Quantity Constraints, Poverty Lines and Poverty Orderings

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Abstract

By limiting the scope for substitution between commodities, other things equal quantity constraints raise the cost of living. Thus, rationed families have higher povery lines than unconstrained ones. This heterogeneity in both resources and poverty lines means that, in principle, bivariate dominance results are required to order distributions in terms of poverty.

Keywords: poverty lines, rationing, poverty orderings, bivariate stochastic dominance.

JEL codes: C_2, C_3, D_6, I_3

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1. Introduction

There are many conceptual problems related to the measurement of poverty. The period over which resources are defined matters because transitory fluctuations tend to average out over time. The exact definition of resources is also important: those who advocate the adoption of an income standard do so on account of an entitlement, or right, to a minimum amount of cash in hand. Others will argue that consumption expenditure is the appropriate standard, since savings may be used to supplement income at times when the pay is low. See Atkinson (1989, ch. 1) for a full account of these issues.

The purpose of this paper is to inquire how quantity constraints may be dealt with when measuring poverty. Quantity constraints may arise in a variety of settings. Involuntary unemployment causes individuals to consume a non-optimal amount of leisure. Necessities such as bread have been publicly provided in several developed countries in times of war in the form of rations. In various parts of India and China today, rationing of necessities still prevails to date (Drèze and Sen, 1995). More generally, public goods whose consumption by the individual cannot be altered may effectively be treated as rationed goods (Deaton and Muellbauer, 1980; p.110).

The economic theory literature is abundant with treatments of optimization problems subject to quantity constraints. Le Chatelier's principle (Samuelson, 1972) relates the comparative statics of quantity constrained and unconstrained problems. At a more specialized level, the theory of choice under rationing is elaborated in Pollak (1969) and Neary and Roberts (1980). In section 2 below we use this conceptual framework to show that poverty lines must be appropriately defined to reflect family specific quantity constraints. In turn, it is shown in section 3 that bivariate dominance results (Atkinson and Bourguignon, 1982) are required to order distributions if families differ according to both resources and quantity constraints. Section 4 provides a conceptual example of a joint distribution of income and poverty lines, with the purpose of illustrating the application of the derived bivariate dominance conditions for the ordering of distributions in presence of quantity restrictions. Section 5 discusses data requirements for the purpose of measuring poverty in presence of rationing, while section 6 concludes the paper.

2. Quantity constraints and the poverty line

Suppose that a family consumes a vector of goods $q = (q_1, ..., q_k)$ at prevailing prices $p = (p_1, ..., p_k)$, and a target level of living u^o. The minimum expenditure required to attain u^o defines a poverty line z^o:

$$z^o = \min_q c(u, p) \qquad s.t. \qquad v(q) = u^o \tag{1}$$

where $c(u, p)$ is the cost function and $v(.)$ defines a preference ordering. Now suppose one good, say the first one, is rationed at a quantity $\bar{q}_1$. This good could denote leisure consumption in a developed country, or some necessity such as grain, or uncontaminated water in a developing country. Because the family faces an additional constraint over the earlier problem (1), reaching the target level u^o will now require an expenditure level $z^R \geq z^o$:

$$z^R = \min_{q_2,\dots,q_k} c(u, p, \bar{q}_1) \qquad s.t. \qquad v(q) = u^o \qquad and \qquad q_1 = \bar{q}_1 \tag{2}$$

The fact that the family has no control over the amount of good 1 consumed will limit the scope for substitution between goods as prices change. This result is known as Le Chatelier's principle (Samuelson, 1972). The related result, that quantity constraints will raise the poverty line, follows from the envelope theorem:

$$c(u^o, p) = \min_{\bar{q}_1} c^R(u^o, p, \bar{q}_1) \tag{3}$$

The case $z^R = z^o$ obtains only in the situation where the family would have optimally chosen to consume the ration $\bar{q}_1$. This may also be noted by writing the relation

$$c^R(u^o, p, \bar{q}_1) = c(u^o, p_1^*, p_2, ..., p_k) + (p_1 - p_1^*)\bar{q}_1 \tag{4}$$

and by noting that if the ration is optimally chosen, p_1^*, the shadow price of q_1 evaluated at $\bar{q}_1$ equals the market price p_1.

When the family is constrained to consume $\bar{q}_1$, the poverty line z^R will generally be above z^o. Note that this result holds whether the family consumes too much, or too little of good 1. Finally, note that families may be rationed on different markets. Alternatively, they may have access to different amounts of the rationed good. Under such circumstances, the family specific poverty line z^R may be continuously distributed in the population. The problem of measuring poverty subject to heterogeneity along the dimensions of resources and needs is discussed in the next section.

3. Quantity constraints and the measurement of poverty

Families typically differ in tastes. However, for a poverty line to be operational we must abstract from such differences in tastes. Under such circumstances, if families all face the same quantity constraint $\bar{q}_1$, poverty accounting proceeds as usual, with a poverty line $z = z^R$ replacing the threshold $z = z^o$. The problem we consider in this section however is the situation where quantity constraints may affect families differently. We thus choose to treat z as a continuous variable. A related problem, examined by Atkinson (1990), is one where needs are a function of family composition. The problem we study in this section however differs from Atkinson's in several respects. Firstly, Atkinson takes z to be a discrete variable, and secondly, he retains for z a finite set of states. Finally, household composition is easily observable, but detecting quantity constraints is far from being a costless exercise (we return to this last point in section 5).

The conceptual framework needed to undertake welfare comparisons in a multi-dimensional context is developed in Atkinson and Bourguigon (1982). The discussion below may certainly be seen as an application of their results in the context of poverty analysis. In what follows however we proceed in some detail, as we have found this body of theory to be fairly complex.

Assume then that $\pi(y, z)$ is a social valuation function where y denotes family income and z is the poverty line. $\pi(.)$ is typically assumed to be increasing in y, decreasing in z and the cross-derivative is taken to be positive. If z is unique, and set at the level z^o, then following Atkinson (1987) a wide class of poverty measures may be written in the form:

$$P = \int_0^{a_y} \pi(y, z^o) f_y(y) dy \tag{5}$$

where for $y \geq z^o$ $\pi(.) = 0$, and f_y is the distribution (i.e. the density function) of income defined over the income range $[0; a_y]$. Written in the form (5), larger values for P indicate a higher level of welfare for the poor; i.e. that poverty is lower. Let Π^- denote the class of poverty measures (5) such that the partial derivative with respect to y is non-negative, $\pi_y \geq 0$, likewise $\pi_z \leq 0$, and $\pi_{yz} \geq 0$ [1]. The exact form of $\pi(.)$ is the subject of much discussion, though the literature has very much evolved in the direction of ordinal analysis that is, finding conditions on f_y which guarantee a qualitative change in poverty for a given class of poverty

[1] In particular, the class Π^- requires that $\pi(y, z)$ be continuous at $y = z$. For this reason, the poverty head-count is not a member of the class Π^-.

indices (see Foster and Shorrocks, 1988). If for instance $F_y(t) = \int_0^t f_y(y)dy$ and $\Delta F_y = F_y^A - F_y^B$ is the change in the cumulative distribution function, poverty is lower under F_y^A, for all members of the class Π^-, if the condition $\Delta F_y \leq 0$ is satisfied everywhere in the income range $0 \leq y < z^o$.

How do these results change if both y and z become family specific? First consider equation (5). If z varies in the population, then we need to integrate $\pi(y,z)$ along both dimensions:

$$P = \int_0^{a_y} \int_{o_z}^{a_z} \pi(y,z) f(y,z) dz dy \tag{6}$$

where z is taken now to be distributed over the range $[o_z; a_z]$ [2]. It is clear in the context of (6) that in order to compare poverty between two populations it is no longer sufficient to plot the cumulative distribution functions F_y^1 and F_y^2 over some specified income range $[0; z^o]$. The distribution of income may be slower at changing than the distribution of z, or vice versa. Because we are in the presence of two dimensions, bivariate stochastic dominance results are now required.

Atkinson and Bourguignon (1982) provide a discussion on the comparison of multi-dimensional distributions in a general setting. Here we proceed along those same lines. Integrating (6) by parts, first with respect to z, we have

$$P = \int_0^{a_y} \left[\pi(y, a_z) \int_{o_z}^{a_z} f(y,z) dz - \int_{o_z}^{a_z} \pi_z(y,z) \int_{o_z}^{z} f(y,t) dt\, dz \right] dy \tag{7}$$

Next, we integrate (7) with respect to y to arrive at

$$\begin{aligned} P \quad = \quad & \pi(a_y, a_z) \int_0^{a_y} \int_{o_z}^{a_z} f(y,z) dz\, dy - \int_0^{a_y} \pi_y(y, a_z) \int_0^{y} \int_{o_z}^{a_z} f(s,t) dt\, ds\, dy \\ & - \int_{o_z}^{a_z} \pi_z(a_y, z) \int_0^{a_y} \int_{o_z}^{z} f(s,t) dt\, ds\, dz \\ & + \int_0^{a_y} \int_{o_z}^{a_z} \pi_{yz}(y,z) \int_0^{y} \int_{o_z}^{z} f(s,t) dt\, ds\, dz\, dy \end{aligned} \tag{8}$$

The first right-hand-side (RHS) term is a constant, and will play no role in comparing two situations. In what follows, it is omitted. Next write $F(y,z)$ as the integral $\int_0^y \int_{o_z}^z f(s,t) dt\, ds$. Then, (8) simplifies to

$$P \quad = \quad - \int_0^{a_y} \pi_y(y, a_z) F(y, a_z)\, dy$$

[2] In particular, we assume that the partial derivative $\pi_z(y,z)$, evaluated at $z = o_z$ is a finite quantity.

$$
\begin{aligned}
&- \int_{o_z}^{a_z} \pi_z(a_y, z) F(a_y, z)\, dz \\
&+ \int_0^{a_y} \int_{o_z}^{a_z} \pi_{yz}(y, z) F(y, z)\, dz\, dy \qquad (9)
\end{aligned}
$$

If $\phi(x_1, x_2)$ is a differentiable function, we may write

$$
\begin{aligned}
\int_0^{a_1} \phi(x_1, \theta) g(x_1, \theta) dx_1 &= \int_0^{a_1} \int_0^{\theta} \phi_{x_2}(x_1, x_2) g(x_1, \theta)\, dx_2\, dx_1 \\
&+ \int_0^{a_1} \phi(x_1, 0) g(x_1, \theta) dx_1 \qquad (10)
\end{aligned}
$$

Using this property, the first RHS term of (9) may be written as

$$
\begin{aligned}
\int_0^{a_y} \pi_y(y, a_z) F(y, a_z)\, dy &= \int_0^{a_y} \int_{o_z}^{a_z} \pi_{yz}(y, z) F(y, a_z) dz\, dy \\
&+ \int_0^{a_y} \pi_y(y, o_z) F(y, a_z)\, dy \qquad (11)
\end{aligned}
$$

Hence, the first and third RHS terms of (9) may be grouped into the following expression:

$$
\begin{aligned}
&\int_0^{a_y} \int_{o_z}^{a_z} \pi_{yz}(y, z) [F(y, z) - F(y, a_z)] dz\, dy \\
&- \int_0^{a_y} \pi_y(y, o_z) F(y, a_z)\, dy \qquad (12)
\end{aligned}
$$

and now (9) can be written as

$$
\begin{aligned}
P &= \int_0^{a_y} \int_{o_z}^{a_z} \pi_{yz}(y, z) [F(y, z) - F(y, a_z)] dz\, dy - \int_0^{a_y} \pi_y(y, o_z) F(y, a_z)\, dy \\
&- \int_{o_z}^{a_y} \pi_z(a_y, z) F(a_y, z)\, dz \qquad (13)
\end{aligned}
$$

Write $K(y, z) \equiv F(y, z) - F(y, a_z)$, and ΔF as the change in the joint distribution function, $F^A(y, z) - F^B(y, z)$. Then we may state the following sufficient conditions for poverty to be lower, or no higher in F^A, in relation to the class Π^-:

$$
\begin{aligned}
\Delta K(y, z) &\geq 0 \quad for \quad all \quad y \quad and \quad z \qquad (14a) \\
\Delta F(a_y, z) &\geq 0 \quad for \quad all \quad z \qquad (14b)
\end{aligned}
$$

Condition (14a) takes care of both the first and second RHS components of (13), since $K(y, o_z) = -F(y, a_z)$. Since we require $\Delta K(y, z) \geq 0$ everywhere, this also necessitates $\Delta F(y, a_z) \leq 0$, a well known first order stochastic dominance condition for the marginal distribution of income.

As the social valuation function $\pi(y, z)$ is non-increasing for $y > z$, we may in fact weaken the conditions (14), by re-stating them as follows:

$$\Delta K(y, z) \geq 0 \quad for \quad all \quad y \leq a_z \quad and \quad z \tag{15a}$$
$$\Delta F(a_z, z) \geq 0 \quad for \quad all \quad z \tag{15b}$$

If we evaluate $\Delta K(y, z)$ at the point (y, o_z), we obtain in (15a) the condition $\Delta F(y, a_z) \leq 0$ for all $y \leq a_z$. Once again, this is the equivalent of the poverty first order stochastic dominance condition of the uni-dimensional context, with the requirement that ΔF_y be negative everywhere in the range $[0; a_z]$, rather than $[0; z^o]$ if a unique poverty line z^o applied to the whole population (see Atkinson, 1987). Condition (15b) reflects the partial-derivative property $\pi_z \leq 0$, namely that the social valuation of income level y to a poor family decreases other things equal when it faces a higher poverty line. A sufficient condition for (15b) to be satisfied is that, for every income level below a_z, there is a transfer of probability mass from a higher to a lower poverty line. To clarify these issues, and to make the bivariate dominance conditions (15) more transparent, we provide next a conceptual example.

4. A conceptual example

Consider an initial situation $f^B(y, z)$ where y and z are jointly distributed as given in table 1, with $\varepsilon = \delta = 0$. The joint cumulative distribution $F^B(y, z)$ is obtained by cumulating the respective elements of table 1, once again setting $\varepsilon = \delta = 0$. Next, consider transferring probability mass from Z_2 to Z_1 (i.e. reducing the share of families subject to the higher poverty line). This is done in table 1 by taking $\varepsilon > 0$ for families with income level Y_1. Likewise, consider a redistributive policy which reduces the share of families with resources Y_1 by transferring a probability mass $\delta > 0$ from the cell (Y_1, Z_1) to the cell (Y_2, Z_1) in table 1. The resulting joint density and cumulative distribution are reported respectively in tables 1 and 2, this time taking ε and δ to be non-zero.

Table 1 : joint distribution of Y and Z

$f(y,z)$	Z_1	Z_2	total
Y_1	$0.25+\varepsilon-\delta$	$0.05-\varepsilon$	$0.30-\delta$
Y_2	$0.25+\delta$	0.15	$0.40+\delta$
Y_3	0.30	0	0.30
total	$0.80+\varepsilon$	$0.20-\varepsilon$	1

Note: $f^B(y,z)$ has $\varepsilon=\delta=0$; $f^A(y,z)$ has $\varepsilon>0$ and $\delta>0$.

Table 2 : cumulative distribution of Y and Z

$F(y,z)$	Z_1	Z_2
Y_1	$0.25+\varepsilon-\delta$	$0.30-\delta$
Y_2	$0.50+\varepsilon$	0.70
Y_3	$0.80+\varepsilon$	1

Note: $F^B(y,z)$ has $\varepsilon=\delta=0$; $F^A(y,z)$ has $\varepsilon>0$ and $\delta>0$

Given that there are two poverty lines, we take $o_z=Z_1$, and $a_z=Z_2$. Let us assume furthermore that $Y_2=a_z$. Turning to the conditions (15), define ΔK as the 2×2 matrix with elements $\Delta K(Y_i,Z_j)=\Delta F(Y_i,Z_j)-\Delta F(Y_i,a_z)$. For the first element for instance, $\Delta K(Y_1,Z_1)=\varepsilon-\delta-(-\delta)=\varepsilon$. Then

$$\Delta K=\begin{bmatrix}\varepsilon & 0\\ \varepsilon & 0\end{bmatrix}$$

so that the dominance condition (15a) is satisfied provided $\varepsilon>0$. Furthermore, when $\delta>0$ the condition for the marginal distribution of Y, namely $\Delta K(Y,o_z)=-\Delta F(Y,a_z)\geq 0$, is satisfied, giving in the present example $-\Delta F(Y,a_z)=\begin{bmatrix}\delta\\ 0\end{bmatrix}$. For the second condition, (15b), we have $\Delta F(a_z,z)=[\varepsilon \quad 0]$, which again will hold if $\varepsilon>0$.

5. Data requirements

How do we go about measuring poverty, or applying the dominance conditions (15) in practice? While cross-sectional surveys provide detailed information on income sources and types of expenditure, allowing the researcher to consider various definitions of the y variable, construction of z–the family level poverty line, is a more challenging task. This is so, because it is difficult in family expenditure surveys to detect quantity constraints. The observed heterogeneity in consumption patterns is typically attributed to taste differences in the first instance, and for this reason little effort has been expanded in the direction of sorting out differences in preferences from limitations on choice.

Going back to section 2, to estimate the cost z^R of attaining a target level of living u^o, we need to observe the quantity constraints $\bar{q}$, and we must estimate demand systems which allow for rationing, in order to be able to recover the structural parameters of the underlying cost function (see for instance Deaton and Muellbauer, 1981).

There are markets in developed countries where quantity constraints are most likely to be binding. These include housing, health and employment [3]. Surveys then should ask specific questions such as *how many hours would you wish to be working at your normal wage?*, or *how much would you want to spend on housing?*, allowing the interviewee to state that the current choice is the desired one.

The above discussion also applies in the developing country context. There, however, the list of rationed goods is likely to be longer. A further problem is that some commodities may be rationed, while totally escaping the attention of the surveyor or public authorities [4]. For this reason, prior specialized surveys in the developing country context with emphasis on needs and aspirations for families may be required in order to charter the territory of rationed commodities.

One situation which is somewhat less demanding in terms of data requirements

[3] If we take an interest in life-cycle aspects of poverty, then clearly the list must also include the credit market.

[4] To give a concrete example, consider the demand for primary schooling. Low enrollment rates have been documented in various parts of the developing world. A recent explanation for this pattern, provided by education scientists, is as follows. National curricula have placed their emphasis on modern types of education, inspired from western models–often foreign to local norms and customs (McGuinn,1997). Parents, on the other hand, would wish their children to become acquainted with more traditional values and customs, that they can relate to, themselves. Absent this type of education, communities in rural areas often arrange for informal types of schooling, resulting in low enrollment of children in the national curriculum (Tawil, 2000).

is worth mentioning. If the social valuation function is additive:

$$\pi(y,z) = \pi_1(y) + \pi_2(z) \tag{16}$$

The cross-derivative π_{yz} vanishes, so that two situations may be ordered without knowledge of the joint distribution function $F(y,z)$; changes in the marginal distributions for y and z being sufficient to investigate the occurrence of poverty orderings. This may be seen by noting that the first RHS term of (13) is zero in the case where $\pi(y,z)$ is of the form (16). In contrast with a situation where the researcher is interested only in ordering distributions, measuring changes in poverty using a given poverty index will however still require knowledge of the value z corresponding to the resources y of each family.

6. Conclusions

By limiting the scope for substitution between commodities, other things equal, quantity constraints will raise the cost of living. Thus, quantity constrained families have higher poverty lines than unconstrained ones. This heterogeneity in both resources and poverty lines means that, in principle, bivariate dominance results are required to order distributions in terms of poverty.

As a country progresses in its economic development, it may be the case that access to certain necessities is improved, thus resulting in a gradual removal of quantity constraints. Such changes may be seen in the perspective of our discussion as beneficial for matters of poverty reduction. Conversely, involuntary unemployment and credit rationing may prevail more frequently in times of recession, thus making matters worse by raising the cost of living for those who already experience income losses. These effects may be easily overlooked by looking only at changes in the distribution of income.

Given that it is difficult in cross-section data to detect quantity constraints, in practice it may be useful to supplement the usual poverty statistics by surveys on the differential access of the population to rationed commodities. This may mean looking at the distribution of scarce necessities in the developing country context, and perhaps examining more closely the labour market in developed countries, looking at pay, but also emphasizing changes in hours worked. Ideally, we would wish to know how incomes and needs are jointly distributed in the population. Short of this, we suggest to look at both distributions separately, but not ignoring the latter at the expense of the former.

7. References

Atkinson, A. (1987): "On the Measurement of Poverty", *Econometrica 55*, 749-764.

Atkinson, A. (1989): *Poverty and Social Security*, Hemel Hempstead, Harvester Wheatsheaf.

Atkinson A. (1990): "Measuring Poverty and Differences in Family Composition", Taxation, Incentives and the Distribution of Income discussion paper 146, Sticerd, London School of Economics.

Atkinson A. and F. Bourguignon (1982): "The Comparison of Multi-Dimensioned Distributions of Economic Status", *Review of Economic Studies* 49, 183-201.

Deaton A. and J. Muellbauer (1980): *Economics and Consumer Behavior*, Cambridge, Cambridge University Press.

Deaton A. and J. Muellbauer (1981): "Functional Forms for Labor Supply and Commodity Demands with and without Quantity Restrictions", *Econometrica* 49, 1521-1532.

Drèze J. and A. Sen (1995): *India: Economic Development and Social Opportunity*, Delhi, Oxford University Press.

Foster J. and A. Shorrocks (1988): "Poverty Orderings", *Econometrica* 56, 173-177.

McGuinn N. (1997): "The Impact of Globalization on National Education Systems", *Prospects* 27,41-54.

Neary J. and K. Roberts (1980): "The Theory of Household Behaviour under Rationing", *European Economic Review* 13, 25-42.

Pollak R. (1969) " Conditional Demand Functions and Consumption Theory", *Quarterly Journal of Economics* 83, 60-78.

Samuelson P. (1972): "Maximum Principles in Analytical Economics", *American Economic Review* 62, 249-62.

Tawil S. (2000): "Household Demand, Basic Education and Exclusion: Focusing on out-of-School Children in Morocco", in Malmberg L., Hansen S. and K. Heino *eds*: *Basic Education for All: A Global Concern for Quality*, Abo Akademi Press.