On the Efficiency of Monetary Exchange: Why Divisibility of Money Matters.*

Aleksander Berentsen†

Economics Departement, University of Bern, Switzerland

Guillaume Rocheteau‡

HEC-DEEP, University of Lausanne, Switzerland

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Abstract

Why is money divisible? To explore this question we introduce a mismatch problem into search-theoretic models of monetary exchange. We use alternative assumptions about the divisibility of goods and money and the ability of agents to use lotteries on money. Our framework potentially generates three types of inefficiencies: the *no-trade* inefficiency where no trade takes place even though it would be socially efficient to trade; the *too-much-trade* and *too-little-trade* inefficiencies where the quantities produced and exchanged are either larger or smaller than what the solution to a social planner's problem would dictate. It is shown that while the no-trade and the too-much-trade inefficiencies are caused by the indivisibility of money, the too-little-trade inefficiency is due to the impatience of the traders and the time-consuming exchange process. Furthermore, we find that the lottery model with indivisible money and divisible goods is qualitatively similar to the divisible money and divisible goods model.

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†Address: University of Bern, Economics Department, Vereinsweg 23, 3012 Bern, Switzerland. E-mail: aleksander.berentsen@vwi.unibe.ch.

‡Address: University of Lausanne, HEC-DEEP, BFSH1, 1015 Lausanne-Dorigny, Switzerland. E-mail: guillaume.rocheteau@hec.unil.ch

1 Introduction

A well-known feature of search-theoretic models of money is that the outcome is generically inefficient in the sense that quantities exchanged and prices differ from the solution to a social planner's problem. Some models predict that prices are too high. Models with this property include the divisible money and divisible goods model of Shi (1997, 1999). Other models find that prices can be too low. For certain parameter values, the divisible goods and indivisible money model of Trejos and Wright (1995) and Shi (1995) display this outcome. Moreover, in the indivisible money and indivisible goods model of Kiyotaki and Wright (1991) in some meetings no trade takes place even though it would be socially efficient to trade.[1]

What lies behind these inefficiencies? Although, each paper in question explains the reason for the observed inefficiencies, the problem is that there is no obvious basis for a comparison of their results. The explanations for these inefficiencies range from the decentralized nature of the price formation (Trejos and Wright, 1995; Shi, 1995), to the time consuming exchange process (Kiyotaki and Wright, 1991; Shi, 1997), or the details of the bargaining protocol.[2] We are, therefore, left with apparent unrelated explanations, which limits our ability to use the search-theoretic approach to evaluate economic policy issues.

This paper explores the nature of these inefficiencies by providing a common framework for assessing the different models. We assume that the economy is composed of a continuum of differentiated commodities identified by points around a circle. Agents have idiosyncratic tastes for these commodities and they choose strategies for determining when (and sometimes how much) to produce, to trade, and to consume. Because of the randomness of the matching process when a buyer and a seller meet, the buyer's preference for the seller's good is represented by a random variable ε. This framework generates a stochastic mismatch problem: Buyers may prefer not to buy a good that would yield a positive consumption utility if their valuation for the good is low, i.e., if ε is low.

[1] The no-trade inefficiency is not explicitely mentioned in Kiyotaki and Wright (1991) but it is discussed in Boldrin et al. (1993).

[2] According to Trejos and Wright (1995, p. 130), the fact that the quantity exchanged is too high is related to the bargaining procedure and the ability of the traders to go on searching while bargaining. Furthermore, even without search while bargaining, if the bargaining power of buyer is sufficiently close to one, the quantity exchanged is above its efficient value (Wright, 1999b).

Basically, the paper's mechanics is to introduce the same stochastic mismatch problem into different random-matching models of money and, then, to compare the outcomes in order to identify the sources of the observed inefficiencies. First, we compare the indivisible goods and indivisible money model of Kiyotaki and Wright (1991, 1993) with the same model when agents use lotteries on indivisible money to determine the terms of trade.[3] Second, we compare the divisible goods and indivisible money model of Trejos and Wright (1995) and Shi (1995), the same model with lotteries on money, and the divisible money and divisible goods model of Shi (1997, 1999). Introducing lotteries on money into these models, as the study of divisible money, will be useful to identify the origin of the inefficiencies that arise in indivisible money models.

Our framework potentially generates three types of inefficiencies. First, there is, what we call, a *no-trade* inefficiency: in some meetings there is no production, exchange, and consumption even though it would be socially efficient to do so. Second, there is a *too-much-trade* inefficiency: in some meetings the quantities produced and exchanged are larger than what the solution to a social planner's problem would dictate. Third, there is a *too-little-trade* inefficiency: production in some matches is too small relative to the social planner's solution.

One advantage of our approach is that these inefficiencies, in contrast to the previously mentioned papers, can arise simultaneously. This facilitates the identification of their origin. For example, in the divisible goods and indivisible money model we find that there is a reservation value $\overline{\varepsilon}$ and a threshold $\widetilde{\varepsilon}$ with $\overline{\varepsilon} < \widetilde{\varepsilon}$ such that in a meeting if the buyer's valuation for the commodity lies below $\overline{\varepsilon}$ buyers and sellers do not trade. If the valuation is in $(\overline{\varepsilon}, \widetilde{\varepsilon})$, the quantities exchanged are too large and if the value is above $\widetilde{\varepsilon}$ they are too small.[4]

The following results emerge from the analysis. First, the no-trade and the too-much-trade inefficiencies are caused by the indivisibility of money. They are neither present in the divisible money model nor in indivisible money models when lotteries are allowed.

[3] In contrast to Berentsen et al. (2000), who consider randomization over both indivisible goods and indivisible money, we will only consider lotteries on money. This restriction allows us to focus on the implications of indivisible money because it introduces a notion of divisible money without affecting the indivisibility of goods.

[4] In contrast, in Trejos and Wright (1995) for some parameter values they find the too-much-trade inefficiency for other parameter values the too-little-trade inefficiency. Moreover, their model does not produce the no-trade inefficiency because all agents have the same valuations for their consumption goods.

Second, the too-little-trade inefficiency appears in all models. This inefficiency is due to the impatience of the traders and the time-consuming nature of the exchange process. Third, the bargaining procedure, particularly, the bargaining power of buyers and sellers, plays a much less important role than we expected. In fact, neither of the inefficiencies can be attributed to a specific bargaining protocol because the *no-trade* and the *too-much-trade* inefficiencies are removed when money is divisible (or when lotteries are allowed) and the *too-little-trade* inefficiency is eliminated in the divisible money model through the Friedman's rule for any bargaining power of buyers.[5]

Our framework makes the divisibility of money *visible.* In our divisible money model, we find a threshold $\widetilde{\varepsilon}$ such that in a single coincidence meeting if a buyer's valuation for the seller's good is below $\widetilde{\varepsilon}$, he only spends a *fraction* of his money holdings and if his valuation is above, he spends his entire money holdings. Moreover, if $\varepsilon < \widetilde{\varepsilon}$, a socially efficient quantity of the good is produced and exchanged and if $\varepsilon \geq \widetilde{\varepsilon}$, an inefficiently low quantity is traded. In contrast, in Shi's (1997, 1999) divisible money model, buyers in each meeting spend their entire money holdings and they always receive inefficiently low quantities of goods in exchange.

Interestingly, we find that the model with indivisible money, divisible goods, and lotteries and the divisible money model yield very similar results. In the lottery model, there is also a threshold $\widetilde{\varepsilon}$ such that if the valuation for the good is below $\widetilde{\varepsilon}$, the indivisible unit of money is exchanged with probability less than one and the socially efficient quantity is produced and exchanged. If the valuation for the good is above this threshold, however, the unit of money is exchanged with certainty and the quantity exchanged is inefficiently low.

Our results suggest that indivisible money models should be used carefully because they introduce inefficiencies into the bargaining that are difficult to distinguish from other modeling assumptions that might introduce distortions into the model. To emphasize this point we show that the indivisibility assumption can generate misleading policy recommendations. In the indivisible goods and indivisible money model inflation has a positive effect on welfare because inflation induces agents to be less choosy and to spend their indivisible money units more quickly. This effect is also present in the divisible goods and indivisible

[5]Note, however, that the too-little trade inefficiency remains in the indivisible money and divisible goods model with lotteries for r close to zero if the bargaining power of buyers is lower than the fraction of buyers in the economy.

money model. In the divisible money model or the indivisible money model with lotteries, however, inflation non-ambiguously reduces welfare.

Our paper is related to search models of money that study the role of the divisibility assumption in monetary exchange. Taber and Wallace (1999) consider the divisibility of money and welfare in the Shi (1995) and Trejos and Wright (1995) setup. They introduce a notion of divisibility by relaxing the assumption of the unit upper bound on money holdings. Money is said to be twice as divisible if both the upper bound on money holdings and the total number of money units are doubled. They show that an increase in the divisibility of money allows agents to smooth consumption and reduce the crowding out effect of money. The paper is also related to the many search models of money that discuss the efficiency of monetary exchange at some point, which include the pioneering papers of Kiyotaki and Wright (1991, 1993), Shi (1995, 1997, 1999) and Trejos and Wright (1995). As far as we know, however, our paper is the first one that makes a comparison across the different types of search models of money.

Section 2 presents the assumptions shared by the different models used in this paper. Section 3 introduces the mismatch problem into the indivisible money models. Section 4 studies the divisible goods and divisible money model and Section 5 concludes.

2 The environment

The economy consists of a continuum of infinitely-lived households who specialize in consumption and production. There are $H \geq 3$ types of goods and H types of households. Households are uniformly distributed among types. For each type, there is a continuum of varieties represented by a circle of circumference 2 denoted by $\mathcal{C}_h$. A household of type $h \in \{1, ..., H\}$ values only goods of type h and produces a good chosen at random on $\mathcal{C}_{h+1}$ (mod H). Denote Q the set of feasible quantities that a household can produce. We will consider two cases: the indivisible goods model where $Q = \{0, 1\}$ and the divisible goods model where $Q = \mathcal{R}^+$.

The mismatch problem is described as follows. Households of type h are uniformly distributed on $\mathcal{C}_h$. The most preferred variety of household $h_i \in \mathcal{C}_h$ is h_i. If we draw at random a variety h_j from $\mathcal{C}_h$, the length l of the arc between h_j and h_i is uniformly distributed on $[0, 1]$. Accordingly, if household h_i consumes only varieties within distance $\bar{l}$ of his most preferred variety h_i, the probability that a randomly chosen variety will be consumed by the household is $\bar{l}$.

The function mapping the distance between the variety that is consumed and the most preferred variety, l, and the quantity consumed, q, into utility is continuous in both arguments, non-increasing in l, and increasing in q. We adopt the following function:[6]

$$\mathcal{U}(l, q) = \varepsilon\left(l\right) u(q)$$

where ε is decreasing and twice differentiable and satisfies $\varepsilon\left(0\right) = \varepsilon_{\sup}$ and $\varepsilon\left(1\right) = 0$. Furthermore, we assume that u is increasing and twice differentiable, and satisfies $u\left[0\right] = 0$, $u\left[1\right] = U$, $u'' < 0$, and $u'\left[0\right] = \infty$. The probability that ε is less than $x \in [0, \varepsilon_{\sup}]$ for a variety chosen at random is equal to:

$$\mathbb{P}\left[\varepsilon\left(l\right) \leq x\right] = \mathbb{P}\left[l \geq \varepsilon^{-1}(x)\right] = 1 - \varepsilon^{-1}(x) \equiv F(x),$$

where $F(.)$ is a cumulative distribution with density f.

A producer incurs disutility $c\left[q\right]$ from producing q units of a good. For the divisible goods model, we will assume that $c\left[q\right] = q$. For the indivisible goods model, let $c\left[1\right] = C$

[6] Our formalization of the mismatch problem is similar to that of Kiyotaki and Wright (1991). Note that the analysis would also hold for any utility function satisfying $\frac{\partial \mathcal{U}(l,q)}{\partial q} > 0$, $\frac{\partial \mathcal{U}(l,q)}{\partial l} < 0$, $\frac{\partial^2 \mathcal{U}(l,q)}{\partial q^2} < 0$, and $\frac{\partial^2 \mathcal{U}(l,q)}{\partial q \partial l} < 0$.

with $C < \varepsilon_{\sup} U$. Goods cannot be stored and production is instantaneous. Finally, the discount factor is $\beta = \frac{1}{1+r}$ with $r > 0$.

In addition to the consumption goods described above there is also an intrinsically worthless, storable object called fiat money. We will again consider two cases: the *indivisible* money model, where each household consists of one individual; the *divisible* money model, where each household consists of a continuum of members. For the indivisible money model, a fraction N of all agents are initially endowed with one unit of money and each agent has a single unit storage capacity. For the divisible money model, each household is initially endowed with M units of divisible money and the household evenly distributes this amount among a fraction N of its members. In both models, therefore, the fraction of agents in the market endowed with money is exogenous and equal to N and the fraction of agent without money is $1 - N$. We call agents with money *buyers* and agents without *sellers*. A buyer attempts to exchange money for consumption goods, and a seller attempts to produce goods for money.

Buyers and sellers meet pairwise and at random. Our assumptions about technology and preferences rule out double coincidence of real wants meetings. Moreover, buyers only buy varieties that lie in the set $A \subseteq [0, \varepsilon_{\sup}]$. This set is determined endogenously. Consequently, the probability of a successful trade for a buyer, i.e., the probability that a buyer of type h meets a seller of type $h-1$ who produces a good in A, is the product of $z = \frac{1}{H}$ and $\Pi = \int_A dF(x)$.

Throughout the paper we assume that in a match the buyer makes a take-it-or-leave-it offer and the seller accepts the offer if made no worse of by accepting. However, in the appendix we show that the results we will present in this paper hold for the more general Nash bargaining too.

3 Indivisible money

The aim of this section is to identify the inefficiencies that are present in indivisible money models. For this purpose we introduce the same mismatch problem in the indivisible money and indivisible goods model of Kiyotaki and Wright (1991, 1993) and the indivisible money and divisible goods model of Shi (1995) and Trejos and Wright (1995). We will also study the outcome in these models when agents are allowed to use lotteries on money to determine the terms of trades. Lotteries on money, as the study of the divisible money model in Section 4, will be useful to identify the origin of the inefficiencies that arise in indivisible money models.

3.1 Indivisible goods without lotteries

Time is continuous, agents cannot hold more than one object at a time, and the terms of trades are exogenous: one unit of money buys one indivisible commodity. Only individuals without money (sellers) are able to produce. Buyers and sellers meet randomly according to a Poisson process with arrival rate $\alpha \in \mathcal{R}^+$. Without loss of generality, we normalize $\alpha = 1$. Thus, the probability per unit of time that a buyer of type h meets a seller of type $h-1$ (a seller who produces a good in $\mathcal{C}_h$) is $z(1-N)$ and the probability per unit of time that a seller of type h meets a buyer of type $h+1$ is zN.

To introduce a notion of inflation, we assume that buyers' money holdings are *confiscated* at rate μ and that sellers receives a lump-sum transfers of money at rate $\delta = \frac{\mu N}{1-N}$.[7] Consequently, the quantity of money is constant and equal to N. The value function of buyers satisfies the following Bellman equation:

$$rV_B = z(1-N)\int_0^{\varepsilon_{\sup}} \max(\varepsilon U + V_S - V_B, 0)\, dF(\varepsilon) + \mu(V_S - V_B) \tag{1}$$

The flow return to a buyer, rV_B, equals the sum of two terms. The first term is the rate at which the buyer meets appropriate sellers, $z(1-N)$, times the expected gain of a match, which equals $\int_0^{\varepsilon_{\sup}} \max(\varepsilon U + V_S - V_B, 0)\, dF(\varepsilon)$. In a single coincidence meeting, either the buyer spends his unit of money, which yields $\varepsilon U + V_S - V_B$, or he remains a buyer, which

[7] This mechanism, which has been proposed by Li (1995), is a proxy for inflation because it reduces the real value of money and because the inflation tax imposed on a money holder is proportional to the lenght of time that she holds the money.

yields no surplus. The second term is the rate at which a buyer's money is confiscated, μ, times the confiscation loss, $V_S - V_B$.

Because a buyer's utility is increasing in ε, the set of acceptable varieties is $A = [\overline{\varepsilon}, \varepsilon_{\text{sup}}]$ where $\overline{\varepsilon}$ is a reservation value for the taste index that satisfies:

$$\overline{\varepsilon}U = V_B - V_S \tag{2}$$

If $\varepsilon \geq \overline{\varepsilon}$, the buyer is willing to buy the consumption good. Denote $\overline{E}$ the reservation value for the average buyer. The probability that a variety of good h is accepted by a buyer of type h chosen at random is $\Pi = 1 - F(\overline{E})$. Accordingly, the value functions of sellers satisfy the following Bellman equation:

$$rV_S = zN\left[1 - F(\overline{E})\right](-C + V_B - V_S) + \delta\left(V_B - V_S\right) \tag{3}$$

The flow return to a seller, rV_S, equals the sum of two terms. The first term is the rate at which the seller of type h meets a buyer of type $h+1$, zN, times the probability that the buyer accepts the trade, $\left[1 - F(\overline{E})\right]$, times the expected gain from trading, producing and becoming a buyer, $-C + V_B - V_S$. The second term is the rate at which the seller receives a unit of money, δ, times the gain of becoming a buyer, $V_B - V_S$.

Existence of a monetary equilibrium requires that sellers are willing to produce for money:

$$V_B - V_S \geq C. \tag{4}$$

In the following we only consider symmetric Nash equilibria where $\overline{\varepsilon} = \overline{E}$.

Definition 1 *For the indivisible goods and indivisible money model, a monetary equilibrium is a triplet $(V_B, V_S, \overline{\varepsilon})$ that satisfies equations (1) - (3) and participation constraint (4).*

To compare the outcome of the decentralized economy with the allocation that a planner would choose in order to maximize welfare, consider the following ex ante Pareto criterion: $W = r\left(NV_B + (1-N)V_S\right)$. Welfare is the expected permanent income of a single agent before money is distributed. Equations (1) and (3) yield

$$W = zN(1-N)\int_{\overline{\varepsilon}}^{\varepsilon_{\text{sup}}} \varepsilon U - CdF(\varepsilon)$$

Welfare is maximized at $\overline{\varepsilon}^* = C/U$. From social point of view buyers should accept any trade with a positive surplus, i.e., with $\varepsilon U - C \geq 0$, because for the society it does not matter who holds the money.

Proposition 1 *Consider the model with indivisible goods and indivisible money. In this model, there exists a critical value ε_0, defined in the proof, such that the following is true:*
(i) If $C/U > \varepsilon_0$, no monetary equilibrium exist;
(ii) If $C/U = \varepsilon_0$, a unique monetary equilibrium exist with $\overline{\varepsilon} = C/U$;
(iii) If $C/U < \varepsilon_0$, there are an odd number of monetary equilibria with $\overline{\varepsilon} > C/U$.

Proposition 1 establishes the existence of a monetary equilibrium in the indivisible goods and indivisible money model if the ratio C/U is below a critical value ε_0. The critical value ε_0 is decreasing in the inflation rate μ, in the rate of time preferences r, and in the measure of buyers N.

The key result is that the frequency of trades is generically too low, i.e., $\overline{\varepsilon} > C/U$. If $\varepsilon \in [C/U, \overline{\varepsilon})$, agents do not trade even though it would be socially efficient to trade. Buyers are too choosy because they fail to internalize the positive effect of spending their money on other market participants. Indeed, a buyer's private gain of a successful trade is $\varepsilon U + V_S - V_B$ whereas the social gain is $\varepsilon U - C$.

As shown in the Appendix, equations (1), (2), and (3) implicitly define a reaction function $\overline{\varepsilon} = \mathcal{R}(\overline{E})$ which is increasing in $\overline{E}$. This illustrate the presence of strategic complementarities: the best response of a single agent is to increase his reservation value if all other agents increase theirs. Consequently, multiple equilibria may occur.

The government can remove the no-trade inefficiency by choosing a sufficiently high inflation rate. In the Appendix A1 we show that a higher inflation rate reduces the reservation value because waiting for a better trading opportunity becomes more costly. This is the "hot potato" effect of inflation mentioned by Li (1995).[8] Welfare maximizing inflation is the value of μ satisfying $\overline{\varepsilon} = C/U$: it is the maximum level that does not destroy the monetary equilibrium.

[8] A similar effect is also mentioned by Trejos (1999) where buyers are less selective about the quality of the goods they buy.

3.2 Indivisible goods with lotteries on money

We now allow agents to use lotteries to determine the terms of trade. In contrast to Berentsen et al. (2000), we will only consider lotteries on money.[9] This restriction allows us to focus on the welfare consequences of the indivisibility of money because it introduces a notion of divisible money without affecting the indivisibility of goods.

Bargaining over lotteries on money means bargaining over the probability τ_ε that the unit of money changes hands. We restrict our attention to take-it-or-leave-it offers (For the generalized Nash solution, see appendix B1). With lotteries on money, the value functions of buyers and sellers satisfy the following generalized versions of (1) and (3):

$$rV_B = z(1-N)\int_{\overline{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon U - \tau_\varepsilon (V_B - V_S)\, dF(\varepsilon) - \mu (V_B - V_S) \tag{5}$$

$$rV_S = zN\int_{\overline{E}}^{\varepsilon_{\sup}} -C + \tau_\varepsilon (V_B - V_S)\ dF(\varepsilon) + \delta (V_B - V_S) \tag{6}$$

When a buyer meets an appropriate seller, the buyer receives the good and delivers his unit of money with some probability τ_ε. Thus, his expected surplus from the trade is $\varepsilon U - \tau_\varepsilon (V_B - V_S)$. The first term of right-hand side of equation (6) is zero. This is a consequence of the take-it-or-leave-it offers that satisfy

$$-C + \tau_\varepsilon (V_B - V_S) = 0, \qquad \forall\varepsilon \tag{7}$$

According to (7), the probability that money changes hands is constant and equal to $\tau = \frac{C}{V_B - V_S}$. Existence of a monetary equilibrium requires that buyers are able to compensate sellers for their production cost, which implies that $\tau \leq 1$. Because a buyer's surplus is increasing in ε, buyers trade if $\varepsilon \geq \overline{\varepsilon}$ where $\overline{\varepsilon}$ is a reservation value that satisfies:

$$\overline{\varepsilon} U - \tau (V_B - V_S) = 0 \tag{8}$$

Definition 2 *For the indivisible goods and indivisible money model with lotteries on money, a monetary equilibrium is a list* $(V_B, V_S, \overline{\varepsilon}, \tau_\varepsilon)$ *satisfying equations (5) - (8).*

From (5) and (6), welfare is maximized if $\overline{\varepsilon} = \frac{C}{U}$. Note that τ_ε is irrelevant: For the society it does not matter how often money changes hand.

[9] Berentsen et al. (2000) allow agents to use lotteries on both money and goods. We have also studied lotteries on both money and goods in our model. Under buyer-takes-all bargaining, we find the same outcome as reported in Proposition 2: For all $\varepsilon \geq \frac{C}{U}$, sellers would deliver their goods with probability one and buyers would give their money unit with some probability less than one.

Proposition 2 *Consider the model with indivisible goods, indivisible money, and lotteries on money. In this model, there is a critical value* ε_0*, defined in the proof, such that the following is true: If* $C/U > \varepsilon_0$*, no monetary equilibrium exist. If* $C/U \leq \varepsilon_0$*, a unique monetary equilibrium exists with* $\overline{\varepsilon} = \frac{C}{U}$*. The probability that the unit of money changes hand is*

$$\tau_\varepsilon = \tau = \frac{(r + \mu + \delta)\, C/U}{z\,(1 - N) \int_{C/U}^{\varepsilon_{\text{sup}}} (1 - F(\varepsilon))\, d\varepsilon}, \qquad \forall \varepsilon \in [C/U, \varepsilon_{\text{sup}}]\,,$$

and $\tau_\varepsilon = 0$ *otherwise.*

The key result is that the monetary equilibrium is efficient because whenever there is a positive surplus in a match, the buyer proposes an offer that exploits the entire surplus and this offer is not refused by the seller.[10] Accordingly, the frequency of trades is at its efficient level, i.e., $\overline{\varepsilon} = \frac{C}{U}$. Moreover, lotteries on money remove the strategic complementarities that are present without lotteries, and hence the possibility of multiple equilibria. Also, in contrast to the model without lotteries, inflation cannot improve the outcome: there is no extensive effect because inflation cannot increase the frequency of trades. Rather, inflation is detrimental because it decreases the critical value ε_0 and, therefore, makes the existence of a monetary equilibrium less likely. Note that the model with lotteries and the same model without lotteries have the same critical value ε_0.

3.3 Divisible goods without lotteries

In this section we explore how the mismatch problem affects the outcome when the terms of trade are endogenized along the lines of Shi (1995) and Trejos and Wright (1995). Suppose, therefore, that goods are divisible but money is still indivisible, and that buyers make a take-it-or-leave-it offer to the seller about the quantity that the seller has to produce for one unit of money. Let q_ε be the amount produced by a seller in exchange for one unit of money when the buyer's valuation for the good is $\varepsilon u(q_\varepsilon)$. Expected lifetime utilities of buyers and sellers obey the following Bellman equations in continuous time:

$$rV_B = z\,(1 - N) \int_0^{\varepsilon_{\text{sup}}} \max\left(\varepsilon u\,[q_\varepsilon] + V_S - V_B, 0\right)\, dF\,(\varepsilon) + \mu\,(V_S - V_B) \tag{9}$$

$$rV_S = zN \int_A (-q_\varepsilon + V_B - V_S)\, dF(\varepsilon) + \delta\,(V_B - V_S)\,, \tag{10}$$

[10] Appendix B1 shows that efficiency is also attained when the terms of trade are determined through generalized Nash bargaining.

where $A \subseteq [0, \varepsilon_{\text{sup}}]$ is the set of varieties that money holders are willing to buy. Equation (9) and (10) have similar interpretations as equations (1) and (3). The main difference is that the first term of right-hand-side of (10) is zero because sellers do not get any benefit from trading with buyers. This is a consequence of the take-it-or-leave-it offers that satisfy

$$q_\varepsilon = V_B - V_S \equiv q, \qquad \forall \varepsilon \in A. \tag{11}$$

Because buyers extract all the surplus from the match and because the surplus of the seller does not depend on the quality of the match, the terms of trade do not depend on the taste index ε.

The surplus of the buyer, $\varepsilon u(q) + V_S - V_B$, is increasing in ε. Consequently, the reservation property holds and the set of acceptable varieties is $A = [\overline{\varepsilon}, \varepsilon_{\text{sup}}]$ where the reservation value $\overline{\varepsilon}$ satisfies

$$\overline{\varepsilon} u\left[q\right] = V_B - V_S = q \tag{12}$$

Definition 3 *For the divisible goods and indivisible money model, a monetary equilibrium is a 4-uplet* $(V_B, V_S, \overline{\varepsilon}, q)$ *that satisfies equations (9) - (12).*

Again, we want to compare the outcome of the decentralized economy with the allocation that a social planner would choose in order to maximize social welfare. Equations (9) and (10) imply that welfare equals:

$$W = z\,(1-N)\,N \int_{\overline{\varepsilon}}^{\varepsilon_{\text{sup}}} \varepsilon u\left[q_\varepsilon\right] - q_\varepsilon \; dF(\varepsilon).$$

By maximizing W with respect to $\overline{\varepsilon}$ and q_ε, we find that the first-best allocation satisfies

$$\begin{aligned} \overline{\varepsilon} &= 0 \\ \varepsilon u'\left[q_\varepsilon\right] &= 1, \qquad \forall \varepsilon \in [0, \varepsilon_{\text{sup}}]\,. \end{aligned}$$

The first equation simply states that buyers should consume all varieties. The second equation states that the quantity produced, exchanged, and consumed should equalize marginal utility of the buyer to marginal cost of the seller. Denote efficient quantities by q_ε^*.

Proposition 3 *Consider the model with divisible goods and indivisible money. In this model, a unique monetary equilibrium exists with* $0 < \overline{\varepsilon} < \varepsilon_{\text{sup}}$. *Furthermore, there is* $\widetilde{\varepsilon} > \overline{\varepsilon}$ *such that:*

(i) If $\varepsilon < \overline{\varepsilon}$ *no trade takes place.*
(ii) If $\overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}$*, the quantity exchanged is too high.*
(iii) If $\widetilde{\varepsilon} < \varepsilon$*, the quantity exchanged is too low.*
Finally, $\widetilde{\varepsilon} < \varepsilon_{\text{sup}}$ *if* $r > r_0$ *where* r_0 *is defined in the proof.*

Proposition 3 establishes the existence of a unique monetary equilibrium which is generically inefficient. There are three types of inefficiencies. First, if $\varepsilon < \overline{\varepsilon}$, no goods are exchanged. Second, if $\overline{\varepsilon} < \varepsilon < \widetilde{\varepsilon}$, the quantities exchanged are inefficiently high (prices are too low). Third, $\widetilde{\varepsilon} < \varepsilon$, the quantity exchanged are inefficiently low (prices are too high). These inefficiencies occur simultaneously if $r > r_0$ where r_0 is negative if μ is large. They are displayed in Figure 1 where the curve labelled q_ε^* displays efficient quantities and the curve labelled q_ε exchanged quantities as functions of ε.

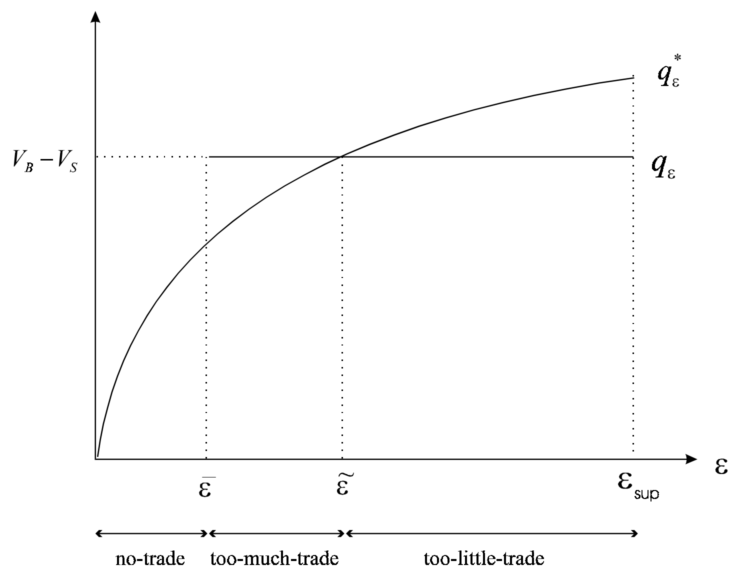

Figure 1: Three types of inefficiencies.

To explain the no-trade and too-much-trade inefficiencies consider a buyer's consumption decision when the seller produces a variety in a small neighborhood around $\overline{\varepsilon}$ (see Figure 1). If $\varepsilon = \overline{\varepsilon}$, the buyer is just indifferent between consuming q units of the good and becoming a seller or remaining a buyer. The seller is also indifferent between producing q units and becoming a buyer or remaining a seller. If ε is slightly below $\overline{\varepsilon}$ no trade takes place because the buyer is strictly worse off when consuming q units and the seller is not willing to produce more. If ε is slightly above $\overline{\varepsilon}$, however, the buyer is strictly better off when consuming q units of the good relative to continuing his search. The consumed quantity q, however, is inefficiently large because of the buyer's low valuation of the good.

In the indivisible goods model of section 3.1, an increase of the inflation rate increases the frequency of trades and welfare by lowering the reservation value $\overline{\varepsilon}$. In the divisible goods model of this section, this effect, which is called the extensive effect of inflation, is also present. In contrast to the indivisible goods model, however, inflation has also an intensive effect, i.e., an increase in the inflation rate reduces the quantities exchanged in each match. In Figure 1, if the inflation rate increases, q is reduced (the intensive effect) and the reservation value $\overline{\varepsilon}$ moves to the left (the extensive effect).

Finally, we want to emphasize that none of these inefficiencies are generated by the assumption of take-it-or-leave-it offers by buyers. Rather, it is a general property of indivisible money models. In Appendix B2, we show that these inefficiencies are also present when the bargaining proceeds according to the generalized Nash bargaining solution. The main difference is that q_ε is not constant but a strictly decreasing function of ε (see Figure B2 in the Appendix). Furthermore, we will show below that the too-much-trade inefficiency is not a consequence of the high bargaining power of buyers as the take-it-or-leave-it-offer might suggest. Indeed, if we allow agents to use lotteries on money to determine the terms of trades this inefficiency vanishes even when the buyer has all the bargaining power.

3.4 Divisible goods with lotteries on money

We now allow agents to use lotteries on money to determine the terms of trade.[11] We restrict our attention to take-it-or-leave-it-offers $(q_\varepsilon, \tau_\varepsilon)$ where q_ε is the amount of goods that the seller produces in an ε-meeting and τ_ε is the probability that the unit of money changes hands. Note that we do not need to introduce a reservation value for the taste index in this model because if a buyer does not want to trade, he can propose $(q_\varepsilon, \tau_\varepsilon) = (0, 0)$.

[11] We only consider lotteries on money because we want to focus on the consequences of the indivisibility of money without affecting how goods are traded. However, allowing agents to also use lotteries on goods would not change our results at all. Indeed, Berentsen et al. (2000) have shown that in equilibrium goods always change hands with probability 1. Appendix B3 shows that the results presented here also hold if terms of trade are determined by the generalized Nash bargaining solution.

With lotteries on money, the expected lifetime utility of buyers and sellers satisfy

$$rV_B = z(1-N)\int_0^{\varepsilon_{\sup}} \varepsilon u[q_\varepsilon] + \tau_\varepsilon (V_S - V_B)\, dF(\varepsilon) + \mu (V_S - V_B) \tag{13}$$

$$rV_S = zN\int_0^{\varepsilon_{\sup}} -q_\varepsilon + \tau_\varepsilon (V_B - V_S)\; dF(\varepsilon) + \delta (V_B - V_S), \tag{14}$$

Under buyer-takes-all bargaining, sellers do not get any benefit from trading with buyers. Hence, the first term of right-hand side of (14) is equal to zero. The take-it-or-leave-it offers satisfy

$$\tau_\varepsilon (V_B - V_S) = q_\varepsilon, \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{15}$$

For a given realization of the taste index ε, the buyer solves

$$\max_{q_\varepsilon, \tau_\varepsilon} \varepsilon u(q_\varepsilon) - \tau_\varepsilon (V_B - V_S) \tag{16}$$

subject to (15) and

$$\tau_\varepsilon \leq 1, \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{17}$$

Definition 4 *For the divisible goods and indivisible money model with lotteries on money, a monetary equilibrium is* $(V_B, V_S, q_\varepsilon, \tau_\varepsilon)$ *such that the value functions satisfy (13) and (14) taking the lottery as given; the lottery solves the maximization problem in (16) taking the value functions as given.*

Note that, as in the model without lotteries, efficiency requires that q_ε satisfies $\varepsilon u'(q_\varepsilon) = 1$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$. Define efficient quantities by q_ε^*.

Proposition 4 *Consider the model with divisible goods, indivisible money, and lotteries on money. In this model, a unique monetary equilibrium exist with* $\widetilde{\varepsilon} > 0$ *such that:*
(i) If $\varepsilon > \widetilde{\varepsilon}$, $0 < q_\varepsilon < q_\varepsilon^*$ *and* $\tau_\varepsilon = 1$.
(ii) If $\varepsilon \leq \widetilde{\varepsilon}$, $q_\varepsilon = q_\varepsilon^*$ *and* $\tau_\varepsilon \leq 1$.
Furthermore, if $r \leq r_1$*, where* r_1 *is defined in the proof, then* $\widetilde{\varepsilon} \geq \varepsilon_{\sup}$ *and individuals always trade efficient quantities.*

The key result is that $0 < q_\varepsilon \leq q_\varepsilon^*$ for all $\varepsilon > 0$. That is, the quantities exchanged are never larger than the efficient quantities and the frequency of trades is at its efficient level. This result suggests that inefficiencies associated with no trade and too much trade are due to the indivisibility of money.

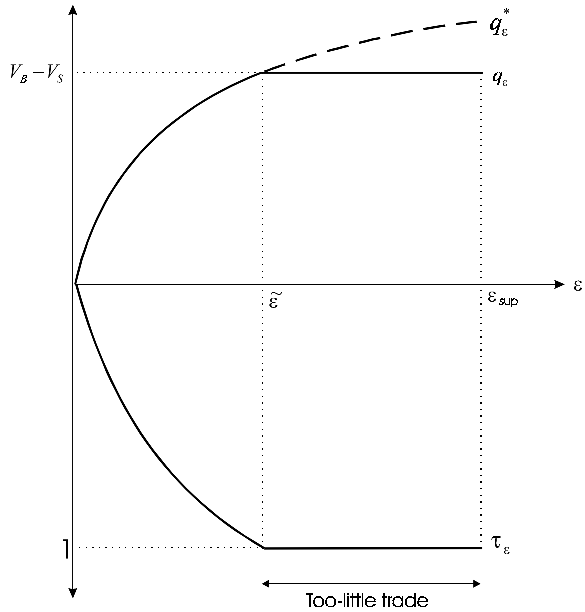

Figure 2: Terms of trade with divisible goods, indivisible money, and lotteries.

The results of Proposition 4 are displayed in Figure 2. In the upper quadrant of Figure 2, the dotted curve labelled q_ε^* plots efficient quantities as a function of ε and the solid curve labelled q_ε displays the quantities that are exchanged in equilibrium. If $\varepsilon < \widetilde{\varepsilon}$, the two curves merge and the traders exchange efficient quantities. The lower quadrant plots the probability that money changes hands, τ_ε, as a function of the taste index.

With lotteries inflation has only an intensive effect, i.e., $\frac{\partial q_\varepsilon}{\partial \mu} < 0$ if $\widetilde{\varepsilon} < \varepsilon$. In Figure 2, if μ increases, the flat part of the q_ε curve decrease and $\widetilde{\varepsilon}$ moves to the left. Notice that all trades are efficient if $\widetilde{\varepsilon} \geq \varepsilon_{\text{sup}}$, respectively, if $r \leq r_1$. If $\mu = \delta = 0$, $r \leq r_1$ is satisfied for r close to 0. Thus, if time is not costly and inflation is low, agents trade efficient quantities in all meetings. In contrast, if μ is large, r_1 can be negative which means that the too-little-trade inefficiency does not vanish even when r approaches zero.

4 Divisible money

In the indivisible money model, the frequency of trades is too low because buyers are too choosy. In the divisible goods and indivisible money model, there are inefficiencies associated with no-trade, too-little-trade, and too-much-trade. What lies behind these inefficiencies?

The models of Section 3 that incorporate lotteries on money suggest that one reason for the no-trade and too-much-trade inefficiencies is the indivisibility of money. To explore this conjecture further, we incorporate the same mismatch problem into the divisible money model of Shi (1997, 1999) who adapted the search-theoretic approach to allow for divisible money.

Most of the description of the environment presented in Section 2 applies to the divisible money environment. The main differences are that time is discrete and that each household consists of a continuum of members normalized to one who carry out different tasks but regard the household's utility as the common objective. When carrying out these tasks household members follow the strategy that has been given to them by their households. In some sense, household members are automaton that just execute household strategies. At the end of each period, they pool their money holdings, which eliminates aggregate uncertainty for households. In the symmetric monetary equilibrium, the distribution of money is degenerate across households.[12]

Household members are grouped into money holders (buyers) and producers (sellers), each performing one task at a time. Buyers attempt to exchange money for consumption goods and sellers attempt to produce goods for money. The fraction of buyers is given by the exogenous constant N.[13] In each period household members are randomly matched in pairs and each household member meets another household member. Hence, the probability that a seller of type h meets a buyer of type $h+1$ is zN, and the probability that a buyer of type h meets a seller of type $h-1$ is $z(1-N)$.

Although households differ in their preferences and production opportunities, they all

[12] By adopting Shi's (1997, 1998, 1999) device, we avoid nondegenerate distributions of money holdings that are difficult to control. With this device, idiosyncratic risks of household members are smoothed out within each household. This facilitates the analysis because we can focus on a representative household.

[13] Shi (1997, 1999) also allows households to choose the fraction of buyers N in each period. Here, we do not consider this choice. Berentsen and Rocheteau (2000) consider the case where each household member is simultaneously a buyer and a seller in a double coincidence environment.

consume and produce the same quantities so that each household and each good can be treated symmetrically. In the following we refer to an arbitrary household as household h_i. Decision variables of this household are denoted by lower-case variables. Capital-case variables denote other households' variables, which are taken as given by the representative household h_i. Furthermore, variables corresponding to the next period are indexed by $+1$.

At the beginning of each period, household h_i has m units of money which he divides evenly among its buyers so that each buyer holds m/N units of money in a match. Then, the household specifies the trading strategies for its members. After this, agents are matched and carry out their exchanges according to the described strategies. After trading, buyers consume the goods they have bought and sellers bring back their receipts of money.[14] At the end of a period, the household receives money transfer τ which is perceived as lump-sum and carries the stock m_{+1} to $t+1$.

As shown by Rauch (2000), the pricing procedure in this model must be described very carefully. We follow Shi (1999) by assuming that the terms of trades are determined through take-it-or-leave-it offers by buyers.[15] Two important features of the bargaining are that (i) bargaining strategies are determined at the household level but are carried out by household members, and (ii) household members observe the match type but cannot observe the marginal value of money and the level of money holdings of their trading partners. Because of (ii) households' strategies depend on the match type and on the distribution of their potential bargaining partners' characteristics. In equilibrium, this distribution is degenerated: all households have the same marginal utility of money and hold the same quantity of money. Because of (i) any strategy aiming to reveal ones true characteristics (if different than the average) is ineffective because the bargaining partner commits to the bargaining strategy chosen by its household.

If $V(m)$ denotes the lifetime expected utility of an household endowed with m units of money, the marginal value of money is $\omega = \beta V_m(m_{+1})$ where V_m is the derivative of V

[14] We consider household members as automaton that have no own will. Accordingly, for the household it does not matter which members consume and which produce. To the extent that one might worry about this, one can assume that the buyers at the beginning of each period are chosen at random among household members. However, it is problematic to depart from the automaton interpretation because this introduces all sorts of commitment problems.

[15] In contrast to Rauch (2000), Shi (1999) does not assume complete information. In Shi's model, buyers' strategies do not depend on the specific characteristics of the sellers they meet, rather they depend on the average characteristics of sellers.

with respect to money holdings m. Furthermore, denote Ω the marginal value of money of all other households.

A buyer's take-it-or-leave-it offer is a pair $(q_\varepsilon, x_\varepsilon)$ where q_ε is the quantity of goods produced by the seller for x_ε units of money. If the seller accepts the offer, the acquired money x_ε will add to her household's money balances at the beginning of the next period, whose value today is Ωx_ε. The cost associated with this trade is q_ε and the seller accepts the offer if $x_\varepsilon \Omega \geq q_\varepsilon$. Thus, any optimal offer satisfies

$$x_\varepsilon \Omega = q_\varepsilon, \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{18}$$

Because a buyer cannot exchange more money than he has, the offer $(q_\varepsilon, x_\varepsilon)$ satisfies

$$x_\varepsilon \leq \frac{m}{N}, \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{19}$$

A household's trading strategy consists of the terms of trade $(q_\varepsilon, x_\varepsilon)$ for each $\varepsilon \in [0, \varepsilon_{\sup}]$ and an acceptance rule for each offer $(Q_\varepsilon, X_\varepsilon)$ by a buyer of another household. Because for other households offers a condition similar to (18) must hold, buyers of other households make offers that are just accepted by sellers of household h_i. Similar to the lottery models, there is no need to introduce a reservation value $\overline{\varepsilon}$ because the household can always set $(q_\varepsilon, x_\varepsilon) = (0, 0)$ if he does not want to trade.

For each period, the household chooses $(m_{+1}, q_\varepsilon, x_\varepsilon)$ to solve the following dynamic programming problem:

$$V(m) = \max_{q_\varepsilon, x_\varepsilon, m_{+1}} \{v - c + \beta V(m_{+1})\} \tag{20}$$

subject to the constraints (18), (19), and

$$v = zN(1-N) \int_0^{\varepsilon_{\sup}} \varepsilon u(q_\varepsilon)\, dF(\varepsilon) \tag{21}$$

$$c = zN(1-N) \int_0^{\varepsilon_{\sup}} Q_\varepsilon dF(\varepsilon) \tag{22}$$

$$m_{+1} - m = \tau + z(1-N)N \int_0^{\varepsilon_{\sup}} X_\varepsilon dF(\varepsilon) - z(1-N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon dF(\varepsilon) \tag{23}$$

The variables taken as given in the above problem are the state variable m and other households' choices. Equality (21) specifies the utility of the household (there is no aggregate uncertainty at the household level). The measure of buyers is N and the probability of meeting an appropriate seller is $z(1-N)$ so that the number of single coincidence meetings involving a buyer of the household h_i in each period is $zN(1-N)$. Aggregate utility is $zN(1-N)\int_0^{\varepsilon_{\sup}} \varepsilon u\left(q_{\varepsilon}\right) dF(\varepsilon)$. Equality (22) specifies the household's disutility of production. Again, the number of single coincidence meeting involving a seller of household h_i is $zN(1-N)$ and the integral specifies the aggregate disutility of production.

Equality (23) describes the law of motion of the household's money balances. The first term on the right-hand side specifies the additional currency the household receives each period: this quantity is perceived as lump-sum by each household. The supply of money increases at rate $(\gamma-1)$, which implies that next period's money supply is $M_{+1}=\gamma M$. The second term specifies sellers' money receipts when selling goods, and the third term specifies buyers' expenses when exchanging money for goods.

Denote λ_{ε} the multipliers associated with constraints (19). Then, the program of the household can be rewritten as follows:

$$V(m)=\max_{(q_{\varepsilon}),(x_{\varepsilon}),\overline{\varepsilon}}\left\{zN(1-N)\left(\int_0^{\varepsilon_{\sup}} \varepsilon u(q_{\varepsilon})dF(\varepsilon)-\int_0^{\varepsilon_{\sup}} Q_{\varepsilon}dF(\varepsilon)\right)+\right.$$

$$\left.+\,zN(1-N)\int_0^{\varepsilon_{\sup}} \lambda_{\varepsilon}\left(\frac{m}{N}-x_{\varepsilon}\right) dF(\varepsilon)+\beta V(m_{+1})\right\}$$

where m_{+1} is given by (23). Note that, according to (18), x_{ε} can be expressed as a function of q_{ε}. The first order conditions and the envelope condition are:

$$\varepsilon u'(q_{\varepsilon})=\frac{1}{\Omega}\left(\lambda_{\varepsilon}+\omega\right), \qquad \forall\varepsilon\in[0,\varepsilon_{\sup}] \tag{24}$$

$$\lambda_{\varepsilon}\left(x_{\varepsilon}-\frac{m}{N}\right)=0, \qquad \forall\varepsilon\in[0,\varepsilon_{\sup}] \tag{25}$$

$$\frac{\omega_{-1}}{\beta}=z(1-N)\int_0^{\varepsilon_{\sup}} \lambda_{\varepsilon}dF(\varepsilon)+\omega \tag{26}$$

Equation (24) states that, for a buyer in a desirable match, the marginal utility of consumption must equal the opportunity cost of the amount of money that must be paid to acquire additional goods. To buy another unit of a good, the buyer must give up $\frac{1}{\Omega}$ units of money (see equation (18)). Increasing the monetary payment has two costs to the buyer. He gives up the future value of money ω and he faces a tighter constraint (19). Together, ω and λ measure the marginal cost of obtaining a larger quantity of goods in exchange for money. Equation (25) is the Kuhn-Tucker condition associated with the multiplier λ_{ε}. Finally, equation (26) describes the evolution of the marginal value of money. It states that the marginal value of money today, $\frac{\omega_{-1}}{\beta}$, equals the discounted marginal benefit of money tomorrow, ω, plus the marginal benefit of relaxing future cash constraints, $z(1-N)\int_0^{\varepsilon_{\sup}} \lambda_\varepsilon dF(\varepsilon)$.

In equilibrium, all households have the same characteristics, which implies that the values for the different variables of the household under consideration equal the values of the same variables of all other households. Consequently, capital-case variables and lower-case variables are equal: $\omega = \Omega$, $m = M$, and $(x_\varepsilon, q_\varepsilon) = (X_\varepsilon, Q_\varepsilon)$ for all ε.

Definition 5 *A steady-state monetary equilibrium is a collection* $\{(q_\varepsilon), (\lambda_\varepsilon), (x_\varepsilon), \omega\}$ *satisfying equations (18) and (24)-(26).*

Social welfare is measured by the lifetime expected discounted utility of a representative household:[16]

$$W = zN(1-N)\int_0^{\varepsilon_{\sup}} \varepsilon u[q_\varepsilon] - q_\varepsilon \, dF(\varepsilon)$$

The social planner maximizes W with respect to q_ε. The first-order conditions imply:

$$\varepsilon u'[q_\varepsilon] = 1, \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}].$$

This equation simply states that the quantities exchanged should equalize the marginal utility of the buyer with the marginal cost of the seller.

Proposition 5 *Consider the divisible goods and divisible money model. If* $\gamma \geq \beta$*, there is a unique monetary equilibrium with* $m\omega > 0$*. Furthermore, there is* $\widetilde{\varepsilon} \leq \varepsilon_{\sup}$ *such that:*
(i) If $\varepsilon > \widetilde{\varepsilon}$*, buyers spend all their money holdings and* $q_\varepsilon = \omega \frac{M}{N} < q_\varepsilon^*$*.*
(ii) If $\varepsilon \leq \widetilde{\varepsilon}$*, buyers spend only a fraction of their money holdings and* $q_\varepsilon = q_\varepsilon^*$*.*
(iii) When γ *tends to* β *then* $\widetilde{\varepsilon}$ *approaches* $\varepsilon_{\sup}$*.*

The key result is that with divisible money $0 < q_\varepsilon \leq q_\varepsilon^*$ for all $\varepsilon \in (0, \varepsilon_{\sup}]$: the quantities exchanged are never larger than the efficient quantity and households consume all varieties. With perfectly divisible money, a household simply spends a small amount of money to acquire a small amount of a low valued variety. Thus, as lotteries, divisible money eliminates the no-trade and the too-much-trade inefficiency that is present in the divisible goods and indivisible money model without lotteries of Section 3. The only inefficiency left is the too-little trade inefficiency that occurs if $\varepsilon > \widetilde{\varepsilon}$.

[16] Welfare as defined here is the expected utility in one period. This expression measure the flow of permant income of a household as in Section 3.

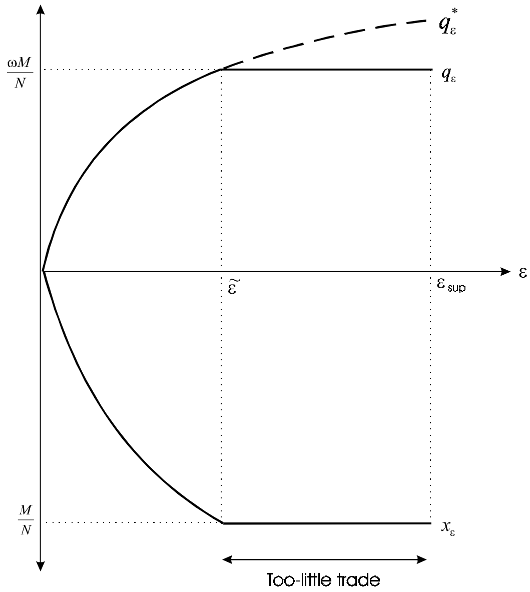

Figure 3: Terms of trade with divisible money.

Figure 3 illustrates Proposition 5. The upper quadrant displays efficient quantity q_ε^* and exchanged quantity q_ε as a function of the taste index ε. If $\varepsilon \leq \widetilde{\varepsilon}$, $q_\varepsilon = q_\varepsilon^*$, i.e., buyer and seller produce and consume efficient quantities. If $\varepsilon > \widetilde{\varepsilon}$, however, $q_\varepsilon < q_\varepsilon^*$, quantities exchanged are inefficiently low. The lower quadrant shows the exchanged quantities of money x_ε as a function of ε. If in a match $\varepsilon \leq \widetilde{\varepsilon}$, $x_\varepsilon \leq M/N$, i.e., the buyer does not spend his entire money holdings. If $\varepsilon > \widetilde{\varepsilon}$, however, $x_\varepsilon = M/N$.

One virtue of our model is that it makes the divisibility of money visible: for low valued varieties buyers spend only a fraction of their money holdings. In contrast, in Shi's (1997, 1999) model buyers always spend their entire money holdings. Moreover, they always exchange inefficiently low quantities. In our version, however, for low valued varieties they exchange efficient quantities.

With divisible money, inflation has no positive "hot potato effect" on welfare because it cannot increase the frequency of trades. In contrast, inflation is always costly because it generates a misallocation of resources. A higher rate of the money supply increases the misallocation because it reduces the set of meetings where agents produce and exchange efficient quantities. To see this, note that in Figure 3, the horizontal line $\frac{\omega M}{N}$ moves upwards as γ decreases. Consequently, the fraction of inefficient trades decreases. In the limit when γ approaches β, almost all trades are efficient, i.e., $\widetilde{\varepsilon}$ approaches ε_{sup}.

Finally, we want to emphasize the similarity of the results of the indivisible money, divisible goods, and lottery model of Section 3 and the divisible money model of this Section. This similarity is revealed when comparing Proposition 4 and 5, respectively Figures 2 and 3. In both models, there is a threshold $\widetilde{\varepsilon}$ such for all $\varepsilon \leq \widetilde{\varepsilon}$ the quantities exchanged are socially efficient whereas for all $\varepsilon > \widetilde{\varepsilon}$ the quantities produced are too low. Moreover, if $\varepsilon > \widetilde{\varepsilon}$ buyers spend all their money holdings in the divisible money model and in the model with lotteries they spend the money unit with probability one. Also, if $\varepsilon \leq \widetilde{\varepsilon}$ buyers spend a fraction of their money holdings only in the divisible money model and in the model with lotteries they spend money with some probability $\tau < 1$.

5 Conclusion

Divisibility is an important characteristic of a medium of exchange. Divisible money enables us to distribute "value according to our varying requirements (Stanley Jevons (1875))." Despite the obvious advantage of having a divisible medium of exchange in real exchanges, in most search models money is an indivisible object.[17] The goal of this paper has been to evaluate the consequences of this assumption.

Our main conclusion is that indivisibility of money is the cause of two kinds of inefficiencies: the *no-trade* inefficiency and the *too-much-trade* inefficiency. The reason why indivisible money generates these inefficiencies is rather intuitive. If money is indivisible, buyers are reluctant to abandon their money holdings for low valuation goods. In order to be willing to give up their money holdings they demand large quantities of these goods in exchange. If a buyer's valuation for a good is very low, the quantity that makes him indifferent between trading and not trading is *larger* than the *largest* quantity that the seller is willing to produce for the money. Consequently, no trade takes place even though it would be socially efficient to trade. We show that this no-trade inefficiency is not present in models with divisible money because if a buyer's valuation for a good is low, he simply spends a small amount of money in exchange for a small (and efficient) amount of the good. The no-trade inefficiency is also absent in models with lotteries because with lotteries our buyer simply delivers the indivisible unit of money with a low probability in exchange for a small (and efficient) amount of the good.

If, in a match, the buyer's valuation for the good is low but not too low, the quantity that makes him indifferent between trading and not trading is *smaller* than the *largest* quantity that the seller is willing to produce for the money. Consequently, an exchange takes place. However, because of the buyer's low valuation for the good the exchanged quantity is larger than the efficient quantity (this is the too-much-trade inefficiency). Again, with divisible money or with lotteries, this inefficiency disappears for the same reason.

If, in a meeting, a buyer's valuation for a good is large, we observe in all models the too-little-trade inefficiency. This inefficiency originates from the fact that agents discount future utilities and that money is only a promise to future consumption. In each model, the too-little-trade inefficiency occurs when the buyer is constrained by his money holdings. He would prefer to buy more but is not able to do so because the purchasing power of his

[17] Notable exceptions are Zhou (1999) and Shi (1997, 1999).

money holdings is too low.

From a methodological perspective, our paper shows that the model with divisible goods, indivisible money, and lotteries on money is qualitatively equivalent to the divisible goods and divisible money model. In either model, the quantities exchanged are efficient for low valuation goods and are inefficiently low for high valuation goods. Furthermore, when the quantities exchanged are efficient, the buyer only gives a fraction of his money holdings in the divisible money model or he delivers his money unit with a probability less than one in the lottery model.

This discussion about the divisibility of money goes beyond a technical issue: indivisible money models have sometimes questionable policy implications. For instance, when money is indivisible, inflation can have a positive "hot potato effect" on welfare because it induces agents to spend their money units more often. In the same model with divisible money (or lotteries), however, inflation is non-ambiguously welfare decreasing. Our view is that a policy recommendation that only relies on the indivisibility of money cannot be justified.

APPENDIX A: Proofs

A1. Proof of Proposition 1

First, note that equations (1), (2), and (3) implicitly define a reaction function $\overline{\varepsilon} = \mathcal{R}(\overline{E})$:

$$(r+\mu+\delta)\,\overline{\varepsilon} + zN\left[1-F(\overline{E})\right](\overline{\varepsilon}-C/U) = z(1-N)\int_{\overline{\varepsilon}}^{\varepsilon_{\sup}} 1-F(\varepsilon)\,d\varepsilon \qquad (27)$$

Second, the critical value $\varepsilon_0 \in (0, \varepsilon_{\sup})$ is the unique value of $\overline{\varepsilon}$ that satisfies

$$(r+\mu+\delta)\,\overline{\varepsilon} = z(1-N)\int_{\overline{\varepsilon}}^{\varepsilon_{\sup}} 1-F(\varepsilon)d\varepsilon. \qquad (28)$$

Indeed, the left-hand side of (28) is strictly increasing in $\overline{\varepsilon}$ whereas the right-hand side is strictly decreasing for all $\overline{\varepsilon} \in (0, \varepsilon_{\sup})$. Furthermore, $LHS(0) < RHS(0)$ and $LHS(\varepsilon_{\sup}) > RHS(\varepsilon_{\sup})$. Note that the left-hand side of (27) is smaller than the left-hand side of (28) if $\overline{\varepsilon} < C/U$ and larger if $\overline{\varepsilon} > C/U$. Accordingly (see Figure A1),

$$\varepsilon_0 \begin{matrix}<\\=\\>\end{matrix} C/U \Rightarrow \mathcal{R}(\overline{E}) \begin{matrix}<\\=\\>\end{matrix} C/U, \qquad \forall \overline{E} \in [0, \varepsilon_{\sup}]\,. \qquad (29)$$

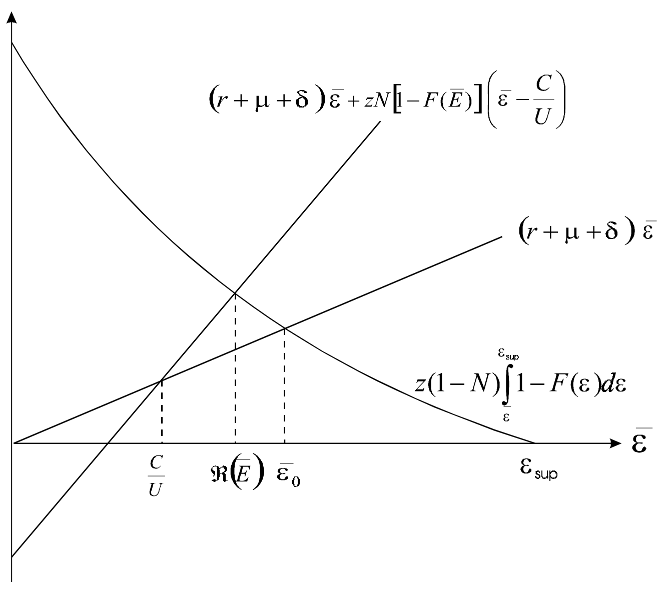

Figure A1: The reaction function.

Existence of a monetary equilibrium requires that inequality (4) holds, i.e., $\overline{\varepsilon} = \mathcal{R}(\overline{E}) \geq C/U$. Thus, (29) implies that (i) if $\varepsilon_0 < \frac{C}{U}$, no monetary equilibrium exist; (ii) if $\varepsilon_0 = C/U$, there is a unique symmetric Nash equilibrium with $\overline{\varepsilon} = \overline{E} = C/U$; (iii) if $\varepsilon_0 > C/U$, a monetary equilibrium exists with $\overline{\varepsilon} = \overline{E} \in (C/U, \varepsilon_{\sup})$ with $\frac{C}{U} < \overline{\varepsilon} = \mathcal{R}(\overline{E}) < \varepsilon_{\sup}$, for all $\overline{E}$.

Finally, by differentiating (27), we find

$$\mathcal{R}'(\overline{E}) = \frac{zNf\left(\overline{E}\right)\left(\mathcal{R}(\overline{E}) - C/U\right)}{r + \mu + \delta + zN\left[1 - F(\overline{E})\right] + z(1-N)\left[1 - F\left(\mathcal{R}(\overline{E})\right)\right]}$$

When $\varepsilon_0 > C/U$ then $\mathcal{R}(C/U) > C/U$, $\mathcal{R}(\varepsilon_{\sup}) < \varepsilon_{\sup}$ and $\mathcal{R}'(\overline{E}) > 0$. Hence, there is an odd number of monetary equilibria.■

A2. Proof of Proposition 2

Inserting equation (7) into equation (8) yields $\overline{\varepsilon}U - C = 0$. Hence, $\overline{\varepsilon} = \frac{C}{U}$.

Equation (7) implies that $\tau_{\varepsilon} = C/\left(V_B - V_S\right) \equiv \tau$. To derive τ use (5) and (6) to get:

$$(r + \mu + \delta)\left(V_B - V_S\right) = z\left(1 - N\right)U \int_{C/U}^{\varepsilon_{\sup}} \left(1 - F(\varepsilon)\right) d\varepsilon$$

Then, equation (7) implies the expression for τ in the proposition. Notice that τ is increasing in C/U. If $C/U \leq \varepsilon_0$, where ε_0 satisfies (28), $\tau \leq 1$. Thus, $C/U \leq \varepsilon_0$ is a necessary and sufficient condition for a monetary equilibrium to exist. Because there is a unique $(V_B - V_S)$ that satisfies the previous equation, if it exists, the monetary equilibrium is unique.■

A3. Proof of Proposition 3

We first determine existence and uniqueness of the equilibrium. Equations (9), (10), and (11) yield

$$(r + \mu + \delta)\, q = z\left(1 - N\right) \int_{\overline{\varepsilon}}^{\varepsilon_{\sup}} \left(\varepsilon u\left[q\right] - q\right) dF\left(\varepsilon\right) \tag{30}$$

From (12), the equilibrium reservation value is the value of $\overline{\varepsilon}$ that solves

$$\left(r + \frac{\mu}{1 - N}\right)\overline{\varepsilon} = z\left(1 - N\right) \int_{\overline{\varepsilon}}^{\varepsilon_{\sup}} \left(1 - F\left(\varepsilon\right)\right) d\varepsilon \tag{31}$$

The left-hand side of (31) is strictly increasing and the right-hand side is strictly decreasing in $\overline{\varepsilon}$ for all $\overline{\varepsilon} < \varepsilon_{\sup}$. Because $LHS(0) = 0$ and $RHS(0) > 0$, and $LHS(\varepsilon_{\sup}) > 0$ and

$RHS(\varepsilon_{\sup}) = 0$, a unique reservation value $0 < \overline{\varepsilon} < \varepsilon_{\sup}$ exists. For a given $\overline{\varepsilon}$, the equilibrium quantity of goods traded in a match is the unique value of q that solves (12). For a given $(\overline{\varepsilon}, q)$, (V_B, V_S) is entirely determined by (9) and (10). Hence, the monetary equilibrium is unique. Note that equation (31) implies that $\frac{\partial \overline{\varepsilon}}{\partial \mu} < 0$ and $\frac{\partial \overline{\varepsilon}}{\partial r} < 0$ and because of equation (12) $\frac{\partial q}{\partial \mu} < 0$ and $\frac{\partial q}{\partial r} < 0$.

We next consider the efficiency of the monetary equilibrium. Define $\widetilde{\varepsilon}$ the value of ε that satisfies $\varepsilon u'(q) = 1$. $\widetilde{\varepsilon}$ is the value of ε such that the quantity produced, q, equals the efficient quantity, $q^*_{\widetilde{\varepsilon}}$. Because of (11), $\widetilde{\varepsilon} u'(V_B - V_S) = 1$. It can be verified that $q > q^*_{\overline{\varepsilon}}$ and, therefore, $\widetilde{\varepsilon} > \overline{\varepsilon}$. From this, it is evident that q is generically inefficient.

Finally, to determine critical value r_0, note that (12) and (31) yield:

$$(r + \mu + \delta) \frac{q}{u[q]} = z(1 - N) \int_{\frac{q}{u[q]}}^{\varepsilon_{\sup}} (1 - F(\varepsilon)) \, d\varepsilon \tag{32}$$

The left-hand side of (32) is increasing in q whereas the right-hand side is decreasing in q. If $\widetilde{\varepsilon} < \varepsilon_{\sup}$, $q^*_{\sup} > q$, respectively $LHS(q^*_{\sup}) > RHS(q^*_{\sup})$. Thus,

$$\left(r + \frac{\mu}{1 - N}\right) q^*_{\varepsilon_{\sup}} > z(1 - N) \, u(q^*_{\varepsilon_{\sup}}) \int_{\frac{q^*_{\varepsilon_{\sup}}}{u\left[q^*_{\varepsilon_{\sup}}\right]}}^{\varepsilon_{\sup}} (1 - F(\varepsilon)) \, d\varepsilon \tag{33}$$

The quantity r_0 is the value of r such that (33) is satisfied with equality.■

A4. Proof of Proposition 4

Let us first characterize the terms of trade. Equality (15) and inequality (17) imply $q_\varepsilon \leq V_B - V_S$ for all $\varepsilon \in [0, \varepsilon_{\sup}]$. Denote λ_ε the multiplier associated with this constraint and rewrite the program (16) to get

$$\max_{q_\varepsilon, \lambda_\varepsilon} \varepsilon u(q_\varepsilon) - q_\varepsilon - \lambda_\varepsilon (V_B - V_S - q_\varepsilon)$$

The first order conditions are

$$\varepsilon u'(q_\varepsilon) = 1 + \lambda_\varepsilon \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{34}$$

$$\lambda_\varepsilon (V_B - V_S - q_\varepsilon) = 0 \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \tag{35}$$

Suppose, first, that inequality (17) is not binding, i.e., that $\lambda_\varepsilon = 0$. This implies that $q_\varepsilon = q^*_\varepsilon = \tau_\varepsilon (V_B - V_S)$ and $\tau_\varepsilon \leq 1$. Suppose, next, that inequality (17) is binding, i.e., $\lambda_\varepsilon > 0$. This implies that $\tau_\varepsilon = 1$ and $q_\varepsilon = V_B - V_S < q^*_\varepsilon$. Finally, define $\widetilde{\varepsilon}$ as the value of ε

such that $\varepsilon u'(V_B - V_S) = 1$. If $\varepsilon \geq \widetilde{\varepsilon}$, $q_\varepsilon = V_B - V_S < q^*_\varepsilon$ and $\tau_\varepsilon = 1$, and if $\varepsilon \leq \widetilde{\varepsilon}$, $q_\varepsilon = q^*_\varepsilon$ and $\tau_\varepsilon < 1$.

Let us, next, show the existence and uniqueness of the equilibrium. Equations (13), (14), and (15) yield

$$(r + \mu + \delta)(V_B - V_S) = z(1 - N)\Psi \tag{36}$$

$$\text{where } \Psi = \int_0^{\widetilde{\varepsilon}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon dF(\varepsilon) + \int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon u(V_B - V_S) - (V_B - V_S) dF(\varepsilon)$$

This equation determines a unique $V_B - V_S$. Indeed, the left-hand side of (36) is a linear function of $V_B - V_S$ and the right-hand side of (36) is an increasing concave function. Furthermore, it can be verified that $LHS(0) < RHS(0)$.

Finally, let us show that if $r \leq r_1$, $\widetilde{\varepsilon} \geq \varepsilon_{\sup}$. Condition $\widetilde{\varepsilon} \geq \varepsilon_{\sup}$ implies $q^*_{\varepsilon_{\sup}} \leq V_B - V_S$, and, therefore, from (36)

$$\left(r + \frac{\mu}{1 - N}\right) q^*_{\varepsilon_{\sup}} \leq z(1 - N) \int_0^{\varepsilon_{\sup}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon dF(\varepsilon) \tag{37}$$

The quantity r_1 is the value of r such that (37) is satisfied with equality.■

A5. Proof of Proposition 5

First, let us characterize the quantity $\widetilde{\varepsilon}$. It is the highest value of the taste index such that the quantity traded is efficient. A buyer who spends all his money in one trade can buy the quantity $\omega M/N$. Hence, $\widetilde{\varepsilon}$ satisfies:

$$\widetilde{\varepsilon} u'\left(\frac{\omega M}{N}\right) = 1$$

Cases (i) and (ii) of the proposition results from equations (24) and (25).

Second, let us turn to the existence and uniqueness of the monetary equilibrium. We determine the unique steady-state value of ωM which can be interpreted as the level of real money balances. According to (24):

$$\varepsilon u'(q_\varepsilon) - 1 = \frac{\lambda_\varepsilon}{\omega}, \qquad \forall \varepsilon > \widetilde{\varepsilon}$$

From (26),

$$\frac{\omega_{-1}}{\omega} = \beta \left\{ z(1 - N) \int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon u'\left(\frac{\omega M}{N}\right) - 1 \, dF(\varepsilon) + 1 \right\}$$

The growth rate of the money supply is $\gamma - 1$. Hence,

$$\frac{M}{M_{-1}} = \gamma \Longrightarrow \frac{\omega_{-1}}{\omega} = \gamma \frac{q_{\varepsilon,-1}}{q_{\varepsilon}}$$

In the steady-state, we must have $q_{\varepsilon,-1} = q_{\varepsilon}$ for all ε and, therefore,

$$\int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon u' \left(\frac{\omega M}{N} \right) - 1 \, dF(\varepsilon) = \frac{\frac{\gamma}{\beta} - 1}{z\,(1-N)}.$$

From $\widetilde{\varepsilon} u' \left(\frac{\omega M}{N} \right) = 1$ we have:

$$u' \left(\frac{\omega M}{N} \right) \int_{\frac{1}{u'\left(\frac{\omega M}{N}\right)}}^{\varepsilon_{\sup}} 1 - F(\varepsilon) \, d\varepsilon = \frac{\frac{\gamma}{\beta} - 1}{z\,(1-N)} \tag{38}$$

The left-hand side of (38) is decreasing in ωM. Furthermore, $LHS\,[0] = +\infty$ and $LHS \left[N u'^{-1} \left(\frac{1}{\varepsilon_{\sup}} \right) \right] = 0$. Hence, if $\frac{\gamma}{\beta} > 1$ then there is a unique $\omega M \in \left(0, N u'^{-1} \left(\frac{1}{\varepsilon_{\sup}} \right) \right)$ that satisfies (38). Furthermore, when $\frac{\gamma}{\beta}$ approaches one, ωM approaches $N u'^{-1} \left(\frac{1}{\varepsilon_{\sup}} \right)$ and $\widetilde{\varepsilon}$ approaches $\varepsilon_{\sup}$.■

APPENDIX B: Generalization of the terms of trade

In Appendix B we show that the claims we make in the text are also true for generalized Nash bargaining.

B1. Indivisible goods and indivisible money with lotteries

In this appendix, we show that lotteries on money remove the no-trade inefficiency even when the terms of trade are determined by the generalized Nash solution.[18] We do not consider here the conditions for the existence and uniqueness of an equilibrium. We take as granted that $V_B - V_S \geq C$, which is a necessary and sufficient condition for a monetary equilibrium.

Let $\theta \in (0,1)$ be the buyers' bargaining power and τ_ε the probability that money changes hands in an ε-meeting. The outcome of a negotiation between a buyer and a seller is $\tau_\varepsilon \in [0,1]$ that maximizes the following Nash product:

$$\tau_\varepsilon = \arg\max \left[\varepsilon U - \tau_\varepsilon \left(V_B - V_S\right)\right]^\theta \left[-C + \tau_\varepsilon \left(V_B - V_S\right)\right]^{1-\theta} \tag{39}$$

$$\begin{aligned} \text{s.t. } 0 &\leq \tau_\varepsilon \leq 1 & (40)\\ \varepsilon U &\geq \tau_\varepsilon \left(V_B - V_S\right) & (41)\\ C &\leq \tau_\varepsilon \left(V_B - V_S\right) & (42)\end{aligned}$$

A necessary condition for (41) and (42) to be satisfied is that $\varepsilon U \geq C$.

The first-order conditions imply that the optimal τ_ε satisfies:

$$\tau_\varepsilon = \begin{cases} \frac{(1-\theta)\varepsilon U + \theta C}{V_B - V_S} & \text{if} \quad \frac{C}{U} \leq \varepsilon \leq \frac{V_B - V_S - \theta C}{(1-\theta)U} \\ 1 & \qquad \varepsilon > \frac{V_B - V_S - \theta C}{(1-\theta)U} \end{cases} \tag{43}$$

[18] As in the text, to focus on the welfare consequences of the indivisibility of money, we only allow lotteries on money. However, we have derived the solution when agents are allowed to use lotteries on both goods and money. In this model we also find that agents trade for all $\varepsilon \geq \frac{C}{U}$, which means that the *no-trade inefficiency* is removed. The main difference, however, is that the goods changes hands with probability less than one if ε is large and if the sellers have sufficient bargaing power. The fact that agents trade goods with probability less than one corresponds to what we call the *too-little trade* inefficiency in the divisible goods, indivisible money, and lottery model of Section 3.3.

The participation constraints (41) and (42) are satisfied for all $\varepsilon \geq \frac{C}{U}$. Indeed, for all $\varepsilon \in \left[\frac{C}{U}, \frac{V_B - V_S - \theta C}{(1-\theta)U}\right]$ we have:

$$\begin{aligned} \varepsilon U - \tau_\varepsilon (V_B - V_S) &= \theta (\varepsilon U - C) \geq 0 \\ -C + \tau_\varepsilon (V_B - V_S) &= (1-\theta)(\varepsilon U - C) \geq 0 \end{aligned}$$

If $\varepsilon > \frac{V_B - V_S - \theta C}{(1-\theta)U}$, the participation constraints are satisfied because of $V_B - V_S \geq C$. Consequently, the reservation value is $\overline{\varepsilon} = \frac{C}{U}$.

B2. Divisible goods and indivisible money

This appendix shows that the no-trade, the too-much-trade, and the too-little-trade inefficiencies identified in the divisible goods and indivisible money model of Section 3 are still present when the terms of trade are determined through generalized Nash bargaining. We will neither demonstrate the existence of a monetary equilibrium nor discuss its uniqueness. We will assume that $V_B - V_S > 0$, which is a necessary condition for the existence of a monetary equilibrium.

The bargaining Suppose that the bargaining power of buyers is $\theta \in]0, 1[$. The outcome of the bargaining between a buyer and the seller is given by the generalized Nash solution. We assume that during a negotiation the bargainers continue to meet other traders. Consequently, the *threat points* of buyers and sellers are V_B and V_S, respectively (see Trejos and Wright, 1995).

Denote q_ε the solution to the general Nash bargaining game when the quality of the match is ε. Then,

$$q_\varepsilon = \arg\max_q \left[\varepsilon u(q) - (V_B - V_S)\right]^\theta \left[-q + V_B - V_S\right]^{1-\theta}$$

The first-order condition is:

$$\theta \varepsilon u'(q_\varepsilon)(-q_\varepsilon + V_B - V_S) = (1-\theta)(\varepsilon u(q_\varepsilon) - (V_B - V_S)) \tag{44}$$

Too-little-trade and too-much-trade inefficiencies. Rewrite equation (44) to get:

$$\theta u'(q_\varepsilon)(-q_\varepsilon + V_B - V_S) = (1-\theta)\left(u(q_\varepsilon) - \left(\frac{V_B - V_S}{\varepsilon}\right)\right)$$

The LHS is decreasing in q_ε whereas the RHS is increasing in q_ε. Furthermore, the RHS is increasing in ε. Consequently, q_ε is a *decreasing* function of ε. This result and the fact that the efficient quantity q_ε^* is increasing in ε show that the too-little-trade and the too-much-trade inefficiencies are also present under generalized Nash bargaining (see Figure B2).[19]

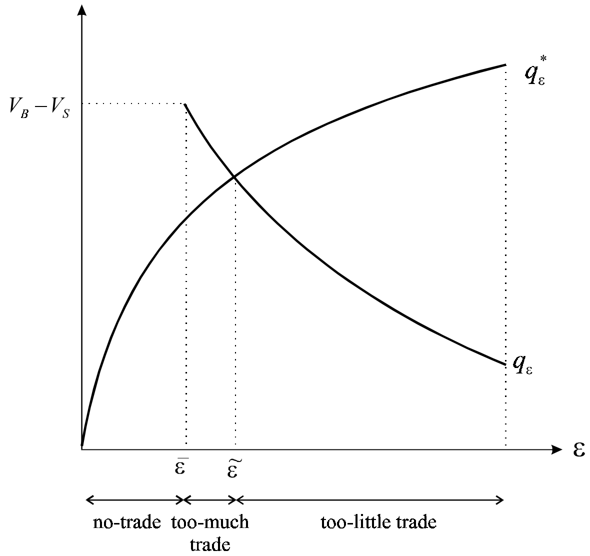

Figure B2.

The reservation value and the no-trade inefficiency From equation (44), the surpluses of buyers and sellers can be expressed as an endogenous fraction of the total surplus, which is $\varepsilon u(q_\varepsilon) - q_\varepsilon$:

$$-q_\varepsilon + V_B - V_S = \frac{1-\theta}{1-\theta+\theta\varepsilon u'(q_\varepsilon)}\left[\varepsilon u(q_\varepsilon) - q_\varepsilon\right] \tag{45}$$

$$\varepsilon u(q_\varepsilon) - (V_B - V_S) = \frac{\theta\varepsilon u'(q_\varepsilon)}{1-\theta+\theta\varepsilon u'(q_\varepsilon)}\left[\varepsilon u(q_\varepsilon) - q_\varepsilon\right] \tag{46}$$

The surplus of the buyer, $\varepsilon u(q_\varepsilon) - (V_B - V_S)$, is increasing in ε. Consequently, the reservation property holds, which implies that there is reservation value $\bar{\varepsilon}$ satisfying

$$\bar{\varepsilon} u(q_{\bar{\varepsilon}}) = V_B - V_S. \tag{47}$$

If $\varepsilon = \bar{\varepsilon}$, a buyer's surplus is zero. Moreover, equation (46) implies that the total surplus of the match is zero too, which because of (45) implies that $q_{\bar{\varepsilon}} = V_B - V_S$.

[19] Equation (44) also implies that $\varepsilon u(q_\varepsilon)$ is increasing in ε. Consequently, when the quality of the match increases, the surpluses of both the buyer and the seller increase.

The too-much-trade inefficiency is always present because of the concavity of the utility function which implies that $q_{\overline{\varepsilon}} > q^*_{\overline{\varepsilon}}$ (see Figure B2). To see this note that $q^*_{\overline{\varepsilon}}$ satisfies $\overline{\varepsilon}u'(q^*_{\overline{\varepsilon}}) = 1$ and $q_{\overline{\varepsilon}}$ satisfies $\overline{\varepsilon}u(q_{\overline{\varepsilon}}) = q_{\overline{\varepsilon}}$. By combining these two equations we find $u'(q^*_{\overline{\varepsilon}}) = \frac{u(q_{\overline{\varepsilon}})}{q_{\overline{\varepsilon}}}$, which can only be satisfied if $q_{\overline{\varepsilon}} > q^*_{\overline{\varepsilon}}$. To see that the no-trade inefficiency is always present, assume, to the contrary, that $\overline{\varepsilon} = 0$. From (47), $V_B = V_S$. Consequently, a monetary equilibrium exists if and only if $\overline{\varepsilon} > 0$.

B3. Divisible goods and indivisible money with lotteries

We will show that with lotteries on money the only inefficiency that remains is the no-trade inefficiency. We will also check under which condition this inefficiency vanishes.

The bargaining The terms of trade in an ε-meeting maximize the Nash product:

$$(q_\varepsilon, \tau_\varepsilon) = \arg\max_{q,\tau} \left[\varepsilon u(q) - \tau_\varepsilon \left(V_B - V_S\right)\right]^\theta \left[-q + \tau_\varepsilon \left(V_B - V_S\right)\right]^{1-\theta} \text{ s.t. } \tau_\varepsilon \leq 1 \tag{48}$$

If the constraint on τ_ε is not binding, then the solution of this program is:

$$q_\varepsilon = q^*_\varepsilon \tag{49}$$

$$\tau_\varepsilon = \frac{(1-\theta)\,\varepsilon u(q^*_\varepsilon) + \theta q^*_\varepsilon}{V_B - V_S} \tag{50}$$

Hence, the constraint on τ_ε (i.e., $\tau_\varepsilon = 1$) is binding if:

$$(1-\theta)\,\varepsilon u(q^*_\varepsilon) + \theta q^*_\varepsilon > V_B - V_S \tag{51}$$

The LHS of (51) is increasing in ε. Consequently, there will be a $\widetilde{\varepsilon}$ satisfying:

$$(1-\theta)\,\widetilde{\varepsilon} u(q^*_{\widetilde{\varepsilon}}) + \theta q^*_{\widetilde{\varepsilon}} = V_B - V_S \tag{52}$$

The terms of trade, $(q_\varepsilon, \tau_\varepsilon)$, when the constraint on τ_ε is binding satisfy $\tau_\varepsilon = 1$ and (44). Equation (44) implies that q_ε is a decreasing function of ε. Moreover, q^*_ε is increasing in ε. But this implies that $q_\varepsilon \leq q^*_\varepsilon$ for all $\varepsilon > \widetilde{\varepsilon}$.

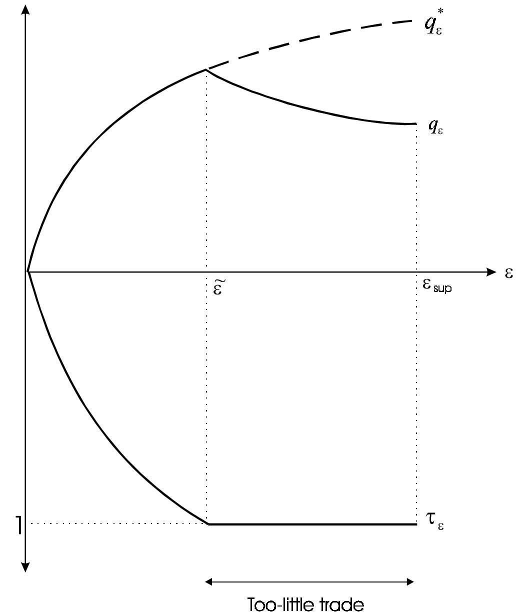

Figure B3

Summary (see Figure B3)**:** For all $\varepsilon < \widetilde{\varepsilon}$ we have $q_\varepsilon = q^*_\varepsilon$ and $\tau_\varepsilon < 1$, for $\varepsilon = \widetilde{\varepsilon}$ we have $q_\varepsilon = q^*_\varepsilon$ and $\tau_\varepsilon = 1$, and for all $\varepsilon > \widetilde{\varepsilon}$, $q_\varepsilon < q^*_\varepsilon$ and $\tau_\varepsilon = 1$. This confirms that the only inefficiency left with lotteries is the too-little inefficiency.

The condition for efficiency Traders exchange efficient quantities in all meetings if $\widetilde{\varepsilon} \geq \varepsilon_{\text{sup}}$. This condition can be rewritten as follows:

$$(1-\theta)\,\varepsilon u(q^*_{\varepsilon_{\text{sup}}}) + \theta q^*_{\varepsilon_{\text{sup}}} < V_B - V_S \tag{53}$$

If $\widetilde{\varepsilon} \geq \varepsilon_{\text{sup}}$, we have, for all ε, $q_\varepsilon = q^*_\varepsilon$ and $\tau_\varepsilon\,(V_B - V_S) = (1-\theta)\,\varepsilon u(q^*_\varepsilon) + \theta q^*_\varepsilon$. Consequently, the value functions of buyers and sellers satisfy:

$$rV_B = z\,(1-N)\,\theta \int_0^{\varepsilon_{\text{sup}}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon dF(\varepsilon) - \mu\,(V_B - V_S) \tag{54}$$

$$rV_S = zN\,(1-\theta) \int_0^{\varepsilon_{\text{sup}}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon\, dF(\varepsilon) + \delta\,(V_B - V_S) \tag{55}$$

Consequently,

$$(r+\mu+\delta)\,(V_B - V_S) = z\left\{(1-N)\,\theta - N\,(1-\theta)\right\} \int_0^{\varepsilon_{\text{sup}}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon dF(\varepsilon) \tag{56}$$

The condition (53) yields:

$$\frac{z\left\{(1-N)\,\theta - N\,(1-\theta)\right\} \int_0^{\varepsilon_{\text{sup}}} \varepsilon u(q^*_\varepsilon) - q^*_\varepsilon dF(\varepsilon)}{r+\mu+\delta} > (1-\theta)\,\varepsilon u(q^*_{\varepsilon_{\text{sup}}}) + \theta q^*_{\varepsilon_{\text{sup}}} \tag{57}$$

First, a necessary condition for efficiency is $(1-N)\,\theta - N\,(1-\theta) > 0$, that is $\theta > N$. This imply that the too-little trade inefficiency will remain for any value of the parameters if the bargaining power of buyers is less than the fraction of buyers in the economy. Second, in the absence of inflation ($\mu = \delta = 0$) and if $\theta > N$, the efficiency condition is satisfied for r close to zero. If agents become infinitely patient and if the bargaining power of buyers is sufficiently high, then the too-little trade inefficiency vanishes.

B4. Divisible goods and divisible money

The game The terms of trades are determined through bargaining games with alternating offers (See Shi, 1999).[20] We consider the bargaining between agent i of household h_i and agent j from any other household. Each period is divided into an infinite number of subperiods of length Δ. If, in a given subperiod, it is agent i's turn to make an offer and agent j rejects the offer, in the following subperiod it is agent j's turn to make a counteroffer. If an offer by a buyer is refused, the negotiation breaks down with probability $\theta\Delta$ ($\theta \in (0,1)$). If an offer by a seller is refused, the negotiation breaks down with probability $(1-\theta)\,\Delta$. We will see that the parameter θ can be interpreted as the bargaining power of buyers. The possibility of an exogenous break-down of the negotiation gives an incentive to traders to agree immediately.[21]

In the alternating offer game, offers and counteroffers converge to the same limiting proposal when Δ goes to zero. Consequently, the first-mover advantage vanishes when Δ goes to zero.[22] Because of this and because, as we will see, it facilitates the derivation of the dynamic equation describing the marginal utility of money, we let members of household h_i make the first offer in all meetings. In equilibrium all households have the same characteristics: as a consequence, first offers of h_i are always accepted. Moreover, because the length of time between two consecutive offers is infinitesimal, these first offers are exactly the counteroffers that would have been made by h_i's partners.

[20] See page 27 for additional details of the bargaining.

[21] This break-down may be explained by the weariness of players when the agreement is postponed. Instead of a break-down of the bargaining, cost of delaying an agreement could arise from the fact that households discount the future (Shi 1998).

[22] This is standard argument in the bargaining literature. See, for example, Muthoo (1999, chapter 3), Osborne and Rubinstein (1990, chaper 3).

The offers Agents i and j bargain over the quantity q_ε that is produced, exchanged, and consumed and over a monetary transfer x_ε. Consider the case when a buyer of household h_i makes a proposal in a given subperiod. The buyer, following the strategy given to him by household h_i, proposes the terms of trade $\left(q_\varepsilon^b, x_\varepsilon^b\right)$ if the quality of the match is ε. In the same way, if a seller of household h_i must make a proposal, he proposes $(q_\varepsilon^s, x_\varepsilon^s)$ if the quality of the match is ε.

Buyers and the sellers in a match have opposite interests. Consequently, $\left(q_\varepsilon^b, x_\varepsilon^b\right)$ makes the seller of another household just indifferent between accepting or rejecting the offer. Denote D_ε^s the expected surplus of the seller if he rejects the offer: it is taken as given by the household of the buyer. Then, the terms of trade $\left(q_\varepsilon^b, x_\varepsilon^b\right)$ must satisfy:

$$-q_\varepsilon^b + x_\varepsilon^b \Omega = D_\varepsilon^s \tag{58}$$

According to (58), the seller is just indifferent between accepting the offer (left-hand side of (58)) or rejecting it (right-hand side of (58)). If the seller rejects the offer, he will make a counteroffer if the negotiation has not been terminated, which happens with probability $(1 - \theta\Delta)$. Thus, the expected surplus of the seller is defined as

$$D_\varepsilon^s = (1 - \theta\Delta)\left[-Q_\varepsilon^s + X_\varepsilon^s \Omega\right] \tag{59}$$

The previous equation simply states that if the seller refuses the current offer, he can make the counteroffer $(Q_\varepsilon^s, X_\varepsilon^s)$ with probability $(1 - \theta\Delta)$. By a similar reasoning, $(q_\varepsilon^s, x_\varepsilon^s)$ makes the buyer of another household just indifferent between accepting or rejecting the offer. If we denote D_ε^b the expected surplus of the buyer if this round of the negotiation fails, which is taken as given by household h_i, then $(q_\varepsilon^s, x_\varepsilon^s)$ must satisfy

$$\varepsilon u\left(q_\varepsilon^s\right) - x_\varepsilon^s \Omega = D_\varepsilon^b \tag{60}$$

where

$$D_\varepsilon^b = \left(1 - (1 - \theta)\Delta\right)\left[\varepsilon u\left(Q_\varepsilon^b\right) - X_\varepsilon^b \Omega\right] \tag{61}$$

The constraints When the household determines the terms of trades for its buyers and sellers, it is subject to two sets of constraints. First, buyers cannot spend more than what they have, i.e., $x_\varepsilon^b \leq \frac{m}{N}$. The multiplier associated with this constraint will be denoted by λ_ε. Second, sellers cannot ask for more than what sellers hold, i.e., $x_\varepsilon^s \leq \frac{M}{N}$. The multiplier associated with this constraint will be denoted by π_ε.

The law of motion of money holdings is given by:

$$m_{+1} - m = \tau + z(1-N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon^s dF(\varepsilon) - z(1-N)N \int_0^{\varepsilon_{\sup}} x_\varepsilon^b dF(\varepsilon) \qquad (62)$$

The first term of the right-hand side of (62) is the transfer received by the household from the government. The second term of the right-hand side of (62) has the following interpretation. With probability zN a seller meets a buyer and there is a single coincidence of wants: the seller makes the offer x_ε^s which is immediately accepted by the buyer. The third term has a similar interpretation.

The program of the household For each period, the household chooses $\left(m_{+1}, q_\varepsilon^b, x_\varepsilon^b, q_\varepsilon^s, x_\varepsilon^s\right)$ s.t. (58), (60), and (62) to solve the following dynamic programming problem: :

$$V(m) = \max_{m_{+1}, q_\varepsilon^b, x_\varepsilon^b, q_\varepsilon^s, x_\varepsilon^s} \left\{ zN(1-N) \int_0^{\varepsilon_{\sup}} \varepsilon u\left(q_\varepsilon^b\right) dF(\varepsilon) - zN(1-N) \int_0^{\varepsilon_{\sup}} q_\varepsilon^s dF(\varepsilon) + \right.$$

$$\left. + zN(1-N) \int_0^{\varepsilon_{\sup}} \lambda_\varepsilon \left(\frac{m}{N} - x_\varepsilon^b\right) dF(\varepsilon) + zN(1-N) \int_0^{\varepsilon_{\sup}} \pi_\varepsilon \left(\frac{M}{N} - x_\varepsilon^s\right) dF(\varepsilon) + \beta V(m_{+1}) \right\} \qquad (63)$$

The first-order conditions and the envelope condition are as follows:

$$\varepsilon u'\left(q_\varepsilon^b\right) = \frac{\lambda_\varepsilon + \omega}{\Omega} \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \qquad (64)$$

$$\varepsilon u'\left(q_\varepsilon^s\right) = \frac{\Omega}{\omega - \pi_\varepsilon} \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \qquad (65)$$

$$\lambda_\varepsilon \left(\frac{m}{N} - x_\varepsilon^b\right) = 0 \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \qquad (66)$$

$$\pi_\varepsilon \left(\frac{M}{N} - x_\varepsilon^s\right) = 0 \qquad \forall \varepsilon \in [0, \varepsilon_{\sup}] \qquad (67)$$

$$\frac{\omega_{-1}}{\beta} = z(1-N) \int_0^{\varepsilon_{\sup}} \lambda_\varepsilon \, dF(\varepsilon) + \omega \qquad (68)$$

There are two additional conditions relative to first-order conditions in the text. Equation (65) states the marginal cost of production must be equal to the marginal utility of the money received by the seller. Condition (65) is the additional Kuhn-Tucker condition.

The terms of trade in equilibrium At the symmetric equilibrium, the values for the different variables of the household h_i are just equal to the values of the same variables for the other households. As a consequence, the capital-case variables and the lower-case variables are equal. Equations (58), (59), (60), and (61) can be rewritten as follows:

$$-q_\varepsilon^b + x_\varepsilon^b\omega = (1-\theta\Delta)\left[-q_\varepsilon^s + x_\varepsilon^s\omega\right] \tag{69}$$

$$\varepsilon u\left(q_\varepsilon^s\right) - x_\varepsilon^s\omega = (1-(1-\theta)\Delta)\left[\varepsilon u\left(q_\varepsilon^b\right) - x_\varepsilon^b\omega\right] \tag{70}$$

First, we show that for any ε, λ_ε and π_ε are either both positive or both zero. Suppose, on the contrary, that $\lambda_\varepsilon > 0$ and $\pi_\varepsilon = 0$. From (64) and (65), $\varepsilon u'(q_\varepsilon^s) = 1$ and $\varepsilon u'(q_\varepsilon^b) > 1$. Hence, $q_\varepsilon^s > q_\varepsilon^b$. Furthermore, according to (66), $\lambda_\varepsilon > 0$ implies that $x_\varepsilon^b = m/N$. From (69) when Δ approaches 0 we have $q_\varepsilon^s - q_\varepsilon^b = \left[x_\varepsilon^s - \frac{m}{N}\right]\omega$. The left-hand side is positive whereas the right-hand side cannot be positive because $x_\varepsilon^s \leq \frac{m}{N}$ $(\pi_\varepsilon = 0)$. A similar contradiction appears when we assume that $\lambda_\varepsilon = 0$ and $\pi_\varepsilon > 0$. Hence, λ_ε and π_ε are either both positive or both zero.

First case: If $\lambda_\varepsilon > 0$ and $\pi_\varepsilon > 0$, the constraints on money holdings are binding and, therefore, $x_\varepsilon^s = x_\varepsilon^b = \frac{m}{N}$. Equations (69) and (70) yield:

$$q_\varepsilon^s - q_\varepsilon^b = -\theta\Delta\left[-q_\varepsilon^s + \frac{m}{N}\omega\right]$$

$$\varepsilon u\left(q_\varepsilon^s\right) - \varepsilon u\left(q_\varepsilon^b\right) = -(1-\theta)\Delta\left[\varepsilon u\left(q_\varepsilon^b\right) - \frac{m}{N}\omega\right]$$

The first equation implies that $\lim_{\Delta\to 0} q_\varepsilon^s = \lim_{\Delta\to 0} q_\varepsilon^b = q_\varepsilon$. Divide the second equation by $\left(q_\varepsilon^s - q_\varepsilon^b\right)$ and take the limit as $\Delta \to 0$ to get

$$\varepsilon u'\left(q_\varepsilon\right) = \frac{(1-\theta)\left[\varepsilon u\left(q_\varepsilon\right) - \frac{m}{N}\omega\right]}{\theta\left[-q_\varepsilon + \frac{m}{N}\omega\right]}$$

Use the first-order condition to get

$$-q_\varepsilon + \frac{m}{N}\omega = \frac{1-\theta}{1+\theta\left(\frac{\lambda_\varepsilon}{\omega}\right)}\left[\varepsilon u\left(q_\varepsilon\right) - q_\varepsilon\right] \tag{71}$$

According to (71), the fraction of the surplus of the match received by the seller is endogenous and depends on the ratio $\lambda_\varepsilon/\omega$. Notice, that the q_ε that solves (71) also maximizes the following Nash product:

$$q_\varepsilon = \arg\max_q \left[\varepsilon u(q) - \frac{m}{N}\omega\right]^\theta \left[-q + \frac{m}{N}\omega\right]^{1-\theta}$$

This confirms our interpretation that θ is the buyer's bargaining power.

Second case: If $\lambda_\varepsilon = 0$ and $\pi_\varepsilon = 0$, equations (64) and (65) imply $q_\varepsilon = q_\varepsilon^*$. Moreover, when $\Delta \to 0$ (69) and (70) imply:

$$-q_\varepsilon + x_\varepsilon \omega = (1-\theta)\left[\varepsilon u(q_\varepsilon) - q_\varepsilon\right] \tag{72}$$

According to (72), the fraction of the total surplus of the match received by the seller equals the probability that the negotiations breaks down if a buyer refuses the offer from a seller.

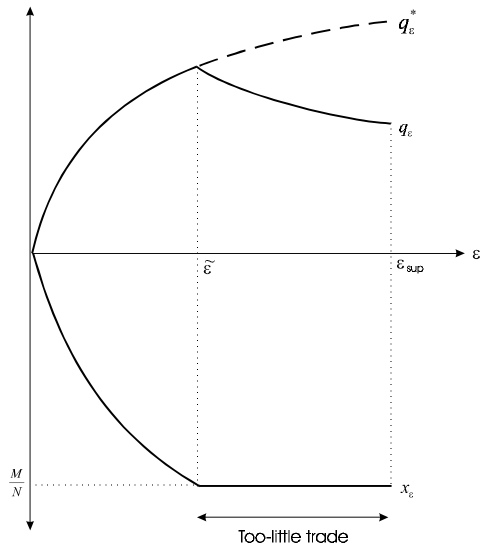

Figure B4

Summary (see Figure B4)**:** For all $\varepsilon < \widetilde{\varepsilon}$ we have $q_\varepsilon = q_\varepsilon^*$ and $x_\varepsilon < \frac{M}{N}$, for $\varepsilon = \widetilde{\varepsilon}$ we have $q_\varepsilon = q_\varepsilon^*$ and $x_\varepsilon = \frac{M}{N}$, and for all $\varepsilon > \widetilde{\varepsilon}$, $q_\varepsilon < q_\varepsilon^*$ and $x_\varepsilon = \frac{M}{N}$. This confirms that the only inefficiency left with divisible money is the too-little trade inefficiency.

Determination of $\widetilde{\varepsilon}$ Denote $\widetilde{\varepsilon}$ the match quality such that for all $\varepsilon \leq \widetilde{\varepsilon}$ sellers and buyers exchange efficient quantities. We must have $q_{\widetilde{\varepsilon}} = q_{\widetilde{\varepsilon}}^*$ and $x_{\widetilde{\varepsilon}} = \frac{m}{N}$. Thus, the pair $(\widetilde{\varepsilon}, q_{\widetilde{\varepsilon}})$ satisfies.

$$\begin{aligned} \widetilde{\varepsilon} u'(q_{\widetilde{\varepsilon}}) &= 1 \\ -q_{\widetilde{\varepsilon}} + \frac{m}{N}\omega &= (1-\theta)\left[\widetilde{\varepsilon} u(q_{\widetilde{\varepsilon}}) - q_{\widetilde{\varepsilon}}\right] \end{aligned}$$

These two equations imply that that $\widetilde{\varepsilon}$ is increasing in the real stock of money, $m\omega$.

The marginal value of money ω in equilibrium Let us determine the marginal value of money ω in equilibrium. Rewrite the envelope condition (68) as follows:

$$\frac{\omega_{-1}}{\beta} = z(1-N)\int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \lambda_\varepsilon \, dF(\varepsilon) + \omega$$

By dividing by $\frac{\omega}{\beta}$ and substituting $\lambda_\varepsilon/\omega$ by its expression given by (64) we have:

$$\frac{\omega_{-1}}{\omega} = \beta\left\{z(1-N)\int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon u'\left(q_\varepsilon\right) - 1 \, dF(\varepsilon) + 1\right\} \tag{73}$$

where q_ε satisfies (71), respectively:

$$-q_\varepsilon + \frac{m}{N}\omega = \frac{1-\theta}{1-\theta+\theta\varepsilon u'\left(q_\varepsilon\right)}\left[\varepsilon u(q_\varepsilon) - q_\varepsilon\right] \tag{74}$$

The left-hand side is decreasing in q_ε and increasing in ωm whereas the right-hand side is increasing in q_ε. Hence, for all $\varepsilon > \widetilde{\varepsilon}$, q_ε is increasing in ωm. The gross growth rate of the money supply is $\gamma = \frac{m}{m_{-1}}$. In a steady state, $q_\varepsilon = q_{\varepsilon,-1}$. Equation (74) implies that $m\omega$ is constant and, therefore, $\frac{\omega_{-1}}{\omega} = \gamma$. From (73), the real stock of money, $m\omega$, is entirely characterized by:

$$\int_{\widetilde{\varepsilon}}^{\varepsilon_{\sup}} \varepsilon u'\left(q_\varepsilon\left(m\omega\right)\right) - 1 \, dF(\varepsilon) = \frac{\frac{\gamma}{\beta} - 1}{z\left(1-N\right)} \tag{75}$$

Efficiency Finally, let us determine the optimal growth rate of the money supply. To do this, we maximizes the expected discounted utility of the representative household, which is

$$W = zN(1-N)\int_0^{\varepsilon_{\sup}} \varepsilon u(q_\varepsilon) - q_\varepsilon \, dF(\varepsilon)$$

The first order condition with respect to $m\omega$ gives:

$$\int_0^{\varepsilon_{\sup}} \left(\varepsilon u'(q_\varepsilon) - 1\right)\frac{\partial q_\varepsilon}{\partial m\omega} \, dF(\varepsilon) = 0$$

Because $\frac{\partial q_\varepsilon}{\partial m\omega} \geq 0$ for all ε, the previous condition is equivalent to $\varepsilon u'(q_\varepsilon) = 1$ for all ε, and then $\widetilde{\varepsilon} = \varepsilon_{\text{sup}}$. In this case, the left-hand side of (75) is zero, and, therefore, $\gamma = \beta$. The Friedman's rule holds.

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