Multistep Predictions for Multivariate GARCH Models: Closed Form Solution and the Value for Portfolio Management[†]

Jaroslava Hlouskova[‡]
Institute for Advanced Studies Vienna
Department of Economics and Finance

Kurt Schmidheiny[§]
Université de Lausanne
Ecole des HEC - DEEP

Martin Wagner[¶]
University of Bern
Department of Economics

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Abstract

In this paper we derive the closed form solution for multistep predictions of the conditional means and covariances for multivariate GARCH models. These predictions are useful e.g. in mean variance portfolio analysis when the rebalancing frequency is lower than the data frequency. In this situation the conditional mean and the conditional covariance matrix of the cumulative higher frequency returns until the next rebalancing period are required as inputs in the mean variance portfolio problem. The closed form solution for this quantity is derived as well. We assess the empirical value of the result by evaluating and comparing the performance of quarterly and monthly rebalanced portfolios using monthly MSCI index data across a large set of GARCH models. The value of using correct multistep predictions is assessed by comparing the performance of the quarterly rebalanced portfolios based on the correct multistep predictions with the quarterly rebalanced portfolios incorrectly based on 1-step predictions and the monthly rebalanced portfolios. Using correct multistep predictions generally results in lower risk and higher returns. Furthermore the correctly computed quarterly rebalanced portfolios exhibit higher returns than monthly rebalanced portfolios. The empirical results thus forcefully demonstrate the substantial value of multistep predictions for portfolio management.

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[‡]Contact address: Jaroslava Hlouskova, Institut für Höhere Studien, Stumpergasse 56, A-1060 Vienna, Austria. Email: hlouskov@ihs.ac.at.

[§]Contact address: Kurt Schmidheiny, Université de Lausanne, Ecole des HEC, BFSH 1, CH-1015 Lausanne, Switzerland. Email: kurt.schmidheiny@hec.unil.ch.

[¶]Contact address: Martin Wagner, Universität Bern, Volkswirtschaftliches Institut, Gesellschaftsstrasse 49, CH-3012 Bern, Switzerland. Email: martin.wagner@vwi.unibe.ch.

1 Introduction

In this paper we derive the closed form solution for multistep predictions of the conditional means and covariances from multivariate GARCH models. These predictions are useful e.g. in mean variance portfolio analysis, when the rebalancing frequency is lower than the data frequency, as in the problem studied in the application: We assess the empirical value of this result by evaluating the performance of quarterly rebalanced portfolios using monthly MSCI index data, and compare their performance with the performance of two corresponding portfolios. The latter are given by: the portfolio incorrectly based on using 1-step predictions in quarterly rebalancing respectively the monthly rebalanced portfolio.

Multistep prediction in GARCH models has been considered previously in e.g. Baillie and Bollerslev (1992), who derive the minimum mean squared error forecasts for the conditional mean and the conditional variance of univariate GARCH processes. We extend their results to the multivariate case and derive closed form representations for the conditional mean and the conditional covariances h-steps ahead. In addition we derive the explicit formula for the conditional covariance of the sum of the conditional means up to h-steps ahead. This corresponds to the conditional variance of the cumulative returns over an h-period horizon, when modelling asset returns.

The result is, as already mentioned, useful for mean variance portfolio analysis, when portfolio reallocations take place at a lower frequency than the data used in estimating the underlying GARCH models. Mean variance portfolio analysis (see section 3.1) requires estimates of expected returns and their covariances. If the rebalancing frequency is lower than the data frequency, the expected returns over the rebalancing interval are given by the cumulative expected returns at the higher data frequency. Hence, the need for the conditional variance of cumulative returns. If the portfolio is adjusted quarterly, as in our empirical application, and GARCH models are estimated using monthly data, the conditional covariance of the cumulative returns can only by computed from the predictions for the conditional means and covariances up to 3-months. The empirical part of our study is closely related to Ledoit, Santa-Clara and Wolf (2003), Nilsson (2002) and Polasek and Pojarliev (2001a, 2001b), who apply 1-step predictions from multivariate GARCH models for portfolio selection using - as we do - MSCI regional indices. By contrast, our study is based on multistep predictions.[1]

[1] Morillo and Pohlman (2002) try to tackle the multistep prediction problem in GARCH models by resorting to Monte-Carlo techniques. The theoretical properties (and the practical implementation) are, however, not

Furthermore, our empirical results are based on a larger set of GARCH models.[2]

The 'value' of the derived multistep predictions for portfolio management is evaluated on monthly data for six regional MSCI indices during the evaluation period January 1992 to December 2003. For a large number of GARCH models (48 to be precise), the minimum variance portfolios are tracked, both for monthly and quarterly rebalancing. In the latter case the quarterly rebalanced portfolios *correctly* based on multistep predictions and those *incorrectly* based on 1-step predictions are evaluated. The following main results are obtained. A majority of the portfolios based on GARCH models, we label them GARCH portfolios, result in lower risk and higher Sharpe ratio than the *naive* portfolio based on the sample mean and covariance. All GARCH portfolios outperform the naive portfolio in terms of return. The multistep prediction is of considerable value for quarterly rebalanced portfolios. For 33 out of 48 models, the portfolios based on the correct predictions show lower risk than the corresponding portfolios based on incorrect predictions. More remarkably, all GARCH portfolios based on correct predictions result in higher returns and Sharpe ratios than those based on incorrect predictions. The average over-performance is 0.24 % return per annum. We also evaluate monthly rebalanced portfolios for the same set of models. The quicker inclusion of information in monthly rebalanced portfolios results in lower risk than the quarterly rebalancing for almost all models. This fact forcefully demonstrates the value of GARCH volatility modelling. However, somewhat surprisingly, monthly rebalancing does not lead to higher returns, even when abstracting from transaction costs. The highest returns are obtained by quarterly rebalanced portfolios based on correct multistep predictions.

The paper is organized as follows: In section 2 the multistep prediction problem is solved. section 3 contains the empirical application in portfolio management. Section 4 briefly summarizes and provides conclusions. In the appendix we describe in detail the variance equations of the implemented GARCH models and present detailed results of the empirical application.

2 Multistep Prediction in Multivariate GARCH Models

In this section we derive the closed form solution for the multistep minimum mean squared error (MSE) prediction of the conditional means, variances and covariances for multivari-

clear.

[2]Contrary to Nilsson (2002), we exclude GARCH-in-Mean models, whose multistep predictions are not covered by our result and are a subject of further research. Polasek and Pojarliev (2001a) use a Bayesian approach to GARCH modelling.

ate GARCH models. Based on these results we also present the solution for the conditional variance of the sum of the predictions over h-periods. To facilitate implementation we furthermore derive recursive formulations for the results. The results of this section can e.g. be used for the prediction of *cumulative* returns in mean-variance portfolio analysis as exemplified in section 3.

2.1 The General Case

Since the seminal contribution of Engle (1982), ARCH and GARCH type models have become standard tools to model financial market data. Modelling and predicting financial data has to take into account the widespread phenomenon of *volatility clustering*, i.e. that periods of sustainedly high volatility and periods of sustainedly low volatility are present. This volatility clustering can e.g. be modelled by ARCH or GARCH type models.[3] During the last two decades an enormous variety of GARCH models has been developed, see e.g. Bollerslev, Engle and Nelson (1994) or Gourieroux (1997) for surveys of some of the models developed.

Multivariate GARCH models consist of two equations. The first one is an ARMA equation for the vector valued observations, $r_t \in \mathbb{R}^n$ say. In portfolio applications r_t is the vector of returns for n assets. Thus, the *mean equation* is of the form $r_t = c + A_1 r_{t-1} + \cdots + A_p r_{t-p} + \varepsilon_t + B_1 \varepsilon_{t-1} + \cdots + B_q \varepsilon_{t-q}$ with $A_i, B_j \in \mathbb{R}^{n \times n}$. The innovation ε_t has time-varying conditional covariance, denoted by $\Sigma_t = var(\varepsilon_t | I_{t-1})$, where I_{t-1} denotes the information set at time $t-1$. The model is called GARCH model, if the *variance* equation, describing the evolution of Σ_t is (appropriately parameterized and vectorized) an ARMA equation in Σ_t and $\varepsilon_t \varepsilon_t'$. For examples see the appendix. If the variance equation reduces to an autoregression, the model is termed an ARCH model.

If in the portfolio optimization problem introduced in section 3.1, the investment horizon is larger than one period, predictions for the *cumulative* returns are needed, which in turn require *multistep* predictions. Consider for example the situation that rebalancing takes place every h months, but monthly data are available. In our empirical application below $h = 3$. The cumulative returns over an h-period horizon, henceforth denoted as $r_{[t+1:t+h]}$, are

[3]An alternative model class is given by stochastic volatility models, see e.g. Harvey, Shephard and Ruiz (1994).

straightforwardly calculated from the single period returns, r_{t+i}, as follows[4]

$$r_{[t+1:t+h]} = r_{t+1} + \cdots + r_{t+h}$$

Thus, the conditional covariance matrix of the cumulative returns $r_{[t+1:t+h]}$ is

$$\begin{aligned} var(r_{[t+1:t+h]}|I_t) &= var(r_{t+1} + \cdots + r_{t+h}|I_t) \\ &= \sum_{i=1}^{h} var(r_{t+i}|I_t) + \sum_{i,j=1,i\neq j}^{h} cov(r_{t+i}, r_{t+j}|I_t) \end{aligned} \tag{1}$$

where I_t denotes the information set at time t. One clearly sees from this equation that the conditional variance matrix of $r_{[t+1:t+h]}$ is composed of the (conditional) variances and covariances of the one-period returns r_{t+i} for $i = 1, \ldots, h$. We thus see from equation (1) that for calculating $var(r_{[t+1:t+h]}|I_t)$ it is necessary to derive the MSE predictors of r_{t+i} for $i = 1, \ldots, h$ and the corresponding conditional variances and covariances. In the context of GARCH models it is important to note that the predictions of the conditional covariances of r_{t+i} in general differ from the predictions of the conditional variances of the residuals ε_{t+i} for $i > 1$.

The general formula for computing the required multistep predictions of the conditional variances of r_{t+i} from multivariate ARMA(p,q)-GARCH(k,l) models is presented below. This result is a generalization of the analogue multistep prediction for *univariate* GARCH models discussed in Baillie and Bollerslev (1992).[5] In the discussion we abstain from deriving also the multistep prediction formula for the conditional variance of the innovations ε_t. Obtaining these is a standard prediction problem in GARCH models. Note that these predictions depend upon the precise formulation of the variance equation, but are easily available if the variance equation is specified. Also note that multistep predictions of the conditional variances of the innovations ε_t are directly available in various software packages, whereas the conditional variances and covariances of multistep predictions of the returns themselves are to our knowledge not implemented in software packages.

Note that the limits for $h \to \infty$ of the derived results for the minimum MSE predictor of the mean and variance are finite only for stationary processes. Furthermore the derivations presented below do not apply to ARCH-in-Mean models, where by construction the prediction of the conditional means is coupled with the prediction of the conditional covariances.

[4]This follows directly from the definition of the 1-period returns, calculated as the logarithmic difference of asset prices.

[5]Alternatively, the temporal aggregation results of Drost and Nijman (1993), derived for a specific class of univariate GARCH models, can be used to obtain multistep predictions.

Let r_t be an n-dimensional ARMA(p,q), $p,\ q \in \mathbb{N}$, process with GARCH errors

$$r_t = c + \sum_{i=1}^{p} A_i r_{t-i} + \varepsilon_t + \sum_{i=1}^{q} B_i \varepsilon_{t-i} \tag{2}$$

distribution? where $c \in \mathbb{R}^n$, $\varepsilon_t \sim WN(0, \Sigma_t)$, with $\Sigma_t = var(\varepsilon_t | I_{t-1})$ and $A_1, \ldots, A_p, B_1, \ldots, B_q \in \mathbb{R}^{n \times n}$. For the derivation of the minimum MSE predictors of r_{t+i} and their conditional co-variances it is convenient to express the model (2) in the following *companion* format:

$$\underbrace{\begin{bmatrix} r_t \\ r_{t-1} \\ \vdots \\ r_{t-p+1} \\ \varepsilon_t \\ \varepsilon_{t-1} \\ \vdots \\ \varepsilon_{t-q+1} \end{bmatrix}}_{R_t} = \underbrace{\begin{bmatrix} c \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E_1 c} + \underbrace{\begin{bmatrix} A_1 & & \ldots & & A_p & B_1 & \ldots & & B_q \\ I & 0 & \ldots & & 0 & 0 & \ldots & & 0 \\ \vdots & & \ldots & & \vdots & \vdots & \ldots & & \vdots \\ 0 & & \ldots & I & 0 & 0 & \ldots & & 0 \\ 0 & & \ldots & & 0 & 0 & \ldots & & 0 \\ 0 & & \ldots & & 0 & I & \ldots & & 0 \\ \vdots & & \ldots & & \vdots & \vdots & \ldots & & \vdots \\ 0 & & \ldots & & 0 & 0 & \ldots & I & 0 \end{bmatrix}}_{\Phi} \underbrace{\begin{bmatrix} r_{t-1} \\ r_{t-2} \\ \vdots \\ r_{t-p} \\ \varepsilon_{t-1} \\ \varepsilon_{t-2} \\ \vdots \\ \varepsilon_{t-q} \end{bmatrix}}_{R_{t-1}} + \underbrace{\begin{bmatrix} \varepsilon_t \\ 0 \\ \vdots \\ 0 \\ \varepsilon_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{E\varepsilon_t} \tag{3}$$

or more compactly as

$$R_t = E_1 c + \Phi R_{t-1} + E \varepsilon_t \tag{4}$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix. The matrices E_j, $j = 1, \ldots, p+q$ denote $(p+q)n \times n$ matrices of $0_{n \times n}$ sub-matrices except for the j-th sub-matrix which equals I. Furthermore, $E = E_1 + E_{p+1}$, $R_t \in \mathbb{R}^{(p+q)n}$ and $\Phi \in \mathbb{R}^{(p+q)n \times (p+q)n}$.

From (3) or equivalently (4) it follows that

$$cov(r_{t+i}, r_{t+j} | I_t) = E_1' cov(R_{t+i}, R_{t+j} | I_t) E_1 \tag{5}$$

Iterating equation (4) $(i-1)$-times leads to

$$R_t = \sum_{j=0}^{i-1} \Phi^j E_1 c + \Phi^i R_{t-i} + \sum_{j=0}^{i-1} \Phi^j E \varepsilon_{t-j} \tag{6}$$

where $i = 1, \ldots, t-1$. For the sake of brevity we introduce the following notation for $i, j \in \mathbb{N}$

$$\begin{aligned} \Sigma^R_{t+i,t} &= var(R_{t+i} | I_t) \\ \Sigma^R_{t+i,t+j,t} &= cov(R_{t+i}, R_{t+j} | I_t) \end{aligned}$$

where $\Sigma^R_{t+i,t}$ and $\Sigma^R_{t+i,t+j,t} \in \mathbb{R}^{(p+q)n \times (p+q)n}$. Note that by definition $\Sigma^R_{t+i,t} = \Sigma^R_{t+i,t+i,t}$ holds.

From the definition of R_t it directly follows that

$$R_t = \sum_{i=0}^{p-1} E_{i+1} r_{t-i} + \sum_{i=0}^{q-1} E_{i+p+1} \varepsilon_{t-i} \tag{7}$$

It then follows from $E = E_1 + E_{p+1}$, (6) and (7) that

$$r_{t+h} = \sum_{i=0}^{h-1} E_1' \Phi^i E_1 c + \sum_{i=0}^{p-1} (E_1' \Phi^h E_{i+1}) r_{t-i} + \sum_{i=0}^{q-1} (E_1' \Phi^h E_{i+p+1}) \varepsilon_{t-i} + \sum_{i=0}^{h-1} (E_1' \Phi^i E) \varepsilon_{t+h-i} \tag{8}$$

Result 1 (Minimum MSE h-step ahead predictor for r_t)

Applying conditional expectations to equation (8)*, the minimum MSE h-step ahead predictor for r_{t+h} is found to be*

$$\mathbb{E}(r_{t+h}|I_t) = \sum_{i=0}^{h-1} E_1' \Phi^i E_1 c + \sum_{i=0}^{p-1} (E_1' \Phi^h E_{i+1}) r_{t-i} + \sum_{i=0}^{q-1} (E_1' \Phi^h E_{i+p+1}) \varepsilon_{t-i} \tag{9}$$

Furthermore, (8) and (9) imply that the forecast error for the $h-$step ahead predictor in (9) is given by

$$e_{t,h} = \sum_{j=0}^{h-1} (E_1' \Phi^j E) \varepsilon_{t+h-j} \tag{10}$$

Result 2 (Conditional variance of minimum MSE predictor)

The above equation (10) *leads to the conditional variance given by*

$$var(r_{t+h}|I_t) = \mathbb{E}(e_{t,h} e_{t,h}'|I_t) = E_1' \sum_{j=0}^{h-1} \Phi^j E \, \Sigma_{t+h-j,t} (\Phi^j E)' E_1 \tag{11}$$

In expression (11) the conditional covariances $\Sigma_{t+i,t}$ for $i = 0, \ldots, h$ show up. For an evaluation, respectively estimation, of this expression therefore i-step ahead predictions of the conditional covariances of the innovations ε_t have to be computed. These, obviously, depend upon the precise specification of the variance equation of the GARCH model.

Using (5) and (6) we obtain

$$\begin{aligned} cov(r_{t+i}, r_{t+j}|I_t) &= E_1' \Sigma^R_{t+i,t+j,t} E_1 \\ &= E_1' cov(\Phi^i R_t + \sum_{k=0}^{i-1} \Phi^k E \varepsilon_{t+i-k}, \Phi^j R_t + \sum_{l=0}^{j-1} \Phi^l E \varepsilon_{t+j-l} | I_t) E_1 \\ &= E_1' \sum_{k=max\{0,i-j\}}^{i-1} \Phi^k E \, \Sigma_{t+i-k,t} (\Phi^{j-i+k} E)' E_1 \end{aligned} \tag{12}$$

An estimate of quantity (12) is, of course, obtained by inserting estimates for the matrices $\Sigma_{t+i,t}$ in this expression.

Result 3 (Conditional variance of cumulative returns)
From (1) and (12) we now obtain the result for the conditional covariance matrix of the aggregated h-period returns:

$$\begin{aligned} var(r_{[t+1:t+h]}|I_t) &= var(r_{t+1} + \cdots + r_{t+h}|I_t) \\ &= E_1' \sum_{i=1}^{h} \left[\sum_{k=0}^{i-1} \Phi^k E\, \Sigma_{t+i-k,t} (\Phi^k E)' \right] E_1 \\ &\quad + E_1' \sum_{i,j=1, i\neq j}^{h} \left[\sum_{k=max\{0,i-j\}}^{i-1} \Phi^k E\, \Sigma_{t+i-k,t} (\Phi^{j-i+k} E)' \right] E_1 \end{aligned} \quad (13)$$

In (13) we see again that the expression for $var(r_{[t+1:t+h]}|I_t)$ depends upon the mean equation and the conditional covariances of the innovations. An estimate of expression (13) is given by substituting all parameters with estimates and by inserting the predictions of the conditional variances of ε_{t+i}.

When rewriting the ARMA mean equation in companion form (4) caution has to be taken in the definition of the quantities R_t, Φ and E, when either the autoregressive order p or the moving average order q are equal to 0. See the following remark:

Remark 1 *In case of an AR(p) mean equation it is more convenient to use (4) with R_t, Φ and E defined by*

$$R_t = (r_t', \ldots, r_{t-p+1}')'$$

$$\Phi = \begin{bmatrix} A_1 & & \ldots & & A_p \\ I & 0 & \ldots & & 0 \\ \vdots & & \ldots & & \vdots \\ 0 & & \ldots & I & 0 \end{bmatrix}$$

and $E = E_1$.

In case of an MA(q) mean equation it is more convenient to use representation (4) with the following quantities:

$$R_t = (r_t', \varepsilon_t', \ldots, \varepsilon_{t-q+1}')'$$

$$\Phi = \begin{bmatrix} 0 & B_1 & \dots & & B_q \\ 0 & 0 & \dots & & 0 \\ 0 & I & \dots & & 0 \\ \vdots & \vdots & \dots & & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}$$

and $E = E_1 + E_2$.

For an actual implementations of the above results concerning the predictions of the conditional variances and covariances a recursive formulation is convenient. A recursion is first derived in equations (14) to (16) for the multistep prediction of the conditional variances and covariances of r_t. Consider the case $i = j$ first, then (4) implies

$$\begin{aligned} \Sigma^R_{t+i,t} = var(R_{t+i}|I_t) &= var(\Phi R_{t+i-1} + E\varepsilon_{t+i}|I_t) \\ &= \Phi\Sigma^R_{t+i-1,t}\Phi' + E\Sigma_{t+i,t}E' \end{aligned} \tag{14}$$

as $cov(E\varepsilon_{t+i}, R_{t+i-1}|I_t) = 0$. Consider next the case $i > j$, then (6) implies

$$\begin{aligned} \Sigma^R_{t+i,t+j,t} &= cov(R_{t+i}, R_{t+j}|I_t) = cov(\Phi^{i-j}R_{t+j} + \sum_{k=0}^{i-j-1}\Phi^k E\varepsilon_{t+i-k}, R_{t+j}|I_t) \\ &= \Phi^{i-j}\Sigma^R_{t+j,t} \end{aligned} \tag{15}$$

as $cov(E\varepsilon_{t+i-k}, R_{t+j}|I_t) = 0$ for $k = 0, \dots, i-j-1$. Finally, let $i < j$. Then using (4) we obtain

$$\begin{aligned} \Sigma^R_{t+i,t+j,t} &= cov(R_{t+i}, R_{t+j}|I_t) = cov(R_{t+i}, \Phi^{j-i}R_{t+i} + \sum_{k=0}^{j-i-1}\Phi^k E\varepsilon_{t+j-k}|I_t) \\ &= \Sigma^R_{t+i,t}(\Phi^{j-i})' \end{aligned} \tag{16}$$

as $cov(R_{t+i}, E\varepsilon_{t+j-k}|I_t) = 0$ for $k = 0, \dots, j-i-1$.

Result 4 (Recursion for the conditional variance)
Equations (14)–(16) in combination with (1), (5) and the definition of $\Sigma^R_{t+i,t+j,t}$ *lead to the following recursion for the conditional variance of the cumulative returns.*

$$\begin{aligned} var(r_{[t+1:t+h]}|I_t) &= E_1'\left(\sum_{i=1}^{h}\Sigma^R_{t+i,t} + \sum_{i=2}^{h}\sum_{j=1}^{i-1}\Phi^{i-j}\Sigma^R_{t+j,t}\right)E_1 + \\ &+ E_1'\left(\sum_{i=1}^{h-1}\sum_{j=i+1}^{h}\Sigma^R_{t+i,t}(\Phi^{j-i})'\right)E_1 \end{aligned} \tag{17}$$

where the conditional covariance matrices $\Sigma^R_{t+i,t}$ *are calculated according to the recursion (14) for* $i = 1, \dots, h$.

2.2 Example: 3-step Ahead Predictions for ARMA(1,1) Mean Equation

In this subsection we derive explicitly the solution for the special case of the above result that we need for the empirical investigations in this paper. We consider 3-month aggregation of returns, which requires 3-step ahead predictions from models for monthly data. The mean equations implemented in our empirical study are AR(1), MA(1) and ARMA(1,1). We thus derive here the solution for the ARMA(1,1) mean equation.

Hence, set $h = 3$ and $p = q = 1$. Then the matrices R_t, Φ and E equal

$$R_t = (r_t', \varepsilon_t')'$$

$$\Phi = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} I \\ I \end{bmatrix}$$

where for notational simplicity we use $A = A_1$, $B = B_1$, $R_t \in \mathbb{R}^{2n}$, $\Phi \in \mathbb{R}^{2n\times 2n}$ and $E \in \mathbb{R}^{2n\times n}$. Note that $E_1 = \begin{bmatrix} I & 0 \end{bmatrix}' \in \mathbb{R}^{2n\times n}$ and thus the following holds

$$E'E_1 = I \tag{18}$$

$$\Phi E = \begin{bmatrix} A+B \\ 0 \end{bmatrix} \tag{19}$$

$$\Phi^2 E = \begin{bmatrix} A(A+B) \\ 0 \end{bmatrix} \tag{20}$$

Using (11) and (18)−(20) we obtain

$$\begin{aligned} var(r_{t+3}|I_t) &= E_1'E\Sigma_{t+3,t}E'E_1 + E_1'\Phi E\Sigma_{t+2,t}E'\Phi'E_1 + E_1'\Phi^2 E\Sigma_{t+1,t}E'(\Phi^2)'E_1 \\ &= \Sigma_{t+3,t} + \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A+B \\ 0 \end{bmatrix} \Sigma_{t+2,t} \, [\, (A+B)\,]', 0\,] \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A(A+B) \\ 0 \end{bmatrix} \Sigma_{t+1,t} \, [\, (A+B)'A', 0\,] \begin{bmatrix} I \\ 0 \end{bmatrix} \\ &= \Sigma_{t+3,t} + \ (A+B)\Sigma_{t+2,t}\ (A+B)' + \ [A\ (A+B)\ ]\Sigma_{t+1,t}\ [A\ (A+B)\ ]' \end{aligned} \tag{21}$$

Similarly it can be shown that

$$var(r_{t+1}|I_t) = \Sigma_{t+1,t} \tag{22}$$

$$var(r_{t+2}|I_t) = (A+B)\Sigma_{t+1,t}\ (A+B)' + \Sigma_{t+2,t} \tag{23}$$

Equations (12), (18)−(20) imply

$$
\begin{aligned}
cov(r_{t+3}, r_{t+2}|I_t) &= E_1' \Phi E \Sigma_{t+2,t} E' E_1 + E_1' \Phi^2 E \Sigma_{t+1,t} (\Phi E)' E_1 \\
&= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A+B \\ 0 \end{bmatrix} \Sigma_{t+2,t} \\
&\quad + \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} A(A+B) \\ 0 \end{bmatrix} \Sigma_{t+1,t} \begin{bmatrix} (A+B)', 0 \end{bmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} \\
&= (A+B)\Sigma_{t+2,t} + A\ (A+B)\Sigma_{t+1,t}\ (A+B)' \qquad (24)
\end{aligned}
$$

Along the same lines it also directly follows that

$$
cov(r_{t+2}, r_{t+1}|I_t) = (A+B)\Sigma_{t+1,t} \qquad (25)
$$

$$
cov(r_{t+3}, r_{t+1}|I_t) = A\ (A+B)\Sigma_{t+1,t} \qquad (26)
$$

and $cov(r_{t+1}, r_{t+2}|I_t) = cov(r_{t+2}, r_{t+1}|I_t)'$, $cov(r_{t+1}, r_{t+3}|I_t) = cov(r_{t+3}, r_{t+1}|I_t)'$, $cov(r_{t+2}, r_{t+3}|I_t) = cov(r_{t+3}, r_{t+2}|I_t)'$. Thus, from (1), (21)−(26) after some algebraic modifications we find the following result:

$$
\begin{aligned}
&var(r_{[t+1:t+3]}|I_t) \\
&= [I + \ (I+A)\ (A+B)\ ]\Sigma_{t+1,t}\ [I + \ (I+A)(A+B)]' \\
&+ (I+A+B)\Sigma_{t+2,t}\ (I+A+B)' + \Sigma_{t+3,t} \qquad (27)
\end{aligned}
$$

The required predictions for $\Sigma_{t+1,t}, \Sigma_{t+2,t}$ and $\Sigma_{t+3,t}$ and the estimates for A and B are in our application for all implemented models directly obtained from the Finmetrics Module in S-Plus, see the discussion below. In the AR(1) case the above result holds with $B = 0$ and for the MA(1) case $A = 0$ has to be inserted.

3 An Empirical Application in Portfolio Management

In the previous section we have shown how multistep predictions are obtained for GARCH models. These become useful for portfolio management if the data frequency is higher than the rebalancing frequency. This situation is often faced by portfolio managers in practice and is also the original motivation for this paper. In this section we assess the practical implications of this result for portfolio selection by comparing the portfolio performance with higher rebalancing frequency (1-month) to lower rebalancing frequency (3-month) using higher frequency (1-month) data. Consequently, the former portfolio selection has to be based on 1-step predictions and the latter on predictions up to three steps ahead. The quantitative importance of correct multistep predictions is evaluated by computing several performance measures of portfolios rebalanced at a 3-month frequency but incorrectly based on 1-step predictions. Note, however, that the interesting exercise of finding an 'optimal' rebalancing interval is beyond the scope of this paper. In the course of this procedure a large number of multivariate GARCH models are implemented, see Table 3 in the appendix for the list of 48 implemented models. This also allows to identify the sets of models leading to the best portfolio performance, according to optimality criteria such as lowest risk, highest return or highest Sharpe ratio.

3.1 Portfolio Optimization

The empirical application is performed within the framework of mean-variance (MV) portfolio analysis (Markowitz, 1952 and 1956). MV analysis assumes that the investor's decisions and hence the optimal portfolio only depend on the expected return and the conditional variance of the portfolio return, the latter measuring risk. Considering n risky assets and an investment horizon of one period the investor faces the following decision problem at time t:

$$\underset{x_t}{Min} \quad \sigma^2_{pt+1} = \sum_{i,j=1}^{n} x_{it} x_{jt} cov(r_{it+1}, r_{jt+1}|I_t)$$

$$s.t. \quad \mathbb{E}(r_{pt+1}|I_t) = \sum_{i=1}^{n} x_{it} \mathbb{E}(r_{it+1}|I_t) = \overline{r}, \quad \sum_{i=1}^{n} x_{it} = 1, \quad x_{it} \geq 0$$

where r_{pt+1} and σ^2_{pt+1} denote the portfolio return and portfolio variance, respectively.[6] Given a fixed value of the expected return, $\mathbb{E}(r_{pt+1}|I_t) = \overline{r}$, the fractions, x_{it}, of wealth invested in an individual asset i, are chosen to minimize the risk of the portfolio return. In addition, we assume nonnegative x_{it}, i.e. short sales are prohibited.[7] $\mathbb{E}(r_{it+1}|I_t)$ and $cov(r_{it+1}, r_{jt+1}|I_t)$ are approximated by predictions (e.g. from GARCH models) of individual asset returns and

[6]Mean-variance portfolio optimization is based on *discrete* returns, which implies that the portfolio return is a weighted average of individual asset returns, as seen in the above equation. The predictions from the multivariate GARCH models are, however, based on *continuous* (log) returns for the following reason: the cumulative returns over multiple periods are linear in the individual period returns when using continuous returns but non-linear (and thus not analytically tractable) for discrete returns. We pursue the following *pragmatic* strategy: we predict continuous returns using the multivariate GARCH models. We then - as is common in the literature, compare e.g. Ledoit, Santa-Clara and Wolf (2003) - use these predictions in the mean-variance optimization to get the optimal portfolio weights. In order to provide a realistic assessment of the portfolio performance, in the evaluation we calculate the *discrete* returns of the portfolios.

[7]Jagannathan and Ma (2003) show that imposing short-sale constraints can improve portfolio performance due to avoiding extreme positions resulting from imprecise covariance estimation.

their covariances over the period from t to $t+1$, given I_t, the information set at t. The above optimization problem leads, by varying $\bar{r}$, to the well-known efficient frontier. The optimal portfolio choice from the set of mean-variance efficient portfolios depends on the investor's preferences and also on the consideration of a potential risk free asset. Omitting the constraint $\mathbb{E}(r_{pt+1}|I_t) = \bar{r}$ leads to the *minimum variance portfolio*, which is independent of expected returns.

It is well-known that MV optimization is very sensitive to errors in the estimated $\mathbb{E}(r_{pt+1}|I_t)$ and $cov(r_{it+1}, r_{jt+1}|I_t)$, see Chopra, Hensel and Turner (1993) or Best and Grauer (1991). Chopra and Ziemba (1997) point out that the asset allocations of efficient portfolios are more sensitive to uncertainty in the expected returns than to uncertainty in their conditional covariances. By focusing in our empirical application on the minimum variance portfolio only, we eliminate thus the impact of the imprecision in the prediction of the returns.

3.2 Return and Risk Predictions from GARCH Models

The required predictions for both the returns and the conditional covariances of the returns are derived in our study from multivariate GARCH models. We implement a large number of GARCH models. The nesting formulation of the mean equations considered in the empirical application is given by the ARMA(1,1) equation:

$$r_t = c + Ar_{t-1} + \varepsilon_t + B\varepsilon_{t-1}$$

Preliminary model selection shows that for our application no higher lags are required. Even in the equations with only one lag many of the coefficients are insignificant. Therefore, we also investigate more parsimonious specifications, where the autoregressive coefficient matrix A, the moving average coefficient matrix B or both are restricted to be diagonal or zero. Note that the significance of coefficients in A or B in the mean equation is a violation of strong market efficiency. Two distributions for ε_t are considered: Normally distributed innovations and t-distributed innovations, where in the latter case the degree of freedom of the innovation distribution is estimated itself. The latter possibility is included in order to allow for stronger leptokurtic behavior. See the upper block of Table 1 for a description of all implemented mean equations. The implemented variance equations are described in detail in the appendix. We consider eight different specifications of orders (1,1), see the lower block of Table 1.

As a benchmark portfolio we consider the *naive* portfolio, where both the return and covariance predictions are given by the sample mean and the sample covariance, respectively,

Table 1: Specifications of implemented GARCH models

Specification of *mean equation*: $r_t = c + Ar_{t-1} + \varepsilon_t + B\varepsilon_{t-1}$			
model	A	B	ε_t
AR(1) diag n	diagonal	0	N(0,Σ_t)
MA(1) diag n	0	diagonal	N(0,Σ_t)
AR(1) full n	unrestr.	0	N(0,Σ_t)
AR(1) diag t	diagonal	0	t-distr.
MA(1) diag t	0	diagonal	t-distr.
ARMA(1,1) full t	unrestr.	unrestr.	t-distr.

Specification of *variance equation*, details in the appendix	
model	description
BEKK(1,1)	
Vector Diag(1,1)	vector diagonal model
Diag GARCH(1,1)	pure diagonal GARCH model
Diag EGARCH(1,1)	pure diagonal exponential GARCH model
Diag PGARCH(1,1)	pure diagonal power GARCH model
CCC GARCH(1,1)	constant conditional correlation GARCH model
CCC EGARCH(1,1)	constant conditional correlation exponential GARCH model
CCC PGARCH(1,1)	constant conditional correlation power GARCH model

over the estimation period. Thus, we need to clarify how we derive multistep predictions for the *naive* portfolio strategy, which is based on sample means and covariances. Since in the quarterly rebalancing the investor is interested in the prediction of the 3-month returns and their covariances, we base our naive predictions for the 3-month return on the sample mean and covariance matrix of the monthly return series aggregated to 3-month returns.[8]

3.3 Portfolio Evaluation

We track internationally diversified portfolios denominated in Swiss francs over the period 1992 to 2003. The portfolio wealth is invested in six world regions. The Morgan Stanley Capital International (MSCI) indices for the United States, Switzerland, Great Britain, Japan, Europe (excluding Great Britain) and Pacific (excluding Japan) are the investment instruments.[9] We use monthly return data from February 1972 to December 2003 for the six indices.

The evaluation with quarterly (respectively monthly) rebalancing proceeds in the following

[8]This seems to be more natural than to simply use the empirical mean and covariance matrix of the returns series at the monthly frequency. The latter are used as incorrect forecasts for the quarterly rebalancing of the naive portfolio.

[9]Note that we do not include a risk free asset in order to focus on the effect of GARCH predicted correlation structures on portfolio performance.

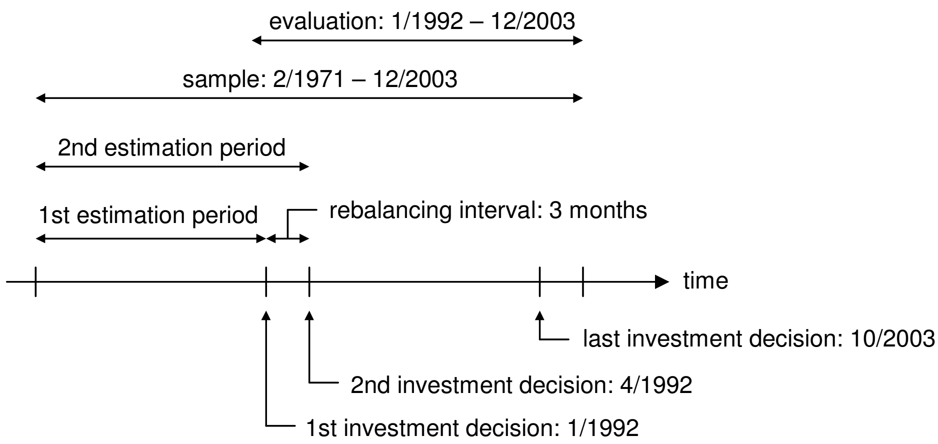

Figure 1: Timing of the evaluation (quarterly rebalancing).

steps (see the timing illustrated in Figure 1):

(1) The monthly return data from February 1972 up to the date of the investment decision are used to predict the covariances of the six regional indices. 49 different predictions are computed: From 48 GARCH models and the naive predictions.

(2) The corresponding minimum variance portfolios are calculated.

(3) The 3- and 1-month returns are calculated.

(4) The investment decision is repeated every 3 (1) months from January 1, 1992 to October 1, (December 1) 2003 and the portfolios are rebalanced accordingly.

3.4 Results

Table 2 exhibits a summary of the results presented in detail in Table 3 in the appendix. The latter table shows the detailed results from the evaluation and juxtaposes the results from monthly rebalancing and quarterly rebalancing for both the correct 3-step prediction and the incorrect 1-step prediction method. In Table 3 we report risk, return as well as the Sharpe ratio of the portfolios based on 48 GARCH models.[10] For comparison, the results obtained from the *naive* portfolio, based on sample means and covariances are displayed too. The Sharpe ratio is defined as excess return (i.e. net return minus riskfree rate)[11] divided by

[10]More detailed tables including further risk adjusted performance measures such as Jensen's alpha, Treynor's measure as well as shortfall are available from the authors upon request.

[11]The 3-month (respectively 1-month) deposit rate is used as riskfree rate. Before December 1996 the deposit rate is approximated by the LIBOR minus five basis points.

Table 2: Main Results of Performance Comparison

	Risk	Return	Sharpe
Average across GARCH portfolios			
- monthly rebalancing	39.00	7.99	0.133
- quarterly rebalancing with 3-step prediction	39.72	8.35	0.135
- quarterly rebalancing with 1-step prediction	39.99	7.89	0.123
Best of GARCH portfolios			
- monthly rebalancing	37.39	9.08	0.167
- quarterly rebalancing with 3-step (correct) prediction	38.47	9.33	0.159
- quarterly rebalancing with 1-step (incorrect) prediction	38.64	9.09	0.156
Naive prediction			
- monthly rebalancing	41.03	7.77	0.120
- quarterly rebalancing with 3-step (correct) prediction	41.38	8.05	0.122
- quarterly rebalancing with 1-step (incorrect) prediction	41.13	7.82	0.117
Number of GARCH models	48	48	48
Comparison of GARCH models			
- quarterly rebal.: correct better than incorrect GARCH prediction	33	48	48
- quarterly (correct prediction) better than monthly rebalancing	1	47	27
Comparison of GARCH models with naive prediction			
- monthly rebal.: GARCH better than 1-step naive prediction	48	36	39
- quarterly rebal.: correct GARCH better than naive prediction	48	40	43
- quarterly rebal.: incorrect GARCH better than naive prediction	48	32	32

This table summarizes the results presented in Table 3 in the appendix. All results apply to quarterly returns. *Return* denotes the mean annualized return of the portfolio. *Risk* denotes the standard deviation of annualized quarterly returns. *Sharpe* ratio is given by excess quarterly return (i.e. return minus riskfree rate) divided by its standard deviation. *correct* means that 3-step predictions for the conditional covariances are used. *incorrect* means that 1-step predictions for the conditional means and covariances are used. *better (best)* means lower risk, higher return and higher Sharpe ratio, respectively.

the standard deviation of the excess return. For comparability of portfolio performance with different rebalancing intervals, the following discussion reports risk and Sharpe ratio based on annualized quarterly returns only.

Let us start by discussing the performance of GARCH predictions used for monthly adjusted portfolios, which requires only 1-step predictions. The portfolio with the lowest risk (3-month standard deviation of 37.39%) is the ARMA(1,1)-CCC-EGARCH(1,1) portfolio. This portfolio also shows above average return (8.78%). However, several portfolios perform better in terms of return, the best being the ARMA(1,1)-CCC-GARCH portfolio with a return of 9.08% per annum. Using the Sharpe ratio as a simple measure for the return risk trade off, the latter also shows the best performance with a Sharpe ratio of 0.167. Note that all GARCH portfolios exhibit lower risk than the naive portfolio. This forcefully demonstrates the substantial value of GARCH modelling. Both, the average across the GARCH portfolios as well as the majority of GARCH portfolios also show higher return (36 out of 48) and Sharpe

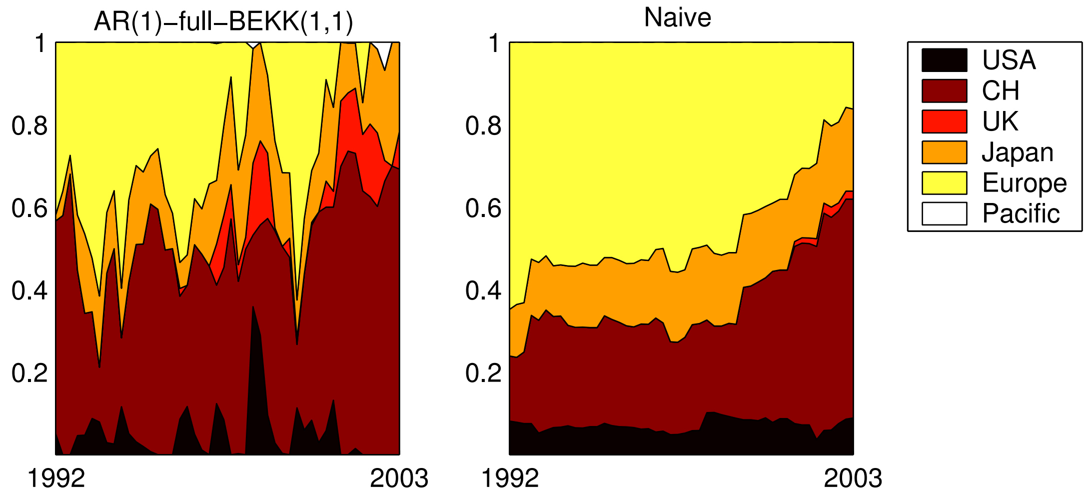

Figure 2: Asset allocations of one GARCH and the naive quarterly rebalanced portfolios.

ratio (39 out of 48) than the naive portfolio. While the average return (7.99%) across all 48 GARCH portfolios is 22 basis points higher than the return of the naive portfolio, the best GARCH portfolio results in a return that is 131 basis points higher than the naive portfolio's return.

Let us now turn to quarterly portfolio rebalancing. The MA(1)-CCC-GARCH(1,1) yields the lowest risk (38.47%). The best portfolio in terms of return (9.33%) and Sharpe ratio (0.159) is the AR(1)-full-BEKK(1,1) portfolio. This compares to the corresponding naive portfolio's risk of 41.38%, a return of 8.05% and Sharpe ratio equal to 0.122. For illustration, the asset allocations corresponding to these two portfolio strategies are displayed in Figure 2. This figure displays clearly a very typical feature of GARCH based portfolios, namely the much larger amount of asset reallocations compared to e.g. the naive portfolio. Again, all GARCH portfolios show lower risk and a majority show higher return (40 of 48) and higher Sharpe ratio (43 of 48) than the naive portfolio. The average return across all GARCH portfolios (8.35%) is now 30 basis points above the naive portfolios return. Thus, the value of GARCH based portfolio selection appears to be substantial at both frequencies.

One might expect that the risks of quarterly rebalanced portfolios are higher and their returns lower than for monthly rebalanced portfolios. This, since with monthly rebalancing new information is incorporated faster. Surprisingly, this relationship is only observed for risk: all but one quarterly rebalanced portfolio result in *higher risk* than the corresponding monthly rebalanced portfolio. However, 47 out of 48 quarterly rebalanced portfolios exhibit *higher returns* than the corresponding monthly rebalanced portfolio. On average, the risk of quarterly rebalanced portfolios is 72 basis points and the return 36 basis points *higher* than that of monthly rebalanced portfolios. Consequently an ambiguous picture emerges when taking the Sharpe ratio as performance measure: the Sharpe ratio of quarterly and monthly rebalanced portfolios are on average as well as for the individual models almost identical.[12]

Let us finally turn to the assessment of the value of correct multistep predictions by comparing the portfolio performance obtained from quarterly rebalancing based on the incorrect 1-step predictions on the one hand and on the correct multistep predictions on the other hand. These results are again contained in Table 3 in the appendix and summarized in Table 2. For the majority (33 of 48) of GARCH portfolios using the correct predictions results in lower portfolio risk. The correct predictions reduce the risk by on average 17 basis points. Note also that for all 48 GARCH portfolios the return is higher with the correct multistep method. The mean difference being 46 basis points. Thus, the correct computation of the predictions is indeed resulting in superior portfolio performance.

4 Summary and Conclusions

In this paper we have derived the closed form solution for multistep predictions of the conditional means and covariances for multivariate GARCH models and have illustrated their value for portfolio management. Multistep predictions of the conditional means and covariances are e.g. needed for mean-variance portfolio analysis when the rebalancing frequency is lower than the data frequency. In order to deal with this problem we have also derived the explicit formula for the conditional covariance matrix of the corresponding cumulative higher frequency returns. The closed form solution for the general ARMA(p,q)-GARCH(k,l) case is provided in section 2 along with a convenient recursive representation.

The practical relevance of the theoretical results is assessed empirically with an application to six regional MSCI indices using a large variety of GARCH models. Based on monthly data, the portfolio performance of monthly and quarterly rebalanced portfolios is investigated. The quarterly rebalancing decision is based on either the (model consistent) correct 3-step predictions or is incorrectly based on 1-step predictions. The evaluation period is January 1992 to December 2003. Several observations emerge: The first observation is that basing

[12]Note that the higher return achieved with lower frequency rebalanced portfolios implies that the results are robust with respect to the consideration of transaction costs.

the quarterly rebalancing decision on correct multistep predictions is advisable for almost all portfolios. For the majority of GARCH models the risk is reduced by using the correct multistep predictions. Furthermore, for all GARCH models the return of the corresponding portfolio is higher when the rebalancing decision is based on the correct multistep predictions. The second main observation is the fact that quarterly rebalanced portfolios based on multistep predictions lead to higher returns than monthly adjusted portfolios, but also increase the risk. This is a surprising result, as a priori one expects that monthly rebalanced portfolios outperform quarterly adjusted portfolios. This conjecture, which is based on the argument that monthly adjusted portfolios incorporate new information faster, is not validated for the returns in our empirical study. The third observation is that by basing the portfolio decision on predictions from GARCH models one can substantially outperform the naive portfolio, a result also found for daily data by Fleming et al. (2001).

An important theoretical question that is left open for future research is the derivation of multistep predictions for multivariate GARCH-in-Mean models. See Karanasos (2001) or Nilsson (2002) for some results concerning prediction for this model class. An important empirical issue that requires further exploration is to assess the value of multistep predictions at higher data frequencies, e.g. to explore the performance of weekly portfolio allocation based on daily data. This might lead to interesting results as the volatility effects are stronger at higher frequencies, which should increase the value of correct conditional multistep predictions of conditional covariances. Exploring the link of these issues to the literature on *realized volatility*, see e.g. Andersen et al. (2003), is left open for further research.

Appendix: Implemented GARCH(1,1) Variance Equations

in this appendix we first describe briefly the implemented specifications for the variance equations and then present detailed results of the evaluation in Table 3.

In the estimation of multivariate GARCH models two aspects have to be considered. Firstly, positive semi-definiteness and symmetry of the estimated conditional covariance matrices has to be guaranteed. Secondly, the number of parameters to be estimated grows rapidly with the number of assets. For circumventing the first problem the literature proposes a variety of multivariate GARCH models that guarantee positive semi-definiteness and symmetry of the estimate of Σ_t. We discuss some of them below. The discussion is in terms of the implemented GARCH(1,1) models only.

The unrestricted GARCH or *diagonal-vec* model (Bollerslev, Engle and Wooldridge (1988)) constitutes the natural starting point for the discussion and is therefore described first. The variance equation of the diagonal-vec(1,1) model is given by:

$$\Sigma_t = P_0 + P_1 \odot (\varepsilon_{t-1}\varepsilon'_{t-1}) + Q_1 \odot \Sigma_{t-1}$$

where $\odot$ denotes the Hadamard (i.e. element-wise) product and $P_0 \in \mathbb{R}^{n\times n}, P_1 \in \mathbb{R}^{n\times n}$ and $Q_1 \in \mathbb{R}^{n\times n}$ for an application with n assets. Taking the symmetry restriction into account, the diagonal-vec model leads to 63 parameters to be estimated in our application with six index returns series. However, in this formulation it also has to be ensured that the resulting estimated Σ_t is positive semi-definite, which complicates the likelihood optimization problem. For this reason we focus on the estimation of alternative formulations of GARCH(1,1) models that incorporate the restrictions that the estimated Σ_t has to be positive (semi-)definite and symmetric and do not consider the diagnonal-vec model further.

One popular formulation in the empirical literature is known as BEKK model (see Engle and Kroner, 1995). The BEKK(1,1) model's variance equation is

$$\Sigma_t = P_0P'_0 + P_1(\varepsilon_{t-1}\varepsilon'_{t-1})P'_1 + Q_1\Sigma_{t-1}Q'_1$$

with P_0, P_1, Q_1 given as above. The BEKK model results in more parameters than the diagonal-vec model, however its formulation incorporates symmetry and positive semi-definiteness of Σ_t and its estimate.

The second implemented version of GARCH(1,1) models is the *vector-diagonal* model

$$\Sigma_t = P_0P'_0 + p_1p'_1 \odot (\varepsilon_{t-1}\varepsilon'_{t-1}) + q_1q'_1 \odot \Sigma_{t-1}$$

with vectors $p_1, q_1 \in \mathbb{R}^n$ and P_0 as above. It is obvious that this formulation reduces the number of estimated parameters while symmetry and positive semi-definiteness of Σ_t remain ensured.

An alternative strategy for parameter reduction consists of transforming the multivariate problem into a set of (essentially) univariate problems. This means that after appropriate transformations the components of the conditional variance series are modelled with standard univariate GARCH type models. We have implemented two variance equations following this strategy: the *constant conditional correlation* (CCC) and the *pure diagonal* models.

In the *constant conditional correlation (CCC)* model (Bollerslev (1990)), the conditional covariance matrix is modelled as

$$\Sigma_t = D_t R D_t$$

where $R \in \mathbb{R}^{n \times n}$ is the constant conditional correlation matrix and $D_t = \mathrm{diag}(\sigma_{1t}, \ldots, \sigma_{nt})$ denotes the diagonal matrix of the conditional standard deviations of the individual returns series. The series σ_{it} are then modelled in our application with univariate GARCH, EGARCH or PGARCH models (see the description below).

Assuming that the returns are conditionally uncorrelated, i.e. that Σ_t is diagonal for all t, one can directly model the individual volatility series with univariate GARCH models. This approach is often termed *pure diagonal* GARCH model. However, one should note that the residuals used in this univariate modelling of the volatilities are derived from the multivariate specification of the mean equation. Hence, the results differ from a completely univariate GARCH analysis, where the mean equations are specified for each of the return series separately.

Let us finally turn to a brief description of the underlying univariate GARCH models used. Hence, from now on we deal only with one volatility series σ_{it} and one residual or innovation series ε_{it}. The basic model is the GARCH model of Bollerslev (1986), which in its GARCH(1,1) form is given by

$$\sigma_{it}^2 = p_i + p_{i1}\varepsilon_{it-1}^2 + q_{i1}\sigma_{it-1}^2$$

with $p_i, p_{i1}, q_{i1} \in \mathbb{R}$. Here, the condition $p_{i1} + q_{i1} < 1$ is necessary for covariance stationarity of the underlying return series. In order to be able to model asymmetric behavior of volatility in response to positive or negative shocks, the standard GARCH specification has been extended in various ways. Two of these extensions have been used in this study, the *exponential* GARCH (EGARCH) model introduced by Nelson (1991) and the *power* GARCH (PGARCH) model, see e.g. Ding, Engle and Granger (1993). The univariate EGARCH(1,1) model has the following variance equation

$$\ln \sigma_{it}^2 = p_i + p_{i1}\frac{|\varepsilon_{it-1}| + \gamma_i \varepsilon_{it-1}}{\sigma_{it-1}} + q_{i1} \ln \sigma_{it-1}^2$$

Finally the variance equation of the PGARCH(1,1) model is given by

$$\sigma_{it}^d = p_i + p_{i1}\left(|\varepsilon_{it-1}| + \gamma_i \varepsilon_{it-1}\right)^{d_i} + q_{i1}\sigma_{it-1}^{d_i}$$

with $\gamma_i, p_i, p_{i1}, q_{i1} \in \mathbb{R}$ and where the parameter $d_i \in \mathbb{R}$ can be estimated as well. Appropriate restrictions to ensure stationarity have to be taken into account.

Table 3: Risk and Return over Evaluation Period 1992/1 to 2003/12

Estimated model		monthly rebalancing				quarterly rebalancing					
Mean	Variance	1-step prediction				3-step prediction (correct)			1-step prediction ("incorrect")		
equation	equation	Risk	Risk	Return	Sharpe	Risk	Return	Sharpe	Risk	Return	Sharpe
		1-month	3-month		3-month	3-month		3-month	3-month		3-month
AR(1) diag n	BEKK(1,1)	57.66	39.01	8.76	0.152	40.64	9.01	0.148	40.37	8.74	0.142
MA(1) diag n	BEKK(1,1)	57.21	38.73	8.67	0.151	40.16	8.87	0.147	40.15	8.57	0.139
AR(1) full n	BEKK(1,1)	57.28	38.90	8.80	0.154	39.95	9.33	0.159	39.87	8.83	0.147
AR(1) diag t	BEKK(1,1)	56.76	38.10	8.87	0.159	39.57	9.18	0.157	39.26	9.09	0.156
MA(1) diag t	BEKK(1,1)	56.81	38.23	8.57	0.150	39.47	8.90	0.150	39.41	8.70	0.145
ARMA(1,1) full t	BEKK(1,1)	56.89	38.13	8.15	0.140	39.16	8.98	0.153	39.21	8.12	0.131
AR(1) diag n	Vector Diag(1,1)	57.23	38.62	7.32	0.117	39.26	7.93	0.126	39.55	7.53	0.115
MA(1) diag n	Vector Diag(1,1)	57.25	38.53	7.29	0.116	39.13	7.91	0.126	39.34	7.50	0.115
AR(1) full n	Vector Diag(1,1)	57.23	38.48	7.45	0.120	39.41	8.19	0.132	39.37	7.84	0.124
AR(1) diag t	Vector Diag(1,1)	56.22	38.03	8.02	0.137	39.07	8.39	0.138	39.24	8.07	0.130
MA(1) diag t	Vector Diag(1,1)	56.09	38.14	7.95	0.134	39.08	8.41	0.139	39.22	8.12	0.131
ARMA(1,1) full t	Vector Diag(1,1)	56.37	37.68	7.88	0.134	39.02	8.72	0.147	38.97	8.37	0.138
AR(1) diag n	Diag GARCH(1,1)	59.04	40.11	8.19	0.134	40.28	8.35	0.133	40.41	8.21	0.129
MA(1) diag n	Diag GARCH(1,1)	59.09	40.15	8.21	0.134	40.35	8.35	0.133	40.45	8.20	0.129
AR(1) full n	Diag GARCH(1,1)	59.16	40.10	8.26	0.136	40.69	8.44	0.134	40.48	8.23	0.130
AR(1) diag t	Diag GARCH(1,1)	58.76	40.18	8.24	0.135	40.52	8.32	0.132	40.65	8.17	0.128
MA(1) diag t	Diag GARCH(1,1)	58.76	40.17	8.23	0.134	40.53	8.30	0.131	40.65	8.15	0.127
ARMA(1,1) full t	Diag GARCH(1,1)	58.83	40.10	8.23	0.135	41.23	8.48	0.133	40.71	8.26	0.130
AR(1) diag n	Diag PGARCH(1,1)	59.67	40.65	7.61	0.118	40.83	7.71	0.116	40.81	7.62	0.114
MA(1) diag n	Diag PGARCH(1,1)	59.62	40.56	7.49	0.115	40.89	7.74	0.116	40.88	7.68	0.115
AR(1) full n	Diag PGARCH(1,1)	59.31	40.36	7.51	0.116	40.85	7.97	0.122	40.56	7.66	0.115
AR(1) diag t	Diag PGARCH(1,1)	59.37	40.56	7.88	0.125	40.67	8.13	0.126	40.68	7.99	0.123
MA(1) diag t	Diag PGARCH(1,1)	59.34	40.52	7.92	0.126	40.60	8.12	0.127	40.74	7.94	0.122
ARMA(1,1) full t	Diag PGARCH(1,1)	59.67	40.53	7.81	0.123	41.14	8.26	0.128	40.70	7.88	0.120
AR(1) diag n	Diag EGARCH(1,1)	59.17	40.24	8.09	0.131	40.44	8.29	0.131	40.51	8.22	0.129
MA(1) diag n	Diag EGARCH(1,1)	59.17	40.24	8.10	0.131	40.45	8.28	0.131	40.51	8.22	0.129
AR(1) full n	Diag EGARCH(1,1)	59.26	40.29	8.11	0.131	40.76	8.46	0.134	40.59	8.28	0.130
AR(1) diag t	Diag EGARCH(1,1)	59.01	40.28	8.06	0.130	40.60	8.23	0.129	40.70	8.07	0.125
MA(1) diag t	Diag EGARCH(1,1)	59.03	40.27	8.05	0.130	40.60	8.21	0.129	40.68	8.05	0.125
ARMA(1,1) full t	Diag EGARCH(1,1)	59.19	40.20	8.12	0.132	41.31	8.52	0.134	40.76	8.23	0.129

Table 3 (continued): Risk and Return over Evaluation Period 1992/1 to 2003/12

Estimated model		monthly rebalancing				quarterly rebalancing					
		1-step prediction				3-step prediction (correct)			1-step prediction (“incorrect”)		
Mean equation	Variance equation	Risk 1-month	Risk 3-month	Return	Sharpe 3-month	Risk 3-month	Return	Sharpe 3-month	Risk 3-month	Return	Sharpe 3-month
AR(1) diag n	CCC GARCH(1,1)	55.92	37.50	7.64	0.128	38.58	7.97	0.129	39.97	6.97	0.100
MA(1) diag n	CCC GARCH(1,1)	56.13	37.56	7.86	0.134	38.47	8.30	0.138	39.88	7.23	0.107
AR(1) full n	CCC GARCH(1,1)	56.23	37.75	7.30	0.119	38.76	8.21	0.135	39.79	6.91	0.099
AR(1) diag t	CCC GARCH(1,1)	55.22	37.72	8.46	0.149	38.96	8.81	0.150	39.87	8.16	0.130
MA(1) diag t	CCC GARCH(1,1)	55.13	37.64	8.32	0.146	38.78	8.84	0.151	39.69	8.10	0.129
ARMA(1,1) full t	CCC GARCH(1,1)	55.48	37.47	9.08	0.167	38.50	9.10	0.159	39.02	8.76	0.148
AR(1) diag n	CCC PGARCH(1,1)	58.01	39.96	6.45	0.091	39.60	6.91	0.099	39.96	6.36	0.085
MA(1) diag n	CCC PGARCH(1,1)	57.96	39.20	6.58	0.096	39.70	6.52	0.089	40.07	5.90	0.073
AR(1) full n	CCC PGARCH(1,1)	57.42	38.47	7.18	0.114	38.59	8.26	0.137	38.64	7.44	0.116
AR(1) diag t	CCC PGARCH(1,1)	56.34	38.47	8.02	0.135	38.88	8.14	0.133	39.22	7.46	0.114
MA(1) diag t	CCC PGARCH(1,1)	56.21	38.16	8.03	0.136	38.86	8.11	0.132	39.46	7.27	0.109
ARMA(1,1) full t	CCC PGARCH(1,1)	56.92	37.99	8.09	0.139	38.60	8.53	0.144	38.66	8.00	0.130
AR(1) diag n	CCC EGARCH(1,1)	57.32	38.36	7.78	0.129	39.02	8.34	0.138	40.36	7.24	0.106
MA(1) diag n	CCC EGARCH(1,1)	57.33	38.42	7.80	0.130	38.94	8.35	0.138	40.33	7.23	0.106
AR(1) full n	CCC EGARCH(1,1)	57.30	38.15	7.46	0.121	39.11	8.35	0.137	39.96	7.08	0.103
AR(1) diag t	CCC EGARCH(1,1)	56.27	37.77	8.42	0.148	38.82	8.65	0.146	39.65	7.99	0.126
MA(1) diag t	CCC EGARCH(1,1)	56.31	37.81	8.37	0.147	38.83	8.64	0.146	39.69	7.90	0.124
ARMA(1,1) full t	CCC EGARCH(1,1)	56.66	37.39	8.78	0.159	38.87	8.86	0.151	40.02	8.13	0.129
Riskfree		2.11	2.11	2.79		2.15	2.95		2.15	2.95	
Naive		59.57	41.03	7.77	0.120	41.38	8.05	0.122	41.13	7.82	0.117

Remarks:

The return and covariance predictions used in the mean-variance optimization are based on monthly data.

The portfolio composition is adjusted every 3 months/1 month from January 1992 to October/December 2003.

The results reported correspond to the evaluation of *minimum variance* portfolios.

Estimated model specifies the *mean equation* and the *variance equation* of the estimated GARCH model.

Risk 1-month/3-month denotes the standard deviation of annualized monthly/quarterly returns.

Return denotes the mean annualized return of the portfolio.

Sharpe ratio is given by excess quarterly return (i.e. return minus riskfree rate) divided by its standard deviation.

3-step prediction means that 3-step predictions for the conditional means and covariances are used.

1-step prediction means that 1-step predictions for the conditional means and covariances are used.

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