences (Vohra, 1999). Moreover, it turns out that *efficiency* and *population-monotonicity* imply *replacement-dominance* and that the former characterization also holds on the domain of symmetric single-peaked preferences and on tree networks (Klaus, 2001). In addition, both aforementioned characterizations hold under much looser assumptions on the set of locations (alternatives) and the domain of preferences (Gordon, 2007a).[6] Finally, if the set of admissible preferences is constrained on attribute-based preference domains,[7] *efficiency* and either one of the two solidarity properties are only compatible on discrete trees, where equivalent characterizations are obtained (Gordon, 2015).

For the location problem on an interval, if the property of *replacement-dominance* is weakened to *ϵ-replacement-dominance*[8] the characterization of target point functions still holds for the domain of single-peaked preferences (Harless, 2015b). However, for the location problem on a circle when a constant set of agents exists, no choice function satisfies *efficiency* and either *replacement-dominance* or *population-monotonicity* on the domain of symmetric single-peaked preferences (Gordon, 2007b).

Regarding choice correspondences, the case of providing a public good at exactly two locations, when one or both of the aforementioned solidarity properties are being considered, has been studied under different settings. On the domain of single-peaked preferences and if the agents compare pairs of locations using the max-extension,[9] the following holds.

[5] Such functions are sometimes called *status quo rules* or *status quo solutions*.

[6] The critical assumptions are: (i) the set of alternatives is fixed, (ii) the preferences are defined over all alternatives, and (iii) the domain of preferences is common to all agents.

[7] Given a finite set of alternatives A, the non-empty and finite family of subsets $\mathcal{H} \subseteq 2^A$ is an attribute space if [for each attribute $H \in \mathcal{H}$, $H \neq \emptyset$ and the complement $H^C \in \mathcal{H}$] and [for each pair $x, y \in A$ with $x \neq y$, there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$].

[8] Agents' solidarity is only required if the change in an agent's preferences are below a certain threshold.

[9] Under the max-extension, an agent prefers set X to set Y if and only if he prefers his best point(s) in set X to his best point(s) in set Y.

For an interval in $\mathbb{R}$ and a constant set of agents, the class of choice functions satisfying *efficiency* and *replacement-dominance* are the "left-peaks choice function" and the "right-peaks choice function"[10] (Miyagawa, 2001). However, if this model is extended to trees, then no choice function satisfies *efficiency* and *replacement-dominance* on the symmetric single-peaked domain (Umezawa, 2012).

For the problem of providing a public good at exactly two locations on an interval, on the domain of single-peaked preferences and if agents compare pairs of locations using the leximin-extension,[11] the following two results have been obtained that consider *population-monotonicity* or *replacement-dominance*. First, the class of choice functions satisfying *efficiency*, *anonymity*, and *population-monotonicity* is the class of "single-plateaued preference choice functions"[12] (Ehlers, 2003); and second, the class of choice functions satisfying *efficiency* and *replacement-dominance* is the class of "single-peaked preference choice functions"[13] (Ehlers, 2002). Note that the model of choosing two locations is rather different from our model of choosing a set of outcomes under complete uncertainty or ignorance (focusing on the best and the worst location in the outcome set does not reduce our model to that of choosing two locations).

In the setting of preference aggregation problems, where agents strictly rank a finite set of alternatives and a (not necessarily strict) social ranking over the alternatives must be chosen, the aforementioned solidarity properties have also been studied. It is shown that on the domain of strict rankings, *efficiency* and *population-monotonicity* characterize the class of "strict status-quo functions"[14] (Bossert and Sprumont, 2014). Moreover, in this result, *population-monotonicity* can be substituted with *adjacent replacement-dominance*.[15] Furthermore, if

[10] The left (right) peaks choice function chooses the two unique left-most (right-most) peaks.

[11] Under the leximin-extension, in the case of sets containing exactly two points, an agent prefers set X to set Y if and only if he either [prefers his best point(s) in set X to his best point(s) in set Y] or [he is indifferent between his best point in set X and his best point in set Y and prefers his second best point in set X to his second best point in set Y].

[12] Each single-plateaued preference choice function is determined by fixed single-plateaued preferences R and plateau $[\underline{r}, \bar{r}]$: if all the agents' peaks lie outside of $[\underline{r}, \bar{r}]$, then the best of the agents' peaks and its indifferent point are chosen (according to R); otherwise, the two locations in the convex hull of the agents' peaks lying closest to $\underline{r}$ and $\bar{r}$ respectively are chosen.

[13] Each single-peaked preference choice function is essentially a single-plateaued preference choice function determined by a fixed single-plateaued preference relation R with the plateau being a point, i.e., $\underline{r} = \bar{r}$.

[14] Each strict status-quo function is determined by a strict ranking R over the alternatives and reaches a unique efficient strict ranking as follows: beginning from R it reverses the order of an adjacently ranked pair of alternatives if all agents prefer the reverse to the initial ranking of the pair.

[15] *Adjacent replacement-dominance* is weaker than *replacement-dominance*: solidarity is only required when

the domain is enlarged to also include weak rankings, *efficiency* and either *population-monotonicity* or *adjacent replacement-dominance* characterize the class of "status-quo functions"[16] (Harless, 2016). Finally, in the binary social choice model (i.e., when there are exactly two alternatives to choose from) and if agents can be indifferent between the two alternatives, a choice function satisfies *replacement-dominance* or *population-monotonicity* if and only if it is a "generalized mixed-consensus rule"[17] (Harless, 2015a).

4.2 Model variations

4.2.1 Symmetric preferences

Preferences $R_i \in \mathcal{R}$ are *symmetric* if for each pair $x, y \in \mathbb{R}$, $|x - p(R_i)| = |y - p(R_i)|$ implies $x \; I_i \; y$. Throughout this paper we assume that preferences are single-peaked. In Klaus and Protopapas (2017) we also show that all our results hold on the domain of symmetric single-peaked preferences.

4.2.2 Chosen sets are not necessarily compact

Although we only study compact subsets of $\mathbb{R}$, the compactness requirement is without loss of generality for the following reasons. First, the agents' peaks being real numbers and Proposition 1(i) imply that unbounded sets are not efficient. Second, concerning not closed (and bounded) sets, after assuming that each agent is indifferent between a set and its *closure*, all our results hold and the target sets of target set correspondences need not be closed. In this case, the second requirement for the efficiency of a set, that is, Proposition 1(ii), would have to be changed to $\text{Conv}(\text{closure}(X)) \cap p(R) \subseteq \text{closure}(X)$; moreover, to accommodate the possibility of sets that are not closed, throughout the text and for each set X, $\text{Conv}(X)$ would have to be changed to $\text{Conv}(\text{closure}(X))$.

an agent reverses a single pair of adjacently ordered alternatives.

[16] Each status-quo function is determined by a ranking $\bar{R}$ over the alternatives and reaches a unique efficient ranking as follows: beginning from $\bar{R}$ it reverses the order of an adjacently ranked pair of single alternatives if all agents prefer the reverse to the initial ranking of the pair. Moreover, it "creates" order in an indifference class (of alternatives) if all agents prefer the alternative moved up in the order to the one (or more) alternatives moved down. Reversals in the order between a single alternative and an indifference class or between two indifference classes occur in a similar way.

[17] Each generalized mixed-consensus rule chooses for each profile either alternative a or alternative b. The only further requirement concerns cases where at least one agent prefers a over b and at least one agent prefers b over a; specifically, either a is selected in all such cases or b is selected in all such cases.

4.2.3 Monotonic preferences

Allowing for agents to have monotonic preferences would correspond to allowing agents' peaks to be drawn from the extended real line $\mathbb{R} \cup \{-\infty, +\infty\}$ and allowing choice correspondences to assign subsets of the extended real like (strictly decreasing preferences correspond to an agents's peak being $-\infty$ and strictly increasing preferences correspond to an agents's peak being $+\infty$). However, if all agents have $-\infty$ or all agents have $+\infty$ as their peak, then by Proposition 1, no efficient set exists in $\mathcal{C}$. Moreover, if unbounded subsets of $\mathbb{R} \cup \{-\infty, +\infty\}$ are considered, then in this case the only efficient sets would be $\{-\infty\}$ (when all agents have $-\infty$ as their peak) or $\{+\infty\}$ (when all agents have $+\infty$ as their peak). However, a policy interpretation for these two sets, as well as other unbounded sets, is not clear and we therefore did not add monotonic preferences to our model.

4.2.4 Closed interval alternative set

All our results hold if preferences are defined on a closed interval $[x, y] \subseteq \mathbb{R}$. By Proposition 1(i), the class of sets considered equals the class of non-empty subsets of $[x, y]$ and closedness is not required (see Subsection 4.2.2). Agents could now have monotonic preferences, i.e., have x or y as peaks, since the policy interpretation of "locating the public good at x (or y)" is straightforward, in contrast to our original model (see Subsection 4.2.3). This restriction on the set of alternatives would facilitate our main characterization proof (Theorem 1) since a profile with x as the minimum peak and y as the maximum peak could be chosen to calibrate the target set (in contrast to our original model, where a profile with $-\infty$ as the minimum peak and $+\infty$ as the maximum peak is not available).

Appendix

A Best-Worst Preferences

We start with a normative justification to focus on best and worst points in our preference extension and introduce the properties of *simple monotonicity* and *independence* that characterize a small class of preference extensions over sets, albeit for a slightly different model than ours (Bossert et al., 2000, Theorem 1). First, we illustrate via two examples why these properties are reasonable to assume in our model. Then, we present the characterization result and discuss its consequences for our model.

We denote preferences defined over $\mathcal{C}$ by $R_i^{\mathcal{C}}$ (if $x \, P_i \, y$, then $\{x\} \, P_i^{\mathcal{C}} \, \{y\}$).

Simple monotonicity. Let $x, y \in \mathbb{R}$. If $x \, P_i \, y$, then

$$\{x\} \, P_i^{\mathcal{C}} \, \{x, y\} \, P_i^{\mathcal{C}} \, \{y\}.$$

Independence. Let $X, Y \in \mathcal{C}$ and $z \in \mathbb{R}$ such that $z \notin X \cup Y$. If $X \, P_i^{\mathcal{C}} \, Y$, then

$$[X \cup \{z\}] \, R_i^{\mathcal{C}} \, [Y \cup \{z\}].$$

Our example pertains to a linear city whose residents own one car each and have single-peaked preferences over where to park.

Example 1. Simple monotonicity: All public parking is located in two (parking) garages at $x, y \in \mathbb{R}$, with $x \neq y$, that we refer to as zone x and y. Neither garage's capacity can accommodate all residents but the joint capacity is sufficient. Initially, a one-zone scheme is in place and all residents are assigned to either zone x or zone y: residents assigned to zone x (zone y) are only allowed to park at garage x (y), which has the capacity to accommodate them. Later, a two-zone scheme is adopted: each resident can use either one of the two garages. Consider a resident i of zone x who prefers x to y. Under the one-zone scheme he always parks at x, while under the two-zone scheme he sometimes parks at y (whenever x is full). We expect resident i to be worse off under the two-zone scheme, that is, if $x \, P_i \, y$, then $\{x\} \, P_i^{\mathcal{C}} \, \{x, y\} \, P_i^{\mathcal{C}} \, \{y\}$ and *simple monotonicity* holds.

Independence: Two single-zone street parking schemes, $X, Y \in \mathcal{C}$, are being considered for adoption. Before a final decision is made, and following a small development project on some previously unused land, an extra single parking garage $z \in \mathbb{R}$ becomes available. Now assume that instead of schemes X and Y, two new schemes are being considered for adoption, $X \cup \{z\}$ and $Y \cup \{z\}$. Suppose resident i initially prefers X to Y. Since space z was unavailable under X and Y and is now available under both $X \cup \{z\}$ and $Y \cup \{z\}$, we expect i to find $X \cup \{z\}$ at least as desirable as $Y \cup \{z\}$. That is, if $z \notin X \cup Y$, and $X \, P_i^{\mathcal{C}} \, Y$, then $[X \cup \{z\}] \, R_i^{\mathcal{C}} \, [Y \cup \{z\}]$ and *independence* holds. □

By the next result, if the two aforementioned properties are required, then an agent with *linear* preferences over outcomes[18] only cares about his best and worst points in each finite set.[19]

[18]A linear preference R^L is a *complete*, *transitive*, *reflexive*, and *antisymmetric* (i.e., for each $x, y \in \mathbb{R}$, $x \, I^L \, y$ implies $x = y$) binary relation. Single-peaked preferences are not antisymmetric.

[19]A similar result using a stronger version of *independence* is shown in Barberà et al. (1984).

Bossert et al. (2000, Theorem 1). If *simple monotonicity* and *independence* are satisfied, then for agent i with *linear* preferences R_i^L, and each finite set $X \in \mathcal{C}$, $X I_i^{\mathcal{C}} \{b_X(R_i^L), w_X(R_i^L)\}$.

In light of this result, two "standard" extensions that could be considered for our model are the *max-min*[20] and the *min-max*[21] preference extensions, both of which fit our parking example since they are *"consistent with the notion of limited rationality which is familiar in the theories of organization and bounded rationality (e.g., March, 1988; March and Simon, 1958), and which suggests that, given a complex decision problem, the agent often seeks to simplify the problem by focusing on only a few salient features of the complex situation"* (Bossert et al., 2000, pp. 300-301). However, given the problem at hand, we prefer to "not choose sides" by adopting either the "optimistic" max-min extension or the "pessimistic" min-max extension (some agents might be optimists and some pessimists). Instead, we opt for the *best-worst* extension of preferences that declares a preference for a set X over a set Y if and only if this preference coincides with the preference of both the min-max extension and the max-min extension. It is straightforward to show that the best-worst extension satisfies *simple monotonicity* and *independence*, not only when based on linear preferences over outcomes but also in our setting of single-peaked preferences over outcomes and sets of alternatives that are not always finite.

Finally, we would like to point out why our results would significantly change when focusing either on optimistic agents or on pessimistic agents.

First, consider very optimistic agents who only focus on the best point in the outcome set (*max* extension, see footnote 9). Then, *efficiency* would require to always assign a set containing all peaks, which for some policy application would not be feasible. Even when considering the less optimistic max-min extension (see footnote 20 for its definition), for some preference profiles, e.g., when agents' peaks are located at exactly two points, *efficiency* could imply the assignment of a set containing all peaks. Therefore, under the optimistic max and max-min preference extensions, target-set correspondences would not be *efficient*.

Second, consider very pessimistic agents who only focus on the worst point in the outcome set (*min* extension). Then, *efficiency* would require to always assign a singleton set: e.g.,

[20] *Max-min extension*: An agent prefers set X to set Y if and only if either [he prefers his best point(s) in set X to his best point(s) in set Y] or [he is indifferent between his best point(s) in both sets and prefers his worst point(s) in set X to his worst point(s) in set Y].

[21] *Min-max extension*: An agent prefers set X to set Y if and only if either [he prefers his worst point(s) in set X to his worst point(s) in set Y] or [he is indifferent between his worst point(s) in both sets and prefers his best point(s) in set X to his best point(s) in set Y].

if an interval $[a, b]$ in $\text{conv}(R)$ is chosen, then choosing the singleton set $\{\frac{a+b}{2}\}$ constitutes a Pareto improvement. Hence, one would only need to study choice functions. Even when considering the less pessimistic min-max extension (see footnote 21 for its definition), for some preference profiles, e.g., when agents' peaks are located at exactly two points, *efficiency* could imply the assignment of a singleton set. Therefore, under the pessimistic min and min-max preference extensions, the target-set correspondences with target sets $[a, b]$, $a < b$, would not be efficient. If agents only focus on the worst point in the outcome set (cautious extension), then *efficiency* would require that no non-singleton set can be chosen (if an interval $[a, b]$ with $a < b$ is chosen, then the set $\{c\}$ such that $a < c < b$ would be a Pareto improvement).

B Proofs of Section 2

Throughout this appendix, for $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$, we assume, without loss of generality, that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. Furthermore, whenever for $X \in \mathcal{C}$ and $i \in N$, $|b_X(R_i)| = 1$ or $|w_X(R_i)| = 1$, we denote the singleton set as unique (best or worst) point.

Lemma 1. *Let $i \in \mathbb{P}$ with $R_i \in \mathcal{R}$ and $X \in \mathcal{C}$.*

(a) *Let $\underline{X} < p(R_i)$, $\underline{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i)$, and $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$. Then, $Y \, R_i \, X$. Moreover, if $w_X(R_i) = \underline{X}$, then $Y \, P_i \, X$.*

(b) *Let $\bar{X} > p(R_i)$, $\bar{x} \in \mathbb{R}$ such that $\bar{X} > \bar{x} \geq p(R_i)$, and $Y = [X \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$. Then, $Y \, R_i \, X$. Moreover, if $w_X(R_i) = \bar{X}$, then $Y \, P_i \, X$.*

(c) *Let $\underline{X} < p(R_i)$, $\bar{X} > p(R_i)$, $\underline{x}, \bar{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i) \leq \bar{x} < \bar{X}$, $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$, and $Z = [Y \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$. Then, $Z \, P_i \, X$.*

Proof. **(a)** By single-peakedness, for each $z \in X \setminus Y$, $\underline{x} \, P_i \, z$. Hence, $Y \, R_i \, X$. If additionally $w_X(R_i) = \underline{X} \notin Y$, then $\bar{X} \, P_i \, w_X(R_i)$ and $\underline{x} \, P_i \, w_X(R_i)$. Since by single-peakedness $w_Y(R_i) \subseteq \{\underline{x}, \bar{X}\}$, it follows that $Y \, P_i \, X$.

(b) Symmetric proof to (i).

(c) By (i), $Y \, R_i \, X$. By (ii), $Z \, R_i \, Y$. Hence, $Z \, R_i \, X$. By single-peakedness, $w_X(R_i) \subseteq \{\underline{X}, \bar{X}\}$ and $w_Z(R_i) \subseteq \{\underline{x}, \bar{x}\}$. By single-peakedness, $\underline{x} \, P_i \, w_X(R_i)$ and $\bar{x} \, P_i \, w_X(R_i)$. Hence, $Z \, P_i \, X$. □

Proof of Proposition 1. Let $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $X \in \mathcal{C}$.

Step 1. Assume by contradiction that $X \in \mathrm{E}(R)$ and $X \nsubseteq \mathrm{Conv}(R)$. Then, $\underline{X} < p(R_1)$ or $\bar{X} > p(R_n)$. By symmetry of arguments, assume that $\underline{X} < p(R_1)$.

Case 1 ($\bar{X} > p(R_n)$). Then, for each $i \in N$, $\underline{X} < p(R_1) \leq p(R_i) \leq p(R_n) < \bar{X}$. Consider $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$ and $Z = [Y \cap (-\infty, p(R_n))] \cup \{p(R_n)\}$. By Lemma 1(c), for each $i \in N$, $Z \; P_i \; X$. Hence, $X \notin \mathrm{E}(R)$; a contradiction.

Case 2 ($\bar{X} \leq p(R_n)$). Then, for each $i \in N$, $\underline{X} < p(R_1) \leq p(R_i)$. Consider $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$. By Lemma 1(a), for each $i \in N$, $Y \; R_i \; X$. Furthermore, $w_X(R_n) = \underline{X}$. Then, by Lemma 1(a), $Y \; P_n \; X$. Hence, $X \notin \mathrm{E}(R)$; a contradiction.

Step 2. Assume by contradiction that $X \in \mathrm{E}(R)$ and $(\mathrm{Conv}(X) \cap p(R)) \nsubseteq X$. By Step 1, $X \subseteq \mathrm{Conv}(R)$. Thus, there exists $j \in N$ such that $p(R_j) \in \mathrm{Conv}(X)$ and $p(R_j) \notin X$.

Let $Y = X \cup \{p(R_j)\}$. By single-peakedness, for each $i \in N$, $w_X(R_i) = w_Y(R_i) \subseteq \{\underline{X}, \bar{X}\}$ and $b_X(R_i) R_i b_Y(R_i)$. Hence, for each $i \in N$, $Y R_i X$. Furthermore, $b_Y(R_j) = p(R_j) P_j b_X(R_j)$. Therefore, $Y \; P_j \; X$. Hence, $X \notin \mathrm{E}(R)$; a contradiction.

Step 3. Let $X \in \mathcal{C}$ such that (i) $X \subseteq \mathrm{Conv}(R)$ and (ii) $(\mathrm{Conv}(X) \cap p(R)) \subseteq X$. Assume by contradiction that $X \notin \mathrm{E}(R)$. Hence, there exists a set $Y \subseteq \mathbb{R}$ such that for each $i \in N$, $Y \; R_i \; X$, and for at least one $j \in N$, $Y \; P_j \; X$.

Case 1 ($p(R_j) \in \mathrm{Conv}(X)$). By condition (ii), $p(R_j) \in X$ and agent j's best point $b_X(R_j) = p(R_j) \in X$ cannot be improved. By single-peakedness, $w_X(R_j) \subseteq \{\underline{X}, \bar{X}\}$; if $w_Y(R_j) \; P_j \; w_X(R_j)$, by single-peakedness, $\underline{X} < \underline{Y}$ or $\bar{X} > \bar{Y}$. By symmetry of arguments, assume $\underline{X} < \underline{Y}$. Considering R_1, by condition (i), $p(R_1) \leq \underline{X} < \underline{Y}$. By single-peakedness, $b_X(R_1) P_1 b_Y(R_1)$ and for agent 1, set Y is not at least as desirable as set X; a contradiction.

Case 2 ($p(R_j) \notin \mathrm{Conv}(X)$). Then, either $p(R_j) < \underline{X}$ or $p(R_j) > \bar{X}$. By symmetry of arguments, assume that $p(R_j) > \bar{X}$. By single-peakedness, $b_X(R_j) = \bar{X}$ and $w_X(R_j) = \underline{X}$. If $b_Y(R_j) P_j b_X(R_j)$, by single-peakedness, $\bar{X} < \bar{Y}$. If $w_Y(R_j) P_j w_X(R_j)$, by single-peakedness, $\underline{X} < \underline{Y}$. Considering R_1, by condition (i), $p(R_1) \leq \underline{X} \leq \bar{X}$. By single-peakedness, $b_X(R_1) = \underline{X}$ and $w_X(R_1) = \bar{X}$. If $\underline{X} < \underline{Y}$, by single-peakedness, $b_X(R_1) \; P_1 \; b_Y(R_1)$. If $\bar{X} < \bar{Y}$, by single-peakedness, $w_X(R_1) \; P_1 \; w_Y(R_1)$. It follows that for agent 1, set Y is not at least as desirable as set X; a contradiction. □

Proof of Corollary 1. Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$.

(a) Follows immediately from Proposition 1.

(b) If $X \in \mathrm{E}(R)$, then by Proposition 1(i), $X \subseteq \mathrm{Conv}(R)$. Let $M \subseteq N$ such that $\mathrm{Conv}(R) = \mathrm{Conv}(R_M)$. Hence, $X \subseteq \mathrm{Conv}(R_M)$. By Proposition 1(ii), $\mathrm{Conv}(X) \cap p(R) \subseteq X$. Since, $p(R_M) \subseteq p(R)$, $\mathrm{Conv}(X) \cap p(R_M) \subseteq X$. By Proposition 1, $X \in \mathrm{E}(R_M)$.

(c) If $X \in \mathrm{E}(R)$, then by single-peakedness, for each $i \in N$ such that $p(R_i) \in \mathrm{Conv}(X)$, $b_{\mathrm{Conv}(X)}(R_i) = p(R_i)$ and by Proposition 1(ii), $\mathrm{Conv}(X) \cap p(R) \subseteq X$. Hence, $b_{\mathrm{Conv}(X)}(R_i) = b_X(R_i)$. By single-peakedness, for each $i \in N$ such that $p(R_i) \notin \mathrm{Conv}(X)$, $b_{\mathrm{Conv}(X)}(R_i) \in \{\underline{X}, \bar{X}\}$. Since $\{\underline{X}, \bar{X}\} \subseteq X$, $b_{\mathrm{Conv}(X)}(R_i) = b_X(R_i)$. Moreover, since $\mathrm{Conv}(X)$ is a closed interval and (trivially) $\mathrm{Conv}(X) = X \cup \mathrm{Conv}(X)$, for each $i \in N$, $\mathrm{Conv}(X)\ I_i\ X$.

Next, let $Y \in \mathcal{C}$ such that for each $i \in N$, $Y\ I_i\ X$. By Proposition 1(i), $X, Y \subseteq \mathrm{Conv}(R)$, hence, $p(R_1) \leq \underline{X} \leq \bar{X}$ and $p(R_1) \leq \underline{Y} \leq \bar{Y}$. By single-peakedness of R_1, $[b_X(R_1) = \underline{X}$ and $b_Y(R_1) = \underline{Y}]$ and $[w_X(R_1) = \bar{X}$ and $w_Y(R_1) = \bar{Y}]$. Since $X\ I_1\ Y$, $b_X(R_1) = b_Y(R_1)$ and $w_X(R_1) = w_Y(R_1)$. Therefore, $\mathrm{Conv}(X) = \mathrm{Conv}(Y)$. □

Proof of Proposition 2. Let $F \in \mathcal{F}^N$ ($N \in \mathcal{P}$, $|N| \geq 3$) satisfy *efficiency* and *replacement-dominance*.

(a) Let $j \in N$ and $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\mathrm{Conv}(\bar{R}) \subseteq \mathrm{Conv}(R)$.

Case 1 ($\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$).

Case 1.1 ($j \in N \setminus \{1, n\}$). By *efficiency*, $F(\bar{R}) \in \mathrm{E}(\bar{R})$ and $F(R) \in \mathrm{E}(R)$. Note that $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R) = \mathrm{Conv}(R_{-j})$, and by Corollary 1(b), $F(\bar{R}), F(R) \in \mathrm{E}(R_{-j})$. By *replacement-dominance*, for each $i \in N \setminus \{j\}$, $F(\bar{R})\ R_i\ F(R)$ or for each $i \in N \setminus \{j\}$, $F(R)\ R_i\ F(\bar{R})$.

If for some $k \in N \setminus \{j\}$ $F(\bar{R})\ P_k\ F(R)$, then for each $i \in N \setminus \{j\}$, $F(\bar{R})\ R_i\ F(R)$, contradicting $F(R) \in \mathrm{E}(R_{-j})$. If for some $k \in N \setminus \{j\}$ $F(R)\ P_k\ F(\bar{R})$, then for each $i \in N \setminus \{1\}$, $F(R)\ R_i\ F(\bar{R})$, contradicting $F(\bar{R}) \in \mathrm{E}(R_{-j})$. Hence, for each $i \in N \setminus \{j\}$, $F(R)\ I_i\ F(\bar{R})$. By Corollary 1(c), $\mathrm{Conv}(F(\bar{R})) = \mathrm{Conv}(F(R))$ and since we represent any efficient set by its convex hull, $F(\bar{R}) = F(R)$.

Case 1.2 ($j \in \{1, n\}$). By symmetry of arguments, assume that $j = 1$. Starting from R, change agent 2's preferences to R_1 and define $R^1 := (R_{-2}, R_1)$ with $\mathrm{Conv}(R^1) = \mathrm{Conv}(R)$. By Case 1.1, $F(R^1) = F(R)$. Next, change agent 1's preferences to $\bar{R}_1$ and define $R^2 := (R^1_{-1}, \bar{R}_1)$ with $\mathrm{Conv}(R^2) = \mathrm{Conv}(R^1)$. By Case 1.1 (with agent 2 in the role of agent 1), $F(R^2) = F(R^1)$. Finally, change agent 2's preferences back to R_2 and obtain $\bar{R} = (R^2_{-2}, R_2)$ with $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R^2)$. By Case 1.1, $F(\bar{R}) = F(R^2)$. Therefore, $F(\bar{R}) = F(R)$.

Case 2 ($\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$). Either (i) $j = 1$ and $p(R_1) < p(R_2)$ or (ii) $j = n$ and $p(R_{n-1}) < p(R_n)$. By symmetry of arguments, we consider Case (i).

Case 2.1 ($\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R_{-1})$). By *efficiency*, $F(\bar{R}) \in \mathrm{E}(\bar{R})$ and $F(R) \in \mathrm{E}(R)$. By Corollary 1(b), $F(\bar{R}) \in \mathrm{E}(R_{-1})$.

Assume that $F(R) \subseteq \text{Conv}(R_{-1})$. Since $F(R) \in \text{E}(R)$, by Proposition 1(ii), $\text{Conv}(F(R)) \cap p(R) \subseteq F(R)$. Hence, $\text{Conv}(F(R)) \cap p(R_{-1}) \subseteq F(R)$ and by Proposition 1, $F(R) \in \text{E}(R_{-1})$. By *replacement-dominance*, for each $i \in N \setminus \{1\}$, $F(\bar{R})\, R_i\, F(R)$ or for each $i \in N \setminus \{1\}$, $F(R)\, R_i\, F(\bar{R})$. If for some $j \in N \setminus \{1\}$ $F(\bar{R})\, P_j\, F(R)$, then for each $i \in N \setminus \{1\}$, $F(\bar{R})\, R_i\, F(R)$, contradicting $F(R) \in \text{E}(R_{-1})$. If for some $j \in N \setminus \{1\}$ $F(R)\, P_j\, F(\bar{R})$, then for each $i \in N \setminus \{1\}$, $F(R)\, R_i\, F(\bar{R})$, contradicting $F(\bar{R}) \in \text{E}(R_{-1})$. Hence, for each $i \in N \setminus \{1\}$, $F(R)\, I_i\, F(\bar{R})$. By Corollary 1(c), $\text{Conv}(F(\bar{R})) = \text{Conv}(F(R))$ and since we always represent any efficient set by its convex hull, $F(\bar{R}) = F(R)$.

Assume that $F(R) \not\subseteq \text{Conv}(R_{-1})$. Then, $\underline{F}(R) < \underline{p}(R_{-1}) \leq \underline{F}(\bar{R}) \leq p(R_n)$. Hence, $w_{F(R)}(R_n) = \{\underline{F}(R)\}$ and $w_{F(\bar{R})}(R_n) = \{\underline{F}(\bar{R})\}$. By single-peakedness, $w_{F(\bar{R})}(R_n)\ P_n\ w_{F(R)}(R_n)$. By *replacement-dominance*, for each $i \in N \setminus \{1\}$, $F(\bar{R})\ R_i\ F(R)$ or for each $i \in N \setminus \{1\}$, $F(R)\, R_i\, F(\bar{R})$. Hence, $F(\bar{R})\, P_n\, F(R)$ and for each $i \in N \setminus \{1\}$, $F(\bar{R})\, R_i\, F(R)$.

Case 2.2 ($\text{Conv}(R_{-1}) \subsetneq \text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$). Hence, $p(R_1) < p(\bar{R}_1) < p(R_2)$. Starting from R, change agent 2's preferences to $\bar{R}_1$ and define $R^1 := (R_{-2}, \bar{R}_1)$ with $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1, $F(R^1) = F(R)$. Next, change agent 1's preferences to $\bar{R}_1$ and define $R^2 := (R^1_{-1}, \bar{R}_1)$ with $\text{Conv}(R^2) = \text{Conv}(R^1_{-1})$. By Case 2.1, for each $i \in N \setminus \{1, 2\}$, $F(R^2)\, R_i\, F(R^1)$. Finally, change agent 2's preferences back to R_2 and obtain $\bar{R} = (R^2_{-2}, R_2)$ with $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$. By Case 1, $F(\bar{R}) = F(R^2)$. Therefore, for each $i \in N \setminus \{1, 2\}$, $F(\bar{R})\ R_i\ F(R)$. In particular, $F(\bar{R})\ R_n\ F(R)$. Since agent n has the largest peak, *efficiency* and single-peakedness imply $\underline{F}(R) \leq \underline{F}(\bar{R})$ and $\bar{F}(R) \leq \bar{F}(\bar{R})$. Hence, either $F(\bar{R}) = F(R)$ or $F(\bar{R})\, P_n\, F(R)$. Then, by *replacement-dominance*, for each $i \in N \setminus \{1\}$ (including agent 2 now), $F(\bar{R})\ R_i\ F(R)$.

(b) Let $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(R) = \text{Conv}(\bar{R})$.

Case 1 (one agent's preferences changed). Then, $F(R) = F(\bar{R})$ follows from Proposition 2(a).

Case 2 (two agents' preferences swapped). Since $|N| \geq 3$, any preference swap between two agents can be constructed sequentially (in at most four steps) via unilateral changes that do not change the convex hull of peaks. Hence, by Case 1 (applied sequentially), $F(R) = F(\bar{R})$.

Case 3 (all remaining cases). Let $\pi : N \to N$ be a permutation of the agents such that $\underline{p}(\bar{R}) = p(\bar{R}_{\pi(1)}) \leq \ldots \leq p(\bar{R}_{\pi(n)}) = \bar{p}(\bar{R})$.

Starting from R, construct R^1 by sequentially replacing preferences such that for each $i \in N$, $R^1_i = \bar{R}_{\pi(i)}$. Since these stepwise changes of the preferences never change the convex hull of peaks, we have $\text{Conv}(R^1) = \text{Conv}(R)$, and by Case 1 (applied sequentially), $F(R^1) = F(R)$. Finally, permute preferences such that each agent $\pi(i)$ obtains the preferences of agent i, i.e., the new profile R^2 is such that for each $i \in N$, $R^2_{\pi(i)} = R^1_i$. Hence, for each $i \in N$, $R^2_{\pi(i)} = \bar{R}_{\pi(i)}$ and $R^2 = \bar{R}$. Since all permutations can be obtained via sequential pairwise swaps, by Case 2, $F(\bar{R}) = F(R)$. $\square$

Proof of Proposition 3. Let $F \in \mathcal{F}^N$ ($N \in \mathcal{P}$, $|N| \geq 3$) satisfy *efficiency* and *replacement-dominance* and assume, by contradiction, that F is not a choice function. Hence, there exists $R \in \mathcal{R}^N$ such that $F(R) = [x, y]$ with $x < y$. By Proposition 2(b), F satisfies *extreme-peaks-onliness*. Assume that $N = \{1, 2, 3\}$ and $p(R_1) \leq p(R_2) \leq p(R_3)$ (by *extreme-peaks-onliness* this is without loss of generality: for larger populations $N' \supsetneq N$, the proof only changes in that agents in $N' \setminus N$ take the same role as agent 2). By *efficiency*, $p(R_1) \leq x < y \leq p(R_3)$.

We divide the interval $[x, y]$ into four equal parts and label the three new points $z_1 = \big(x + \frac{1}{4}(y - x)\big)$, $z_2 = \big(x + \frac{1}{2}(y - x)\big)$, and $z_3 = \big(x + \frac{3}{4}(y - x)\big)$. By *extreme-peaks-onliness*, it is without loss of generality to assume that $p(R_2) = z_2$.

Starting from R, change agent 3's preferences to R^1_3 such that $p(R^1_3) = z_3$, and define $R^1 := (R_{-3}, R^1_3)$ with $\text{Conv}(R^1) \subsetneq \text{Conv}(R)$. Denote $F(R^1) = [\tilde{x}, \tilde{y}]$. By Proposition 2(a), for $i = 1, 2$, $F(R^1)\, R_i\, F(R)$. Then, $b_{F(R)}(R_1) = \{x\}$, $w_{F(R)}(R_1) = \{y\}$, and $b_{F(R)}(R_2) = \{p(R_2)\}$, together with *efficiency* and single-peakedness, imply that

$$p(R_1) \leq \tilde{x} \leq x < z_1 < p(R_2) = z_2 \leq \tilde{y} \leq z_3 < y \leq p(R_3).$$

Starting from R, change agent 1's preferences to R^2_1 such that $p(R^2_1) = z_1$, and define $R^2 := (R_{-1}, R^2_1)$ with $\text{Conv}(R^2) \subsetneq \text{Conv}(R)$. Denote $F(R^2) = [\hat{x}, \hat{y}]$. By Proposition 2(a), for $i = 2, 3$, $F(R^2)\, R_i\, F(R)$. Then, $b_{F(R)}(R_3) = \{y\}$, $w_{F(R)}(R_3) = \{x\}$, and $b_{F(R)}(R_2) = \{p(R_2)\}$, together with *efficiency* and single-peakedness, imply that

$$p(R_3) \geq \hat{y} \geq y > z_3 > p(R_2) = z_2 \geq \hat{x} \geq z_1 > x \geq p(R_1).$$

Hence,

$$\tilde{x} \leq x < z_1 \leq \hat{x} \leq z_2 \leq \tilde{y} \leq z_3 < y \leq \hat{y}. \tag{1}$$

Let $\tilde{R}^1$ be such that $\tilde{R}^1_1 = R^1_1$, $[p(\tilde{R}^1_2) = z_1$ and $\tilde{x}\, \tilde{P}^1_2\, \tilde{y}]$, $[p(\tilde{R}^1_3) = z_3$ and $\hat{y}\, \tilde{P}^1_3\, \hat{x}]$, and $\text{Conv}(\tilde{R}^1) = \text{Conv}(R^1)$. By *extreme-peaks-onliness*, $F(\tilde{R}^1) = F(R^1) = [\tilde{x}, \tilde{y}]$. By Inequality (1),

$b_{F(\tilde{R}^1)}(\tilde{R}^1_2) = p(\tilde{R}^1_2) = \{z_1\}$,	$w_{F(\tilde{R}^1)}(\tilde{R}^1_2) = \{\tilde{y}\}$,
$b_{F(\tilde{R}^1)}(\tilde{R}^1_3) = \{\tilde{y}\}$,	$w_{F(\tilde{R}^1)}(\tilde{R}^1_3) = \{\tilde{x}\}$.

Let $\hat{R}^2$ be such that $\hat{R}^2_1 = R_3$, $\hat{R}^2_2 = \tilde{R}^1_2$, $\hat{R}^2_3 = \tilde{R}^1_3$, and $\text{Conv}(\hat{R}^2) = \text{Conv}(R^2)$. By *extreme-peaks-onliness*, $F(\hat{R}^2) = F(R^2) = [\hat{x}, \hat{y}]$. By Inequality (1),

$b_{F(\hat{R}^2)}(\tilde{R}^1_2) = \{\hat{x}\}$,	$w_{F(\hat{R}^2)}(\tilde{R}^1_2) = \{\hat{y}\}$,
$b_{F(\hat{R}^2)}(\tilde{R}^1_3) = p(\tilde{R}^1_3) = \{z_3\}$,	$w_{F(\hat{R}^2)}(\tilde{R}^1_3) = \{\hat{x}\}$.

Note that $\tilde{R}^1_{-1} = \hat{R}^2_{-1}$ and by *replacement-dominance* either for each $i = 2, 3$, $F(\tilde{R}^1)\ \tilde{R}^1_i\ F(\hat{R}^2)$ or for each $i = 2, 3$, $F(\hat{R}^2)\ \tilde{R}^1_i\ F(\tilde{R}^1)$. However, on the one hand side,

$$b_{F(\tilde{R}^1)}(\tilde{R}^1_2) = z_1\ \tilde{R}^1_2\ \hat{x} = b_{F(\hat{R}^2)}(\tilde{R}^1_2) \text{ and } w_{F(\tilde{R}^1)}(\tilde{R}^1_2) = \tilde{y} = \tilde{P}^1_2 \hat{y} = w_{F(\hat{R}^2)}(\tilde{R}^1_2),$$

and on the other hand side,

$$b_{F(\hat{R}^2)}(\tilde{R}^1_3) = z_3\ \tilde{R}^1_3\ \tilde{y} = b_{F(\tilde{R}^1)}(\tilde{R}^1_3) \text{ and } w_{F(\hat{R}^2)}(\tilde{R}^1_3) = \hat{x}\ \tilde{P}^1_3\ \tilde{x} = w_{F(\tilde{R}^1)}(\tilde{R}^1_3).$$

Thus, $F(\tilde{R}^1)\ \tilde{P}^1_2\ F(\hat{R}^2)$ and $F(\hat{R}^2)\ \tilde{P}^1_3\ F(\tilde{R}^1)$; a contradiction. □

Proof of Proposition 4. Let $F \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*.

(a) Let $N, M \in \mathcal{P}$, $N \subseteq M$, and $R \in \mathcal{R}^M$. By *efficiency*, $F(R) \in \text{E}(R)$ and $F(R_N) \in \text{E}(R_N)$. By *population-monotonicity*, for each $i \in N$, $F(R)\ R_i\ F(R_N)$ or for each $i \in N$, $F(R_N)\ R_i\ F(R)$. If for each $i \in N$, $F(R)\ R_i\ F(R_N)$, then (since $F(R_N) \in \text{E}(R_N)$) for each $i \in N$, $F(R_N)\ I_i\ F(R)$. Therefore, for each $i \in N$, $F(R_N)\ R_i\ F(R)$. In particular, if $\text{Conv}(R_N) = \text{Conv}(R)$, then by $F(R) \in \text{E}(R)$ and Corollary 1(b), $F(R) \in \text{E}(R_N)$. Since for each $i \in N$, $F(R_N)\ R_i\ F(R)$, and $[F(R) \in \text{E}(R_N)$ and $F(R_N) \in \text{E}(R_N)]$, for each $i \in N$, $F(R_N)\ I_i\ F(R)$. By Corollary 1(c), $\text{Conv}(F(R_N)) = \text{Conv}(F(R))$, and since we always represent any efficient set by its convex hull, $F(R_N) = F(R)$.

For the remainder of the proof let $N \in \mathcal{P}$, $|N| \geq 3$, and $F \in \mathcal{F}^N$.

(b) Let $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R))$.

Case 1 ($\text{Conv}(\bar{R}) = \text{Conv}(R)$).

Case 1.1 ($j \in N \setminus \{1, n\}$). Since $\text{Conv}(\bar{R}) = \text{Conv}(R) = \text{Conv}(R_{-j})$, (a), $F(R_{-j}) = F(R)$ and $F(R_{-j}) = F(\bar{R})$. Therefore, $F(\bar{R}) = F(R)$.

Case 1.2 ($j \in \{1, n\}$). By symmetry of arguments, assume that $j = 1$. Starting from R, change agent 2's preferences to R_1 and define $R^1 := (R_{-2}, R_1)$ with $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1.1, $F(R^1) = F(R)$. Next, change agent 1's preferences to $\bar{R}_1$ and define $R^2 := (R^1_{-1}, \bar{R}_1)$ with $\text{Conv}(R^2) = \text{Conv}(R^1)$. By Case 1.1 (with agent 2 in the role of agent 1), $F(R^2) = F(R^1)$. Finally, change agent 2's preferences back to R_2 and obtain $\bar{R} = (R^2_{-2}, R_2)$ with $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$. By Case 1.1, $F(\bar{R}) = F(R^2)$. Therefore, $F(\bar{R}) = F(R)$.

Case 2 ($\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$). Either (i) $j = 1$ and $p(R_1) < p(R_2)$ or (ii) $j = n$ and $p(R_{n-1}) < p(R_n)$. By symmetry of arguments, we consider Case (i).

Case 2.1 ($\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$). Starting from R, remove agent 1 from profile R to obtain profile R_{-1}. By (a), for each $i \in N \setminus \{1\}$, $F(R_{-1})\, R_i\, F(R)$. Next, add agent 1 with preferences $\bar{R}_1$ to obtain profile $\bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$, by (a), $F(\bar{R}) = F(R_{-1})$. Therefore, for each $i \in N \setminus \{1\}$, $F(\bar{R})\, R_i\, F(R)$.

Case 2.2 ($\text{Conv}(R_{-1}) \subsetneq \text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$). Hence, $p(R_1) < p(\bar{R}_1) < p(R_2)$. Starting from $\bar{R}$, change agent 2's preferences to $\bar{R}_1$ and define $\bar{R}^1 := (\bar{R}_{-2}, \bar{R}_1)$ with $\text{Conv}(\bar{R}^1) = \text{Conv}(\bar{R})$. By Case 1, $F(\bar{R}^1) = F(\bar{R})$. Furthermore, since $\text{Conv}(\bar{R}^1) = \text{Conv}(\bar{R}^1_{-1})$, by (a), $F(\bar{R}^1) = F(\bar{R}^1_{-1})$.

Starting from R, change agent 2's preferences to $\bar{R}_1$ and define $R^2 := (R_{-2}, \bar{R}_1)$ with $\text{Conv}(R^2) = \text{Conv}(R)$. By Case 1, $F(R^2) = F(R)$. Furthermore, by (a), for each $i \in N \setminus \{1\}$, $F(R^2_{-1})\, R^2_i\, F(R^2)$. Since $\bar{R}^1_{-1} = R^2_{-1}$, for each $i \in N \setminus \{1\}$, $F(\bar{R}) = F(\bar{R}^1) = F(\bar{R}^1_{-1}) = F(R^2_{-1})\, R^2_i\, F(R^2) = F(R)$. Since for all $i \in N \setminus \{1, 2\}$, $R^2_i = R_i$, we have $F(\bar{R})\, R_i\, F(R)$ and now only need to show that $F(\bar{R})\, R_2\, F(R)$ (or that $F(\bar{R}) = F(R)$).

We also know that $F(\bar{R})\, R^2_2\, F(R)$ and $F(\bar{R})\, R^2_n\, F(R)$. Since agent n has the largest peak at R and $\bar{R}$, *efficiency* and single-peakedness imply $\underline{F}(R) \leq \underline{F}(\bar{R}) \leq p(R^2_n)$ and $\bar{F}(R) \leq \bar{F}(\bar{R}) \leq p(R^2_n)$. By *efficiency*, $p(R^2_2) \leq \underline{F}(\bar{R})$.

Case 2.2.1 ($\underline{F}(R) < p(R^2_2) < \bar{F}(\bar{R})$). Assume that R^2_2 was chosen such that $w_{F(R)}(R^2_2) = \{\underline{F}(R)\}\, P^2_2\, \{\bar{F}(\bar{R})\} = w_{F(\bar{R})}(R^2_2)$ instead of $R^2_2 = \bar{R}_1$. Then, we would have gotten the same results, in particular $F(\bar{R})\, R^2_2\, F(R)$; a contradiction.

Case 2.2.2 ($\underline{F}(R) < p(R^2_2) = \bar{F}(\bar{R})$). Then, $\underline{F}(R) \leq \bar{F}(R) \leq p(R^2_2) = \underline{F}(\bar{R}) = \bar{F}(\bar{R})$. Since, $p(R_2) > p(R^2_2)$, by *efficiency* and single-peakedness, $F(\bar{R})\, R_2\, F(R)$.

Case 2.2.3 ($p(R^2_2) \leq \underline{F}(R)$). Then, $F(\bar{R})\, R^2_2\, F(R)$ implies $p(R^2_2) \leq \underline{F}(\bar{R}) \leq \underline{F}(R)$ and $p(R^2_2) \leq \bar{F}(\bar{R}) \leq \bar{F}(R)$. Since $\underline{F}(R) \leq \underline{F}(\bar{R}) \leq p(R^2_n)$ and $\bar{F}(R) \leq \bar{F}(\bar{R}) \leq p(R^2_n)$, we then have $F(R) = F(\bar{R})$.

(c) The proof proceeds exactly as the proof of Proposition 2(b), the only difference being that in Case 1, (b) is used instead of Proposition 2(a).

(d) Let $j \in N$, and $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $p(R_j) \leq p(\bar{R}_j)$.

Case 1 ($\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$). Hence, $\underline{p}(R) \leq \underline{p}(\bar{R}) \leq \bar{p}(R) = \bar{p}(\bar{R})$ and, without loss of generality, $j \in N \setminus \{n\}$. By (b), for each $i \in N \setminus \{j\}$, $F(\bar{R})\, R_i\, F(R)$. Since $p(R_n) = \bar{p}(R) = \bar{p}(\bar{R})$, by $F(R)\, R_n\, F(\bar{R})$, *efficiency*, and single-peakedness, we have $\underline{F}(R) \leq \underline{F}(\bar{R}) \leq p(R_n)$ and $\bar{F}(R) \leq \bar{F}(\bar{R}) \leq p(R_n)$.

Case 2 ($\mathrm{Conv}(R) \subseteq \mathrm{Conv}(\bar{R})$). Hence, $\underline{p}(R) = \underline{p}(\bar{R}) \leq \bar{p}(R) \leq \bar{p}(\bar{R})$ and, without loss of generality, $j \in N \setminus \{1\}$. By a symmetric argument to Case 1 (with agent 1 in the role of agent n), $\underline{F}(R) \leq \underline{F}(\bar{R})$ and $\bar{F}(R) \leq \bar{F}(\bar{R})$.

Case 3 ($\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) < \bar{p}(\bar{R})$). Starting from R, change agent j's preferences to R^1_j such that $p(R^1_j) = \underline{p}(\bar{R})$, and define $R^1 := (R_{-j}, R^1_j)$. By Case 1, $\underline{F}(R) \leq \underline{F}(R^1)$ and $\bar{F}(R) \leq \bar{F}(R^1)$. Next, change agent j's preferences to $\bar{R}_j$ and obtain $\bar{R} = (R^1_{-j}, \bar{R}_j)$. By Case 2, $\underline{F}(R^1) \leq \underline{F}(\bar{R})$ and $\bar{F}(R^1) \leq \bar{F}(\bar{R})$. Hence, $\underline{F}(R) \leq \underline{F}(\bar{R})$ and $\bar{F}(R) \leq \bar{F}(\bar{R})$.

(e) Let $j \in N$, and $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$. By (c) and (d), $F \in \mathcal{F}^N$ satisfies *extreme-peaks-onliness* and *peak-monotonicity*. By symmetry of arguments, we prove (i).

Let $[p(R_j) < \underline{F}(R)$ and $p(\bar{R}_j) \leq \underline{F}(R)]$ or $[p(R_j) > \underline{F}(R)$ and $p(\bar{R}_j) \geq \underline{F}(R)]$. Then, $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ or $\mathrm{Conv}(\bar{R}) \supsetneq \mathrm{Conv}(R)$.

Case 1 ($p(R_j) < \underline{F}(R)$ and $p(\bar{R}_j) \leq \underline{F}(R)$). Hence, $p(R_j) \neq \bar{p}(R)$.

Case 1.1 ($\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$). By *extreme-peaks-onliness*, $F(R) = F(\bar{R})$.

Case 1.2 ($\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$). Hence, $j = 1$ and $\underline{p}(R) = p(R_1) < \underline{p}(\bar{R}) \leq p(\bar{R}_1) \leq \underline{F}(R)$. By *peak-monotonicity*, $\underline{F}(R) \leq \underline{F}(\bar{R})$ and $\bar{F}(R) \leq \bar{F}(\bar{R})$. Starting from $\bar{R}$, change agent 2's preferences to $\bar{R}^1_2$ such that $p(\bar{R}^1_2) = \underline{p}(\bar{R})$, and define $\bar{R}^1 := (\bar{R}_{-2}, \bar{R}^1_2)$ with $\mathrm{Conv}(\bar{R}^1) = \mathrm{Conv}(\bar{R})$. By *extreme-peaks-onliness*, $F(\bar{R}^1) = F(\bar{R})$. Furthermore, since $\mathrm{Conv}(\bar{R}^1) = \mathrm{Conv}(\bar{R}^1_{-1})$, by (a), $F(\bar{R}^1) = F(\bar{R}^1_{-1})$.

Starting from R, change agent 2's preferences to $R^2_2 = \bar{R}^1_2$ and define $R^2 := (R_{-2}, R^2_2)$ with $\mathrm{Conv}(R^2) = \mathrm{Conv}(R)$. By *extreme-peaks-onliness*, $F(R^2) = F(R)$. Furthermore, by (a), for each $i \in N \setminus \{1\}$, $F(R^2_{-1}) \, R^2_i \, F(R^2)$. Since $\bar{R}^1_{-1} = R^2_{-1}$, for each $i \in N \setminus \{1\}$, $F(\bar{R}) = F(\bar{R}^1) = F(\bar{R}^1_{-1}) = F(R^2_{-1}) \, R^2_i \, F(R^2) = F(R)$. In particular, $F(\bar{R}) \, R^2_2 \, F(R)$. Since, $p(R^2_2) \leq \underline{F}(R)$, by *efficiency* and single-peakedness $\underline{F}(\bar{R}) \leq \underline{F}(R)$ and $\bar{F}(\bar{R}) \leq \bar{F}(R)$. Hence, $F(R) = F(\bar{R})$.

Case 1.3 ($\mathrm{Conv}(\bar{R}) \supsetneq \mathrm{Conv}(R)$). Hence, $j = 1$ and $\underline{p}(\bar{R}) = p(\bar{R}_1) < \underline{p}(R) = p(R_1) \leq \underline{F}(R)$. By *peak-monotonicity*, $\underline{F}(\bar{R}) \leq \underline{F}(R)$ and $\bar{F}(\bar{R}) \leq \bar{F}(R)$.

If $p(R_1) \leq \underline{F}(\bar{R})$, then the proof proceeds as in Case 1.2 with the roles of R and $\bar{R}$ switched. Assume that $\underline{F}(\bar{R}) < p(R_1)$. Then, starting from $\bar{R}$, change agent 2's preferences to $\bar{R}^1_2$ such that $p(\bar{R}^1_2) = p(R_1)$ and $\underline{F}(\bar{R}) \, \bar{P}^1_2 \, \underline{F}(R)$, and define $\bar{R}^1 := (\bar{R}_{-2}, \bar{R}^1_2)$ with $\mathrm{Conv}(\bar{R}^1) = \mathrm{Conv}(\bar{R})$. By *extreme-peaks-onliness*, $F(\bar{R}^1) = F(\bar{R})$.

Starting from R, change agent 2's preferences to $R^2_2 = \bar{R}^1_2$ and define $R^2 := (R_{-2}, R^2_2)$ with $\mathrm{Conv}(R^2) = \mathrm{Conv}(R)$. By *extreme-peaks-onliness*, $F(R^2) = F(R)$. Since $\bar{R}^1_{-1} = R^2_{-1}$ and $\mathrm{Conv}(R^2) \subsetneq \mathrm{Conv}(\bar{R}^1)$, by (b), for each $i \in N \setminus \{1\}$, $F(R^2) \, \bar{R}^1_i \, F(\bar{R}^1)$. In particular, $F(R) \, \bar{R}^1_2$

$F(\bar{R})$, which implies $b_{F(R)}(\bar{R}_2^1)\ \bar{R}_2^1\ b_{F(\bar{R})}(\bar{R}_2^1)$. However, $\underline{F}(\bar{R})\ \bar{P}_2^1\ \underline{F}(R)$ implies $b_{F(\bar{R})}(\bar{R}_2^1)\ \bar{P}_2^1\ b_{F(R)}(\bar{R}_2^1) = \underline{F}(R)$; a contradiction.

Case 2 ($p(R_j) > \underline{F}(R)$ and $p(\bar{R}_j) \geq \underline{F}(R)$). Hence, $p(R_j) \neq \underline{p}(R)$.

Case 2.1 ($\text{Conv}(\bar{R}) = \text{Conv}(R)$). By *extreme-peaks-onliness*, $F(R) = F(\bar{R})$.

Case 2.2 ($\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$). Hence, $j = n$ and $\underline{F}(R) \leq p(\bar{R}_n) \leq \bar{p}(\bar{R}) < p(R_n) = \bar{p}(R)$. By *peak-monotonicity*, $\underline{F}(\bar{R}) \leq \underline{F}(R)$ and $\bar{F}(\bar{R}) \leq \bar{F}(R)$. Starting from $\bar{R}$, change agent 2's preferences to $\bar{R}_2^1$ such that $p(\bar{R}_2^1) = \bar{p}(\bar{R})$, and define $\bar{R}^1 := (\bar{R}_{-2}, \bar{R}_2^1)$ with $\text{Conv}(\bar{R}^1) = \text{Conv}(\bar{R})$. By *extreme-peaks-onliness*, $F(\bar{R}^1) = F(\bar{R})$. Furthermore, since $\text{Conv}(\bar{R}^1) = \text{Conv}(\bar{R}^1_{-n})$, by (a), $F(\bar{R}^1) = F(\bar{R}^1_{-n})$.

Starting from R, change agent 2's preferences to $R_2^2 = \bar{R}_2^1$ and define $R^2 := (R_{-2}, R_2^2)$ with $\text{Conv}(R^2) = \text{Conv}(R)$. By *extreme-peaks-onliness*, $F(R^2) = F(R)$. Furthermore, by (a), for each $i \in N \setminus \{1\}$, $F(R^2_{-n})\ R_i^2\ F(R^2)$. Since $\bar{R}^1_{-n} = R^2_{-n}$, for each $i \in N \setminus \{n\}$, $F(\bar{R}) = F(\bar{R}^1) = F(\bar{R}^1_{-n}) = F(R^2_{-n})\, R_i^2\, F(R^2) = F(R)$. In particular, $F(\bar{R})\, R_2^2\, F(R)$. Since, $p(R_2^2) \geq \underline{F}(R)$, by *efficiency* and single-peakedness $\underline{F}(\bar{R}) \geq \underline{F}(R)$. Hence, $\underline{F}(R) = \underline{F}(\bar{R})$.

Case 2.3 ($\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$). Hence, $j = n$ and $\underline{F}(R) \leq p(R_n) = \bar{p}(R) < p(\bar{R}_n) = \bar{p}(\bar{R})$. By *peak-monotonicity*, $\underline{F}(R) \leq \underline{F}(\bar{R})$ and $\bar{F}(R) \leq \bar{F}(\bar{R})$.

If $p(R_n) \geq \underline{F}(\bar{R})$, then the proof proceeds as in Case 2.2 with the roles of R and $\bar{R}$ switched. Assume that $\underline{F}(\bar{R}) > p(R_n)$. Then, starting from $\bar{R}$, change agent 2's preferences to $\bar{R}_2^1$ such that $p(\bar{R}_2^1) = p(R_n)$ and $\underline{F}(\bar{R})\ \bar{P}_2^1\ \underline{F}(R)$, and define $\bar{R}^1 := (\bar{R}_{-2}, \bar{R}_2^1)$ with $\text{Conv}(\bar{R}^1) = \text{Conv}(\bar{R})$. By *extreme-peaks-onliness*, $F(\bar{R}^1) = F(\bar{R})$.

Starting from R, change agent 2's preferences to $R_2^2 = \bar{R}_2^1$ and define $R^2 := (R_{-2}, R_2^2)$ with $\text{Conv}(R^2) = \text{Conv}(R)$. By *extreme-peaks-onliness*, $F(R^2) = F(R)$. Since $\bar{R}^1_{-n} = R^2_{-n}$ and $\text{Conv}(R^2) \subsetneq \text{Conv}(\bar{R}^1)$, by (b), for each $i \in N \setminus \{1\}$, $F(R^2)\ \bar{R}_i^1\ F(\bar{R}^1)$. In particular, $F(R)\, \bar{R}_2^1\, F(\bar{R})$, which implies $b_{F(R)}(\bar{R}_2^1)\, \bar{R}_2^1\, b_{F(\bar{R})}(\bar{R}_2^1)$. However, $\underline{F}(\bar{R})\, \bar{P}_2^1\, \underline{F}(R)$, which implies $b_{F(\bar{R})}(\bar{R}_2^1)\ \bar{P}_2^1\ b_{F(R)}(\bar{R}_2^1) = \underline{F}(R)$; a contradiction. □

C Proof of Theorem 1

Throughout this appendix, for $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$, we assume, without loss of generality, that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$.

The following lemma is crucial in the proof of Theorem 1.

Lemma 2. *Let $F \in \mathcal{F}$ satisfy efficiency and population-monotonicity. Let $N \in \mathcal{P}$, $|N| \geq 3$, and $F \in\in \mathcal{F}^N$. Then, for each $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, if $F(R) = F^{a,b}(R)$, then $F(\bar{R}) = F^{a,b}(\bar{R})$.*

***Proof*.** Let $F \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*. Let $N \in \mathcal{P}$ and $|N| \geq 3$. Then, by Proposition 4(c,e), $F \in\in \mathcal{F}^N$ satisfies *extreme-peaks-onliness* and *uncompromisingness*. It is easy to see that target set correspondence $F^{a,b} \in \mathcal{F}^N$ satisfies *efficiency*, *extreme-peaks-onliness* and *uncompromisingness* as well.[22] Let $R, \bar{R} \in \mathcal{R}^N$ such that $F(R) = F^{a,b}(R)$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$.

Case 1 ($\text{Conv}(\bar{R}) = \text{Conv}(R)$). By *extreme-peaks-onliness* and the definition of $F^{a,b}$, $F(\bar{R}) = F(R) = F^{a,b}(R) = F^{a,b}(\bar{R})$.

Case 2 ($\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)$). By *extreme-peaks-onliness*, it is without loss of generality to assume that $\underline{p}(R) = p(R_1)$, $\underline{p}(\bar{R}) = p(\bar{R}_1)$, and for all $i \in N \setminus \{1\}$, $p(R_i) = \bar{p}(R)$ and $p(\bar{R}_i) = \bar{p}(\bar{R})$. Hence, $R_{-1} = \bar{R}_{-1}$ and $p(R_1) < p(\bar{R}_1) \leq \bar{p}(\bar{R}) = \bar{p}(R)$. By *efficiency* and Proposition 1(i), $p(R_1) \leq \underline{F}(R) \leq \bar{F}(R) \leq \bar{p}(R)$ and $p(\bar{R}_1) \leq \underline{F}(\bar{R}) \leq \bar{F}(\bar{R}) \leq \bar{p}(\bar{R})$.

Case 2.1 ($p(\bar{R}_1) \leq \underline{F}(R)$). Then, $p(R_1) < p(\bar{R}_1) < \underline{F}(R) = \underline{F}^{a,b}(R) = a$. By *uncompromisingness*, $F(\bar{R}) = F(R) = F^{a,b}(R)$. If $a \leq \bar{p}(R) = \bar{p}(\bar{R})$, then $F^{a,b}(R) = [a, b] \cap \text{Conv}(R) = [a, b] \cap \text{Conv}(\bar{R}) = F^{a,b}(\bar{R})$. If $a > \bar{p}(R) = \bar{p}(\bar{R})$, then, $F^{a,b}(R) = \{\bar{p}(R)\} = F^{a,b}(\bar{R})$. Therefore, $F(\bar{R}) = F^{a,b}(\bar{R})$.

Case 2.2 ($\underline{F}(R) < p(\bar{R}_1) \leq \bar{F}(R)$). Then, $p(R_1) \leq \underline{F}(R) < \bar{F}(R)$. By *uncompromisingness*, $\bar{F}(\bar{R}) = \bar{F}(R)$. By *efficiency* and Proposition 1(i), $p(\bar{R}_1) \leq \underline{F}(\bar{R})$. Next, assuming that $\underline{F}(R) < p(\bar{R}_1) < \underline{F}(\bar{R})$ results in a contradiction as follows: since $p(\bar{R}_1) < \underline{F}(\bar{R})$ and $p(R_1) < \underline{F}(\bar{R})$, by *uncompromisingness*, $\underline{F}(R) = \underline{F}(\bar{R}) \neq \underline{F}(R)$, a contradiction. Hence, $\underline{F}(\bar{R}) = p(\bar{R}_1)$ and thus, $F(\bar{R}) = [p(\bar{R}_1), \bar{F}(R)]$. Since $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and $F(R) = [a, b] \cap \text{Conv}(R)$, $F(\bar{R}) = F(R) \cap \text{Conv}(\bar{R}) = [a, b] \cap \text{Conv}(\bar{R})$. Therefore, by the definition of $F^{a,b}$, $F(\bar{R}) = [a, b] \cap \text{Conv}(\bar{R}) = F^{a,b}(\bar{R})$.

Case 2.3 ($\underline{F}(R) \leq \bar{F}(R) < p(\bar{R}_1)$). By the definition of $F^{a,b}$, $a \leq b < p(\bar{R}_1)$. Next, assuming that $\bar{F}(R) < p(\bar{R}_1) < \bar{F}(\bar{R})$ results in a contradiction as follows: since $p(\bar{R}_1) < \bar{F}(\bar{R})$ and $p(R_1) < \bar{F}(\bar{R})$, by *uncompromisingness*, $\bar{F}(R) = \bar{F}(\bar{R}) \neq \bar{F}(R)$, a contradiction. Hence, $\bar{F}(\bar{R}) = p(\bar{R}_1)$ and since $p(\bar{R}_1) \leq \underline{F}(\bar{R}) \leq \bar{F}(\bar{R})$, $F(\bar{R}) = \{p(\bar{R}_1)\}$. Since $a \leq b < p(\bar{R}_1)$, by the definition of $F^{a,b}$, $F(\bar{R}) = \{p(\bar{R}_1)\} = F^{a,b}(\bar{R})$.

[22] Alternatively, we show in the first part of the proof of Theorem 1 that $F^{a,b} \in \mathcal{F}^N$ satisfies *population-monotonicity*. Then, by Proposition 4(c,e), $F^{a,b}$ satisfies *extreme-peaks-onliness* and *uncompromisingness*.

Case 3 ($\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) = \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$). By a symmetric proof to Case 2, $F(\bar{R}) = F^{a,b}(\bar{R})$.

Case 4 ($\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$). Let profile $R^1 \in \mathcal{R}^N$ such that $\underline{p}(R^1) = \underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(R^1) = \bar{p}(R)$. By Case 2, $F(R^1) = F^{a,b}(R^1)$. Next, since $\underline{p}(\bar{R}) = \underline{p}(R^1)$ and $\bar{p}(\bar{R}) < \bar{p}(R^1)$, by Case 3, $F(\bar{R}) = F^{a,b}(\bar{R})$. □

Proof of Theorem 1. If part. By Proposition 1, all target set correspondences satisfy *efficiency*; we next show that they also satisfy *population-monotonicity*. Let $F^{a,b} \in \mathcal{F}$ be a target set correspondence with the same target set $[a, b]$ for all populations. Let $N \in \mathcal{P}$ such that $|N| \geq 2$ and $R \in \mathcal{R}^N$. We prove *population-monotonicity* of $F^{a,b}$ by showing that if $j \in N$ leaves, then for each $i \in N \setminus \{j\}$, $F^{a,b}(R_{-j})\ R_i\ F^{a,b}(R)$.

Case 1 ($\text{Conv}(R_{-j}) = \text{Conv}(R)$). Then, $F^{a,b}(R_{-j}) = F^{a,b}(R)$.

Case 2 ($\text{Conv}(R_{-j}) \subsetneq \text{Conv}(R)$). Then, either (i) $j = 1$ and $\underline{p}(R) = p(R_1) < p(R_2) = \underline{p}(R_{-1})$ or (ii) $j = n$ and $\bar{p}(R_{-1}) = p(R_{n-1}) < p(R_n) = \bar{p}(R)$. By symmetry of arguments, we consider Case (i).

Case 2.1 ($a \leq b < p(R_2)$). Then, $F^{a,b}(R_{-1}) = \{p(R_2)\}$. Furthermore, if $b \leq p(R_1)$, then $F^{a,b}(R) = \{p(R_1)\}$; if $a \leq p(R_1) < b$, then $F^{a,b}(R) = [p(R_1), b]$; and if $p(R_1) < a \leq b$, then $F^{a,b}(R) = [a, b]$. Hence, for each $i \in N \setminus \{1\}$, $b_{F^{a,b}(R_{-1})}(R_i) = w_{F^{a,b}(R_{-1})}(R_i) = \{p(R_2)\}$, $b_{F^{a,b}(R)}(R_i) \in \{p(R_1), b\}$, and $w_{F^{a,b}(R)}(R_i) \in \{p(R_1), a\}$. Thus, for each $i \in N \setminus \{1\}$, $b_{F^{a,b}(R)}(R_i) < b_{F^{a,b}(R_{-1})}(R_i) \leq p(R_i)$ and $w_{F^{a,b}(R)}(R_i) < w_{F^{a,b}(R_{-1})}(R_i) \leq p(R_i)$. By single-peakedness, for each $i \in N \setminus \{1\}$, $F^{a,b}(R_{-1})\ P_i\ F^{a,b}(R)$.

Case 2.2 ($a < p(R_2) \leq b$). Then, $\underline{F}^{a,b}(R) < \underline{F}^{a,b}(R_{-1}) = p(R_2)$ and $\bar{F}^{a,b}(R) = \bar{F}^{a,b}(R_{-1})$. Thus, for each $i \in N \setminus \{1\}$, $\underline{F}^{a,b}(R) < \underline{F}^{a,b}(R_{-1}) \leq p(R_i)$. If $\bar{F}^{a,b}(R_{-1}) < p(R_i)$, then $b = b_{F^{a,b}(R)}(R_i) = b_{F^{a,b}(R_{-1})}(R_i) < p(R_i)$ and $w_{F^{a,b}(R)}(R_i) < w_{F^{a,b}(R_{-1})}(R_i) < p(R_i)$. If $\bar{F}^{a,b}(R_{-1}) \geq p(R_i)$, then $b_{F^{a,b}(R)}(R_i) = b_{F^{a,b}(R_{-1})}(R_i) = \{p(R_i)\}$ and $w_{F^{a,b}(R_{-1})}(R_i) \in F^{a,b}(R_{-1}) \subseteq F^{a,b}(R)$. In both cases, by single-peakedness, $b_{F^{a,b}(R_{-1})}(R_i)\ R_i\ b_{F^{a,b}(R)}(R_i)$ and $w_{F^{a,b}(R_{-1})}(R_i)\ R_i\ w_{F^{a,b}(R)}(R_i)$. Hence, for each $i \in N \setminus \{1\}$, $F^{a,b}(R_{-1})\ R_i\ F^{a,b}(R)$.

Case 2.3 ($p(R_2) \leq a \leq b$). Then, $F^{a,b}(R_{-1}) = F^{a,b}(R)$.

Only if part. Let choice correspondence $F \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*.

Step 1. Let $N \in \mathcal{P}$ and $|N| \geq 3$. Then, by Lemmas 4(c,d), $F \in\in \mathcal{F}^N$ satisfies *extreme-peaks-onliness* and *uncompromisingness*. For each pair of points $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, define $R^{\alpha,\beta} \in \mathcal{R}^N$ such that $\alpha = \underline{p}(R^{\alpha,\beta}) = p(R_1^{\alpha,\beta}) \leq \ldots \leq p(R_n^{\alpha,\beta}) = \bar{p}(R^{\alpha,\beta}) = \beta$. By *efficiency* and Proposition 1(i), $\alpha \leq \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) \leq \beta$.

We prove that there exists a target set correspondence $F^{a,b} \in \mathcal{F}^N$ such that for each profile $R \in \mathcal{R}^N$, $F(R) = F^{a,b}(R)$.

Case 1 (there exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) < \beta$). Define $a := \underline{F}(R^{\alpha,\beta})$ and $b := \bar{F}(R^{\alpha,\beta})$. Since $F(R^{\alpha,\beta}) = [a,b] = [a,b] \cap \text{Conv}(R^{\alpha,\beta})$, by the definition of $F^{a,b}$, $F(R^{\alpha,\beta}) = F^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Starting from $R^{\alpha,\beta}$, change agent 1's preferences to R_1^1 such that

$$p(R_1^1) = \begin{cases} \alpha & \text{if } \alpha \leq \underline{p}(R) \\ \underline{p}(R) & \text{if } \alpha > \underline{p}(R), \end{cases}$$

and define $R^1 := (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $\alpha = p(R_1^{\alpha,\beta}) < \underline{F}(R^{\alpha,\beta})$ and $p(R_1^1) < \underline{F}(R^{\alpha,\beta})$, by *uncompromisingness*, $F(R^1) = F(R^{\alpha,\beta}) = [a,b]$.

Next, change agent n's preferences to R_n^2 such that

$$p(R_n^2) = \begin{cases} \beta & \text{if } \beta \geq \bar{p}(R) \\ \bar{p}(R) & \text{if } \beta < \bar{p}(R), \end{cases}$$

and define $R^2 := (R_{-n}^1, R_n^2)$. Since $\beta = p(R_n^1) > \bar{F}(R^1)$ and $p(R_n^2) > \bar{F}(R^1)$, by *uncompromisingness*, $F(R^2) = F(R^1) = [a,b]$. Since $F(R^2) = [a,b] = [a,b] \cap \text{Conv}(R^2)$, by the definition of $F^{a,b}$, $F(R^2) = F^{a,b}(R^2)$. Since, $F(R^2) = F^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 2, $F(R) = F^{a,b}(R)$.

Case 2 (there exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) < \beta$, and for each $\bar{\alpha} \leq \alpha$ and its associated $R^{\bar{\alpha},\beta} \in \mathcal{R}^N$, $\bar{\alpha} = \underline{F}(R^{\bar{\alpha},\beta}) \leq \bar{F}(R^{\bar{\alpha},\beta}) < \beta$).

Case 2.1 ($\alpha = \underline{F}(R^{\alpha,\beta}) < \bar{F}(R^{\alpha,\beta}) < \beta$). Define $a := -\infty$ and $b := \bar{F}(R^{\alpha,\beta})$. Since $F(R^{\alpha,\beta}) = [\underline{p}(R^{\alpha,\beta}), b] = [a,b] \cap \text{Conv}(R^{\alpha,\beta})$, by the definition of $F^{a,b}$, $F(R^{\alpha,\beta}) = F^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Starting from $R^{\alpha,\beta}$, change agent 1's preferences to R_1^1 such that

$$p(R_1^1) = \begin{cases} \alpha & \text{if } \alpha \leq \underline{p}(R) \\ \underline{p}(R) & \text{if } \alpha > \underline{p}(R), \end{cases}$$

and define $R^1 := (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $\underline{p}(R^1) \leq \alpha$ and $\bar{p}(R^1) = \beta$, as specified in Case 2, and by *extreme-peaks-onliness*, $\underline{p}(R^1) = \underline{F}(R^1)$. Since $\alpha = p(R_1^{\alpha,\beta}) < \bar{F}(R^{\alpha,\beta})$ and $p(R_1^1) < \bar{F}(R^{\alpha,\beta})$, by *uncompromisingness*, $\bar{F}(R^1) = \bar{F}(R^{\alpha,\beta}) = b$. Hence, $F(R^1) = [\underline{p}(R^1), b]$.

Next, change agent n's preferences to R_n^2 such that

$$p(R_n^2) = \begin{cases} \beta & \text{if } \beta \geq \bar{p}(R) \\ \bar{p}(R) & \text{if } \beta < \bar{p}(R), \end{cases}$$

and define $R^2 := (R^1_{-n}, R^2_n)$. Since $\beta = p(R^1_n) > \bar{F}(R^1)$ and $p(R^2_n) > \bar{F}(R^1)$, by *uncompromisingness*, $F(R^2) = F(R^1) = [\underline{p}(R^2), b]$. Since $F(R^2) = [\underline{p}(R^2), b] = [a, b] \cap \text{Conv}(R^2)$, by the definition of $F^{a,b}$, $F(R^2) = F^{a,b}(R^2)$. Since $F(R^2) = F^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 2, $F(R) = F^{a,b}(R)$.

Case 2.2 ($\alpha = \underline{F}(R^{\alpha,\beta}) = \bar{F}(R^{\alpha,\beta}) < \beta$). Define $a := -\infty$ and $b := -\infty$. Since $b < \underline{p}(R^{\alpha,\beta}) = \alpha$ and $F(R^{\alpha,\beta}) = \{\alpha\}$, by the definition of $F^{a,b}$, $F(R^{\alpha,\beta}) = F^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Starting from $R^{\alpha,\beta}$, change agent 1's preferences to R^1_1 such that

$$p(R^1_1) = \begin{cases} \alpha & \text{if } \alpha \leq \underline{p}(R) \\ \underline{p}(R) & \text{if } \alpha > \underline{p}(R), \end{cases}$$

and define $R^1 := (R^{\alpha,\beta}_{-1}, R^1_1)$. Since $\underline{p}(R^1) \leq \alpha$ and $\bar{p}(R^1) = \beta$, as specified in Case 2, and by *extreme-peaks-onliness*, $F(R^1) = \{\underline{p}(R^1)\}$.

Next, change agent n's preferences to R^2_n such that

$$p(R^2_n) = \begin{cases} \beta & \text{if } \beta \geq \bar{p}(R) \\ \bar{p}(R) & \text{if } \beta < \bar{p}(R), \end{cases}$$

and define $R^2 := (R^1_{-n}, R^2_n)$. Since $\beta > p(R^1_n) > \bar{F}(R^1)$ and $p(R^2_n) > \bar{F}(R^1)$, by *uncompromisingness*, $F(R^2) = F(R^1) = \{\underline{p}(R^2)\}$. Since $b < \underline{p}(R^2)$, by the definition of $F^{a,b}$, $F(R^2) = F^{a,b}(R^2)$. Since $F(R^2) = F^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 2, $F(R) = F^{a,b}(R)$.

Case 3 (there exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) = \beta$, and for each $\bar{\beta} \geq \beta$ and its associated $R^{\alpha,\bar{\beta}} \in \mathcal{R}^N$, $\alpha < \underline{F}(R^{\alpha,\bar{\beta}}) \leq \bar{F}(R^{\alpha,\bar{\beta}}) = \bar{\beta}$). The proof of this case is symmetric to Case 2.

Case 4 (for each $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) = \beta$). Define $a := -\infty$ and $b := \infty$. Since for each $\alpha, \beta \in \mathbb{R}$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{F}(R^{\alpha,\beta}) \leq \bar{F}(R^{\alpha,\beta}) = \beta$, by *extreme-peaks-onliness*, for each $R \in \mathcal{R}^N$, $F(R) = \text{Conv}(R)$. Therefore, since $a < \underline{p}(R) \leq \bar{p}(R) < b$, by the definition of $F^{a,b}$, $F(R) = F^{a,b}(R)$.

Step 2. Let $M \in \mathcal{P}$ such that $|M| \geq 3$. By Step 1, for each $R \in \mathcal{R}^M$, $F = F^{a_M,b_M} \in \mathcal{F}^M$. Define points $a := a_M$ and $b := b_M$.

We show that for each $N \in \mathcal{P}$ and each $\bar{R} \in \mathcal{R}^N$, $F(\bar{R}) = F^{a,b}(\bar{R})$. We do so by showing that for each $N \in \mathcal{P}$, each $\bar{R} \in \mathcal{R}^N$, and each $R \in \mathcal{R}^M$, if $\text{Conv}(\bar{R}) = \text{Conv}(R)$, then $F(\bar{R}) = F^{a,b}(R) = F^{a,b}(\bar{R})$ (the latter equality follows by the definition of $F^{a,b}$).

Let $R \in \mathcal{R}^M$ and $\bar{R} \in \mathcal{R}^N$. Recall that $F(R) = F^{a,b}(R)$. Starting from $R \in \mathcal{R}^M$, add the population $N \setminus M$ with profile $\bar{R}_{N\setminus M}$ such that $R^1 \in \mathcal{R}^{M\cup N}$ and $R^1 = (R, \bar{R}_{N\setminus M})$. Since $\text{Conv}(R^1) = \text{Conv}(R)$, by *population-monotonicity* and Proposition 4(a), $F(R^1) = F(R)$. Next, change the preferences of each agent $i \in N$ to $\bar{R}_i$ and define $R^2 := (R^1_{M\setminus N}, \bar{R}) \in \mathcal{R}^{M\cup N}$. Since $\text{Conv}(R^2) = \text{Conv}(R^1)$, by *population-monotonicity* and Proposition 4(a), $F(R^2) = F(R^1)$. Finally, remove the population $M \setminus N$ from R^2 such that $R^2_N = \bar{R} \in \mathcal{R}^N$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$, by *population-monotonicity* and Proposition 4(a), $F(\bar{R}) = F(R^2)$. Hence, $F(\bar{R}) = F^{a,b}(R) = F^{a,b}(\bar{R})$. □

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