Philosophy of Mathematics

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Philosophical perspectives on the exact sciences and their history

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Outline

Classical Positions in Philosophy of Mathematics

Platonism

Platonism

Idea that mathematical objects are abstract entities that exist independently of our minds. Abstract: the objects do not enter into causal relations, are not located spatiotemporally.

- Metaphysical thesis
- Seems to be a good idea: why should the fact that 2+2=4, which obviously is a relation between numbers, depend on the existence of human beings?
- Platonism makes sure that mathematics is 'robust'.

Platonism: A Problem

Contra Platonism

1. Mathematical objects are neither constructions of our minds nor do they enter into causal relations.
2. We only gain knowledge about the world by causal interactions.
3. how do we know about mathematical objects?

- Epistemological problem
- several ways out: mathematical objects do enter into causal relationships, or are mind-dependent (or say that the causal theory of knowledge is false)

formulated in Benacerraf (1973): 'Mathematical Truth', in Benacerraf and Putnam

Nominalism

Nominalism

Idea that there are no such things as abstract, especially non-causal mathematical objects. The word ‘nominalism’ signifies that mathematical objects exist only as names or nomina.

- Natural opponent of Platonism
- Comes in many varieties such as Formalism, Fictionalism, ...
- Also a good idea: ’empiricist’, i.e. it does not postulate a dubious realm of abstract, inaccessible mathematical objects.

Nominalism: A Problem

Contra Nominalism

All of a sudden mathematical theorems depend on what the world is like: If the world contains only finitely many objects, then all mathematical theorems depending on infinite structures such as the natural numbers become dubious (not to speak of set theory).

Interlude: Nominalization

How can a Nominalist deal with the problem that scientific theories that make reference to mathematical objects all the time?

Nominalization

Eliminate reference to all mathematical objects, represent all mathematical properties in some physical space. (This is called Nominalization, has been carried out by Hartry Field.)

Two problems with Nominalization:

1. Not naturalistic, i.e. it does not conform with current scientific practice.
2. Field showed how to nominalize Newtonian mechanics, but there are some problems with other theories, e.g. phase space formulations.

Field (1980): 'Science without Numbers: A defense of nominalism'

Is Mathematics about Objects?

Mathematicians seem to be sympathetic to the view that structures rather than objects are the subject matter of mathematics: groups, rings, fields, etc. What about natural numbers?

1. Assume that the natural numbers are nothing but a certain set. The natural numbers could be one of the two following sets:
   - von Neumann ordinals: $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \ldots$
   - Zermelo ordinals: $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \ldots$
2. But there is only one natural number structure (or at least that is our intuitive idea).
3. Also, the above sets have properties that numbers don't have, and they differ in that respect.
4. Therefore: The natural numbers cannot be sets – although we can use sets to represent the natural numbers.

Benacerraf (1965): 'What Numbers Could Not Be', in Benacerraf and Putnam

Structuralism a la Shapiro: Ante Rem

One way out of this dilemma is structuralism as proposed by Stewart Shapiro (1997):

Structuralism

Idea that the unique natural number structure is exactly what all things like the above representations, or instantiations of this structure have in common. What they have in common is that the elements stand in a certain relation to other elements, e.g. the number 0 has no predecessor etc. The natural numbers exist before or independently of their instantiations. This form of structuralism is therefore called ante rem, and it is a form of platonism.

see Shapiro (2008): 'Identity, Indiscernibility and Ante Rem Structuralism: The Tale of +i and -i'. Philosophia Mathematica, and Shapiro (1997): 'Philosophy of Mathematics', Oxford University Press.

Metaphysics and the Working Mathematician

(How) Do these issues concern the working mathematician?

- It is often claimed that most mathematicians are platonists of some sort. Ultimately, the question of what mathematicians think about the status of mathematical objects is empirical.
- On the one hand, it seems safe to say that if you work in some field, then it is natural to have a realistic attitude towards the object of your studies.
- On the other hand, people working in the foundations of mathematics, or logic, or set theory sometimes have a more formalistic, 'reductionist' perspective.
- In the end, doing mathematics and the metaphysical issues just presented are largely orthogonal.

Shaken To The Core: Foundational Crisis

A Plea for the History of Mathematics (and Science)

Why should we care about the history of mathematics, or any science?

- History can help you understand contemporary science. It shows that scientific laws, and mathematical definitions, theorems etc. did not fall from heaven, but developed, were changed, discarded etc.
- Reading original text brings you closer to the people who actually made the discoveries. Don't rely too much on the polished version of science presented in textbooks.
- It's a nice thing to see geniuses struggle and fail, because if they did, then so can we.
- If you are interested in philosophy of science, then consulting historical sources is an advantage, because many philosophers of science don't and are therefore often wrong.

Discovering Set Theory

- Georg Cantor worked on questions of existence and uniqueness of trigonometric series
- Started to think about sets of points on the real line in which series might not agree, or diverge:
    1. Finitely many points in finite intervals
    2. Infinitely many points with one limit point – set of limit points is finite.
    3. Infinitely many points with infinitely many limit points – set of limit points is infinite with one limit point.
    4. Infinitely many such that the iteration of the above never leads to an empty set.
- Here, we have a transfinite iteration in the 'index' of 'derived sets'. Somewhen, Cantor started considering 'indexes' as 'proper numbers'; this lead to a new perspective on e.g. the status of infinity.

See e.g. Dauben's classic 1979 monograph on Cantor

The Role of Set Theory

- Set theory is not only an important mathematical discipline, it has a special status in all of mathematics in that it is often used as a tool to formulate most other mathematical theories, and to express mathematical concepts more clearly.
- Starting from a technical question, Cantor had laid the foundation for a clearer understanding of concepts such as order, infinity, functions, relations, ...
- Other mathematicians were also working on foundational issues. Especially David Hilbert had begun early on to reduce mathematical theories to others, e.g. geometry to analysis. He was very enthusiastic about the advent of set theory.

Trouble In Paradise?

Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben koennen. Hilbert (1926): 'On the Infinite'

But why would anyone do such a horrible thing?

Around the turn of the century, several paradoxes surrounding set theory in its original form were discovered, the most famous one being Russell's paradox.

Russell's paradox

Russell's paradox

1. Assume (as it was common in naive set theory) that for any concept whatsoever the set of elements that fall under this concept can be formed.
2. Consider the concept 'sets that are not members of themselves'. Can we form the set of elements that fall under this concept?
3. If we can, then either this set is a member of itself or not.
4. If this set is a member of itself, then it must not be a member of itself.
5. If it is not a member of itself, then it must be a member of itself.
6. Contradiction.

From Russell's Paradox to Hilbert's Program

This paradox, and others, were a threat to the idea that mathematics could be grounded in set theory: Anything follows from a contradiction. Therefore, Hilbert formulated a program to prevent accidents such as Russell's paradox. His program consisted roughly of the following ideas:

- All of mathematics can be done in formal, axiomatic systems in a rigorous manner, it is not more than a formal play with signs.
- All talk of infinities can somehow be reduced to finite elements of mathematics to which we have immediate and unproblematic access, e.g natural numbers.
- Proofs are not more than finite strings, and they can be carried out in the mentioned axiomatic systems
- If we can show how to reduce mathematical systems to some well-understood mathematical theory such as arithmetic, and prove that arithmetic is consistent (without using any dubious, infinitary stuff), then we should be on the safe side.

All Hope Lost? Goedel 1931

Hilbert's dreams were shattered when the young mathematician Kurt Goedel proved his famous incompleteness theorems in 1930/1931, showing that Hilbert's program cannot be carried out (Goedel actually attempted to contribute to Hilbert's program):

Goedel's Theorems

1. In a consistent, nice theory sufficiently strong to formalize arithmetic, e.g. Peano Arithmetic, we can find arithmetical sentences that cannot be proved nor refuted in that theory.
2. If (a system like) PA is consistent, then we cannot prove the consistency of PA in PA.

Especially the second incompleteness theorem shows that the consistency of interesting systems have to be proven by other means.

The Interpretation of Incompleteness

- One important caveat concerning the second theorem: it does not say that we cannot prove the consistency of PA tout court. There are consistency proofs of PA, e.g, by Gentzen, although not in PA itself.
- We should be very moderate in our interpretation of these results, especially when taking them out of their natural, formal habitat, and applying the to minds, physics, and general mathematics.

The Story Does Not End Here

The philosophy of set theory is very much alive; several new books on the philosophy of set theory have appeared recently:

- Michael Potter: Set Theory and its Philosophy. A Critical Introduction. Oxford University Press, 2004.
- Penelope Maddy: Defending the Axioms. On the Philosophical Foundations of Set Theory. Oxford University Press, 2011.

Explanations in Mathematics: Pure and Applied

What is a Good Proof?

There seem to be two different kinds of proofs in mathematics:

1. Some proofs merely demonstrate *that* a certain theorem is true. Examples: Proofs that use ‘brute force techniques’, proofs by contradiction.
2. Other proofs do more: they also show us *why* a theorem is true.

It has been proposed, by philosophers but also by mathematicians, that this is the distinction between explanatory and non-explanatory proofs. The distinction could be important because it drives research in mathematics (reproving of theorems), it connects nicely with other debates in philosophy of science, ... on the other hand, the debate is underdeveloped ... and is badly in need of good examples.

Mathematics in Physical Explanations

How can mathematics help in the explanation of physical phenomena? Two toy examples:

- Bridges of Koenigsberg. Is it possible to cross all the bridges of Koenigsberg exactly once and return to the starting point? Euler translated this problem into a graph-theoretic question, and solved it.
- Periodic cicadas. Certain kinds of american cicadas have a prime-numbered life cycle, some 13, some 17 years. Why prime-numbered? The commonly accepted explanation is that prime-numbered life cycles minimize both exposure to predators and hybridization, and uses a number-theoretic result.

Proposals for Special Projects

Role of Mathematics in (early) GR

- Starting point: What is the role of mathematics in describing, explaining etc. phenomena in the world? (see above)
- Plan: Look at a real life example, also taking into account the historical process of discovery.
- One candidate: the application of previously known techniques (absolute differential calculus) by Einstein in the discovery of general relativity
- If there's an expert in the audience...

Financial Engineering: Models vs. The World

- All is not well in the world of finance (aka financial crisis).
- Seems that one of the aspects of the problem is, again, the application of mathematical techniques and models to the world.
- One example: Pricing of derivatives using stochastic processes, Black-Scholes model. (Can also be other economic models.)
- Task: You already have good knowledge of one such model, then you have the chance to reflect on it.

Further References

The following entries from the SEP were used:

- Philosophy of Mathematics
- The Early Development of Set Theory
- Russell's Paradox
- Hilbert's Program
- Explanation in Mathematics