Contracts versus Salaries in Matching: A General Result

Jan Christoph Schlegel*
Faculty of Business and Economics, University of Lausanne, Switzerland
jschlege@unil.ch

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Abstract

It is shown that a matching market with contracts may be embedded into a matching market with salaries under weaker assumptions on preferences than substitutability. In particular, the result applies to the recently studied problem of cadet-to-branch matching. As an application of the embedding result, a new class of mechanisms for matching markets with contracts is defined that generalize the firm-proposing deferred acceptance algorithm to the case where firms have unilaterally substitutable preferences. *JEL-classification:* C78
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The matching with contracts framework of Hatfield and Milgrom (2005) (see also Roth, 1984 and Fleiner, 2003) is a key model in recent market design research. It has been successfully applied to the matching of cadets to branches in the United States Military Academy (Sönmez and Switzer, 2013) and the Reserve Officer Training Corps (Sönmez, 2013) as well as to the design of affirmative action matching mechanisms in school choice (Kominers and Sönmez, 2013). Despite this practical success, some theoretical questions about the model have not been satisfyingly answered so far. In particular, it is not clear (Echenique, 2012) to what extend the model is really

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more general than previous matching models, like the job market matching framework of Kelso and Crawford (1982).

Echenique (2012) shows that the Hatfield-Milgrom model can be "reduced" to a job matching model with salaries in the following sense: Under the assumption of substitutability in markets with contracts, there exists an embedding that assigns to each market with contracts a corresponding market with salaries such that the set of stable allocations of the market remains invariant under the embedding. Moreover, the gross substitutability condition that is the key assumption of the analysis of Kelso and Crawford (1982) is satisfied in the market with salaries. This result was extended by Kominers (2012) to many-to-many models of matching with contracts. The embedding results show that, under the assumption of substitutability, the matching with contracts model is essentially not more general than a matching model with salaries. Nevertheless, substitutable preferences are not the most general domain of preferences for which the key results of the theory of many-to-one matching with contracts hold. Hatfield and Kojima (2010) have pointed out that the essential results of the theory can be proved under the strictly weaker assumption of unilateral substitutes, respectively under the even weaker assumption of bilateral substitutes.[1] Furthermore, these weaker substitutes conditions play a crucial role in recent market design applications (Sönmez and Switzer, 2013; Sönmez, 2013; Kominers and Sönmez, 2013).

In this paper, we extend the result of Echenique (2012) and show that a market with unilaterally substitutable preferences can be embedded into a market with salaries where firms' demands are gross substitutable. In particular, the result applies to the cadet-to-branch matching problem as formulated by Sönmez and Switzer (2013). Furthermore, we show that, under a weaker notion of embeddability, an embedding is possible even for bilaterally substitutable preferences. For this purpose we introduce a new condition for firms' demands in market with salaries, *weak gross substitutability*, that guarantees the convergence of a descending auction to a stable allocation. We then show that a market with bilaterally substitutable preferences can be embedded into a market with salaries where firms' demands are weakly gross substitutable. Both results are the most general one can hope for. We show that, for the embedding method proposed in this paper, unilateral substitutes are also necessary for an embedding into a market with salaries

[1] Unilateral substitutes (respectively unilateral substitutes and the law of demand) are sufficient for the existence of a worker-optimal stable allocation, the rural hospitals theorem, group-strategy-proofness and weak Pareto optimality of the cumulative offer mechanism for the workers. Bilateral substitutes guarantee the existence of stable allocations.

where firms' demands are gross substitutable and that bilateral substitutes are necessary for an embedding into a market with salaries where firms' demands are weakly gross substitutable. In this sense, the results of this paper clarify to what extend the model with contracts is more general than the model with salaries.

In this paper we propose a new embedding technique that differs from that of Echenique (2012) and that of Kominers (2012) by not relying on "Pareto separability" of firms' preferences, i.e. the property that a firm's ranking of contracts with a given worker is independent of its contracts with other workers. As Hatfield and Kojima (2010) have pointed out, Pareto separability in many-to-one markets with contracts is implied by substitutability but not by unilateral substitutability. Thus, an embedding for unilaterally substitutable preferences cannot use the embedding construction employed by Echenique (2012). The underlying observation behind our embedding is that firms' preferences can be modified such that they become (essentially) Pareto separable while the set of efficient allocations as well as their blocking properties remain invariant under the modification of preferences.[2]

Independently of establishing an equivalence between matching with contracts and with salaries, the result also gives new insight into the matching model with contracts itself. For unilaterally substitutable choices we introduce a new class of mechanisms (corresponding to the salary-adjustment process of Kelso and Crawford (1982)) that can be interpreted as the hospital-proposing deferred-acceptance mechanism of Hatfield and Milgrom (2005) applied to the modified firm preferences.

In addition, we also provide new insights in the structure of the set of stable allocations in markets where unilateral substitutability holds. Under unilaterally substitutable preferences the set of stable allocations forms an upper semi-lattice with respect to workers' preferences (compare Hatfield and Kojima, 2010). We show that the semi-lattice can be completed to a lattice by altering firms' preferences in a very particular way. It turns out that there exist substitutable firm preferences that differ from the original preferences only in so far as they alter the ranking among allocations that match the same agents under different contracts such that every stable allocation under the original preferences is still stable under the modified preferences. Modifying the preferences in this way may extend the set of stable allocations. But only allocations are added to the set that match the

[2]The approach taken here resembles the approach of Martínez et al. (2012) who also study preferences changes in many-to-one matching markets that leave the set of stable allocations invariant. One difference is that Martínez et al. (2012) consider classical matching markets without contracts.

same agents as a previously stable allocation but under different contracts that make every worker weakly worse off. This provides some of the intuition why an embedding is still possible under unilateral substitutability.

1 Model

1.1 Matching with Contracts

The following model is due to Hatfield and Milgrom (2005). Let F and W be two finite disjoint sets of agents. We call members of F **firms** and members of W **workers**. A **matching market with contracts** is a tuple $(X, (u_i)_{i\in F\cup W})$ consisting of a finite set X of **contracts**, where each contract $x \in X$ is between one firm $x_F \in F$ and one worker $x_W \in W$, and utility functions $(u_i)_{i\in F\cup W}$ describing the agents' preferences over different feasible sets of contracts. Here, a **feasible** set of contracts or an **allocation** is a set of contracts $Y \subseteq X$ such that Y contains at most one contract with each worker. We now introduce some notation of which we make repeated use throughout this paper: For an agent $i \in F \cup W$ and a set of contracts $Y \subseteq X$ we let Y_i denote the set of contract in Y that involve i. For a set of contracts $Y \subseteq X$ we denote the set of firms involved in contracts in Y by Y_F and the set of workers involved in contracts in Y by Y_W.

We make the following assumption on preferences: We assume that workers do not care what contracts other workers sign. Thus, with the above notation, we may represent a worker w's preferences by a utility function $u_w : X_w \cup \{\emptyset\} \to \mathbb{R}$. Similarly, we assume that firms do not care what contracts other firms sign. Thus, we may represent a firm f's preferences by a utility function $u_f : 2^{X_f} \to \mathbb{R}$. Note that, for ease of notation, firms' utility functions are also defined for infeasible sets of contracts. As in the original contribution of Hatfield and Milgrom (2005), we assume that agents' utilities are strict, i.e. workers are never indifferent between different contracts and firms are never indifferent between different sets of contracts.[3] Thus, for every firm f the utility function u_f induces a **choice function** $C_f : 2^{X_f} \to 2^{X_f}$ that selects from every set of contracts the utility maximizing sets of contracts

$$C_f(Y) = \mathrm{argmax}_{Z\subseteq Y_f} u_f(Z).$$

[3] As an alternative framework Aygün and Sönmez (2013) propose to work with choice functions instead of utility functions as a primitive of the model. In Section 3 we will come back to this point and discuss how the results in the present paper can be extended to the Aygün-Sönmez-framework.

The following conditions on firms' choice behavior, due to Hatfield and Milgrom (2005) and Hatfield and Kojima (2010), will play a crucial role in the subsequent discussion. The first property requires that firms see contracts as substitutes.

Definition 1 (Hatfield and Milgrom, 2005)**.** C_f satisfies **substitutability** if for all $Y \subseteq X_f$, $x, z \in X \setminus Y$, we have

$$x \in C_f(Y \cup \{x, z\}) \Rightarrow x \in C_f(Y \cup \{x\}).$$

It is easy to see (compare Hatfield and Kojima, 2010) that substitutability implies that (in individually rational allocations) a firm always ranks contracts with a given worker in the same way. This property, called Pareto separability, is important for the embedding results of Echenique (2012) and Kominers (2012).

Definition 2 (Hatfield and Kojima, 2010)**.** C_f satisfies **Pareto separability** if for two contracts $x, x' \in X$ with the same worker $x_W = x'_W$ and allocations $Y, Z \subseteq X$ with $x_W = x'_W \notin Y_W, Z_W$ we have

$$x \in C_f(Y \cup \{x, x'\}) \Rightarrow x' \notin C_f(Z \cup \{x, x'\}).$$

The following two conditions allow for certain complementarities between contracts but maintain that workers are substitutes for firms. Unilateral substitutability is the stronger of the two properties in the sense that bilateral substitutability allows for more complementarities and imposes less structure on choice functions. It can be shown (see, Hatfield and Kojima, 2010) that a choice function is substitutable if and only if it unilaterally substitutable and Pareto separable.

Definition 3 (Hatfield and Kojima, 2010)**.** C_f satisfies **unilateral substitutability** if for all $Y \subseteq X_f$, $x, z \in X \setminus Y$ with $x_W \notin Y_W$ we have

$$x \in C_f(Y \cup \{x, z\}) \Rightarrow x \in C_f(Y \cup \{x\}).$$

Definition 4 (Hatfield and Kojima, 2010)**.** C_f satisfies **bilateral substitutability** if for all $Y \subseteq X_f$, $x, z \in X \setminus Y$ with $x_W, z_W \notin Y_W$ we have

$$x \in C_f(Y \cup \{x, z\}) \Rightarrow x \in C_f(Y \cup \{x\}).$$

Next we define solution concepts for matching markets with contracts. An allocation Y is **individually rational** if for every firm f we have

$Y_f = C_f(Y)$ and for every contract $y \in Y$ we have $u_{y_W}(y) \geq u_{y_W}(\emptyset)$. An allocation Y is **blocked** if there is a firm f and an allocation $Z \neq Y$, such that $Z = C_f(Y \cup Z)$ and for every $z \in Z$ we have $u_{z_W}(z) \geq \max_{y \in Y \cup \{\emptyset\}} u_{z_W}(y)$. An allocation that is individually rational and not blocked is called **stable**. We denote the set of all stable allocations in $(X, (u_i)_{i \in F \cup W})$ by $\mathcal{S}(X, (u_i)_{i \in F \cup W})$. An allocation Y is **(Pareto)-dominated** by an allocation Z if for every agent $i \in F \cup W$ we have $u_i(Z_i) \geq u_i(Y_i)$ with at least one strict inequality. We call an allocation **(Pareto)-efficient** if it not dominated by any allocation. We denote the set of all efficient allocations in $(X, (u_i)_{i \in F \cup W})$ by $\mathcal{P}(X, (u_i)_{i \in F \cup W})$. Note that $\mathcal{S}(X, (u_i)_{i \in F \cup W}) \subseteq \mathcal{P}(X, (u_i)_{i \in F \cup W})$.

1.2 Matching with Salaries

The following model is based on the job matching model of Kelso and Crawford (1982). We consider a set of firms F and a set of workers W. A **matching market with salaries** is a tuple $(S, (v_i)_{i \in F \cup W})$ consisting of a set $S \subseteq \mathbb{R}_+$ of **salaries**, a utility function $v_f : 2^{W \times S} \to \mathbb{R}$ for every firm $f \in F$ and a utility function $v_w : (F \times S) \cup \{\emptyset\} \to \mathbb{R}$ for every worker $w \in W$. Throughout this paper we only consider markets with finite salary sets: $|S| < \infty$. We make the following assumptions on utilities: For every worker w the utility function v_w is increasing in salaries, i.e. for every firm $f \in F$ and salaries $s, s' \in S$ we have

$$s < s' \Rightarrow v_w(f, s) < v_w(f, s').$$

For every firm f the utility function v_f is non-increasing in salaries, i.e. for every set of workers $A \subseteq W$ and salaries $\mathbf{s} = (s_w)_{w \in A}, \mathbf{s}' = (s'_w)_{w \in A} \in S^A$ we have[4]

$$\mathbf{s} \leq \mathbf{s}' \Rightarrow v_f\left(A, \mathbf{s}'\right) \leq v_f\left(A, \mathbf{s}\right).$$

Assuming that utility functions are weakly but not necessarily strictly decreasing in salaries will be crucial for the embedding result.

Furthermore, we assume that for every firm f the utility function is strict over sets of workers holding salaries fixed, i.e. for every vector of salaries $\mathbf{s} = (s_w)_{w \in W}$ and sets of workers $A, B \subseteq W$ we have

$$A \neq B \Rightarrow v_f(A, \mathbf{s}) \neq v_f(B, \mathbf{s}).$$

[4]Here and in the following, the notation $\mathbf{s} \leq \mathbf{s}'$ means $s_w \leq s'_w$ for every $w \in A$ and the notation $\mathbf{s} < \mathbf{s}'$ means that furthermore at least one of the inequalities is strict.

Under this assumption a utility function v_f induces a **demand function** $D_f : S^W \to 2^W$ that assigns to a vector of salaries $\mathbf{s} = (s_w)_{w \in W} \in S^W$ the utility maximizing set of workers

$$D_f(\mathbf{s}) := \text{argmax}_{A \subseteq W} v_f(A, (s_w)_{w \in A}).$$

Next we introduce the gross substitutability condition that will play a crucial role in the subsequent discussion. The condition requires that workers are substitutes for firms in the sense that if we raise the salary for some workers while keeping other salaries fixed, a firm still demands all workers with unchanged salaries that it has previously demanded. In the model of Kelso and Crawford (1982) and the modification of the model that we consider here, gross substitutes guarantee the existence of a stable allocation.

Definition 5 (Kelso and Crawford, 1982)**.** D_f satisfies **gross substitutability** if for all $\mathbf{s} \leq \mathbf{s}'$ and every worker with $s_w = s'_w$ we have

$$w \in D_f(\mathbf{s}) \Rightarrow w \in D_f(\mathbf{s}').$$

An **allocation** in the market $(S, (v_i)_{i \in F \cup W})$ is a **matching** $\mu : W \to F \cup \{\emptyset\}$ together with salaries $\mathbf{s} = (s_w)_{w : \mu(w) \neq \emptyset}$ where s_w is the salary between w and $\mu(w)$. In the following we use the notation that for an allocation $(\mu, \mathbf{s})$, $v_f(\mu, \mathbf{s}) := v_f(\mu^{-1}(f), (s_w)_{w \in \mu^{-1}(f)})$ and $v_w(\mu, \mathbf{s}) := v_w(\mu(w), s_w)$. An allocation $(\mu, \mathbf{s})$ is **individually rational** if for every firm f and $A \subseteq \mu^{-1}(f)$ we have $v_f(\mu, \mathbf{s}) \geq v_f(A, \mathbf{s})$ and for every worker w with $\mu(w) \neq \emptyset$ we have $v_w(\mu, \mathbf{s}) \geq v_w(\emptyset)$. An allocation $(\mu, \mathbf{s})$ is **blocked** if there is a firm f, a set of workers $A \subseteq W$ and a salaries $\mathbf{s}' = (s'_w)_{w \in A}$ such that $v_f(A, \mathbf{s}') \geq v_f(\mu, \mathbf{s})$ and for every worker $w \in A$ we have $v_w(f, s'_w) \geq v_w(\mu, \mathbf{s})$ with a strict inequality for at least one agent. An allocation that is individually rational and not blocked is called **stable**. We denote the set of all stable allocations in $(S, (v_i)_{i \in F \cup W})$ by $\mathcal{S}(S, (v_i)_{i \in F \cup W})$. An allocation $(\mu, \mathbf{s})$ is **(Pareto)-dominated** by an allocation $(\mu', \mathbf{s}')$ if for every agent $i \in F \cup W$ we have $v_i(\mu', \mathbf{s}') \geq v_i(\mu, \mathbf{s})$ with a strict inequality for at least one agent. We call an allocation **(Pareto)-efficient** if it is not dominated by any allocation. We denote the set of all efficient allocations in $(S, (v_i)_{i \in F \cup W})$ by $\mathcal{P}(S, (v_i)_{i \in F \cup W})$. Note that $\mathcal{S}(S, (v_i)_{i \in F \cup W}) \subseteq \mathcal{P}(S, (v_i)_{i \in F \cup W})$.

2 Embedding

The embedding of Echenique (2012) depends crucially on the observation that substitutable choice functions in many-to-one matching markets with

$\succeq_f$	$\succeq_{w_1}$	$\succeq_{w_2}$
$\{x, z\}$	x	z'
$\{x', z\}$	x'	z
$\{x, z'\}$	$\emptyset$	$\emptyset$
$\{x', z'\}$		
$\{x\}$		
$\{x'\}$		
$\{z\}$		
$\{z'\}$		
$\emptyset$		

x	$g(x)$
x	$(f, w_1, 1)$
x'	$(f, w_1, 2)$
z	$(f, w_2, 1)$
z'	$(f, w_2, 2)$

$\succeq_f$	$\succeq_{w_1}$	$\succeq_{w_2}$
$\{(w_1, 1), (w_2, 1)\}$	$(f, 1)$	$(f, 2)$
$\{(w_1, 2), (w_2, 1)\}$	$(f, 2)$	$(f, 1)$
$\{(w_1, 1), (w_2, 2)\}$	$\emptyset$	$\emptyset$
$\{(w_1, 2), (w_2, 2)\}$		
$(w_1, 1)$		
$(w_1, 2)$		
$(w_2, 1)$		
$(w_1, 2)$		
$\emptyset$		

Table 1: A market with one firm f and two workers w_1 and w_2 and contracts $X = \{x, x', z, z'\}$ where $x_W = x'_W = w_1$ and $z_W = z'_W = w_2$. To simplify notation, preferences are given by lists $\succeq_f, \succeq_{w_1}, \succeq_{w_2}$ instead of utility functions. The preferences of the firm induce substitutable choices. Note that the contract x' is dominated by the contract x. The second and third table represent the embedding as defined by Echenique (2012). The second table shows the correspondence between contracts and firm-worker-salary triples. The third table represents the preferences in the market with salaries. Note that w_1 prefers $(f, 1)$ to $(f, 2)$. Thus monotonicity in salaries is violated.

contracts are Pareto separable. Thus we can define a marginal utility ranking of contracts with any given worker that is independent of the firm's contract with other workers. The marginal utility ranking allows to define a dominance relation over contracts and the set of all efficient (i.e. undominated) contracts can be parameterized to define an embedding of a market with contracts into a market with salaries. Dominated contracts are mapped to arbitrary high salaries. Similarly, Kominers (2012) requires Pareto separability for his embedding result for many-to-many matching markets. An example for the embedding technique of Echenique is given in Table 1.

In the absence of Pareto separability, the notion of a dominated contract does not necessarily make sense. It can be the case that a firm sometimes prefers one contract with a worker and sometimes an other contract with the same worker depending on which contracts with other workers are available. In the following, we show that Pareto separability is not necessary for an embedding by providing a different embedding method. For this purpose, we introduce the following notion of dominance for matching markets with contracts.

Definition 6. An allocation Z **dominates** an allocation Y **through a change of contract-terms** if both allocations match the same agents, i.e. for every firm f we have $(Y_f)_W = (Z_f)_W$, and Z Pareto-dominates Y. We call an allocation that is not dominated through a change of contract terms by any allocation **contract-term efficient**.

Echenique defines an embedding where workers' utility functions are strictly increasing in salaries for salaries corresponding to efficient contracts and firms' utility functions are strictly decreasing in salaries for salaries corresponding to efficient contracts. For his embedding however (compare Table 1), utility functions are not necessarily monotonic for all salaries; a worker's utility can decrease if one passes from a lower salary to a higher salary and a firm's utility can increase if one passes from lower salaries to higher salaries in the case that the higher salaries correspond to dominated contracts.[5] In the following we will construct an embedding without Pareto separability. The crucial difference to the embedding of Echenique and that of Kominers (2012) is that we now only require monotonicity for salaries corresponding to *contract-term efficient* allocations (as discussed before, we cannot require monotonicity for efficient contracts because the very notion of an efficient contract might not be well-defined without Pareto-separability). An example of the new embedding method can be found in Table 2.

To construct the market with salaries, note that without Pareto separability, even though a firm f does not always rank contracts with a given worker w in the same way, w has still a unique ranking of his contracts with f which is simply given by w's preferences over the contracts with f. We can use this ranking to parameterize the contracts of the worker-firm pair and interpret the parameter as a salary. Thus for every pair (f, w), we denote the set of contracts between them by X_{fw} and enumerate the elements in $X_{fw} = \left\{x^1_{fw}, x^2_{fw}, \ldots, x^{|X_{fw}|}_{fw}\right\}$ in increasing order ranked by w's utility, i.e. such that $u_w(x^i_{fw}) < u_w(x^{i+1}_{fw})$. For every contract $x^s_{fw} \in X_{fw}$ we define its image under the embedding to be $g(x^s_{fw}) := (f, w, s)$. We let $\bar{s} := \max_{f\in F, w\in W} |X_{fw}| + 1$ be a high salary that does not correspond to any contract and define the set of feasible salaries by $S := \{0, 1, \ldots, \bar{s}\}$.[6] We have obtained a well defined injective map $g : X \to F \times W \times S$ that

[5] In general, by the possibly multidimensional nature of the set of contracts, an embedding into a market where both firms' and workers' utility functions are everywhere monotone in salaries is impossible. Requiring monotonicity of utility functions on their whole domain will create new stable allocations in the market with salaries. In Section 2.3, we will discuss in more detail, what will happen if we require monotonicity everywhere.

[6] The purpose of the highest salary, as it will be evident when we define the utility functions in the market with salaries, is to guarantee that for every worker we can always raise the salary to a level such that he is no longer demanded.

$\succeq_f$	$\succeq_{w_1}$	$\succeq_{w_2}$
$\{x', z'\}$	x	z
$\{x', z\}$	x'	z'
$\{x, z\}$	$\emptyset$	$\emptyset$
$\{x, z'\}$		
$\{x'\}$		
$\{x\}$		
$\{z'\}$		
$\{z\}$		
$\emptyset$		

x	$g(x)$
x	$(f, w_1, 2)$
x'	$(f, w_1, 1)$
z	$(f, w_2, 2)$
z'	$(f, w_2, 1)$

$\succeq_f$	$\succeq_{w_1}$	$\succeq_{w_2}$
$\{(w_1, 1), (w_2, 1)\}$	$(f, 3)$	$(f, 3)$
$\{(w_1, 1), (w_2, 2)\}$	$(f, 2)$	$(f, 2)$
$\{(w_1, 2), (w_2, 2)\}$	$(f, 1)$	$(f, 1)$
$\sim \{(w_1, 2), (w_2, 1)\}$	$\emptyset$	$\emptyset$
$(w_1, 1)$		
$(w_1, 2)$		
$(w_2, 1)$		
$(w_1, 2)$		
$\emptyset$		

Table 2: The preferences of the firm induce unilaterally substitutable choices that are not Pareto separable. f prefers z to z' if it signs x but z' to z if it signs x'. The allocation $\{x, z\}$ dominates $\{x, z'\}$ through a change of contract-terms. Note that in the market with salaries preferences are not strictly monotonic at the salaries corresponding to the contract-term inefficient allocation $\{x, z'\}$. The firm is indifferent between $\{(w_1, 2), (w_2, 1)\}$ and $\{(w_1, 2), (w_2, 2)\}$. We also could have defined preferences without indifferences by letting the firm strictly prefer $\{(w_1, 2), (w_2, 2)\}$ to $\{(w_1, 2), (w_2, 1)\}$ and otherwise defining the preferences in the same way. In the example, $\bar{s} = 3$ is a high salary that does not correspond to any contract.

assigns to any contract a unique firm-worker-salary triple. In the other direction, for a firm f, a set of workers $A \subseteq W$ and salaries $\mathbf{s} = (s_w)_{w \in A} \in S^A$ we have a corresponding allocation in the market with contracts which we denote by $X^{\mathbf{s}}_{f,A} := g^{-1}(\{(f, w, s_w) : w \in A\})$. If $A = W$ we will drop the subscript and simply write $X^{\mathbf{s}}_f$. Similarly, for an allocation $(\mu, \mathbf{s})$, with $s_w \leq |X_{\mu(w),w}|$ for every w, we write $X^{\mu,\mathbf{s}}$ for the corresponding allocation $\left\{x^{s_w}_{\mu(w),w} : w \in W \text{ such that } \mu(w) \neq \emptyset\right\}$.

The crucial difference to the construction of Echenique and that of Kominers is the way in which we define utility functions in the market with salaries: For workers, the utilities will be the same for every contract and the corresponding firm salary pair, i.e. for every worker w and contract $x^s_{fw} \in X_{fw}$ with firm f we define $v_w(f, s) := u_w(x^s_{fw})$. Similarly, we define the utility of the outside option by $v_w(\emptyset) := u_w(\emptyset)$. For firms, the utilities will be the same for each set of contract and the corresponding set of workers and salary vector, as long as the set of contracts is *contract-term efficient*. Thus for a firm f, workers $A \subseteq W$ and salaries $\mathbf{s} = (s_w)_{w \in A} \in S^A$ with $s_w \leq |X_{fw}|$ for every $w \in A$ we define $v_f(A, \mathbf{s}) := u_f(X^{\mathbf{s}}_{f,A})$ whenever $X^{\mathbf{s}}_{f,A}$ is contract-term efficient. Now, if $X^{\mathbf{s}}_{f,A}$ is not contract-term efficient, we proceed differently as the previous literature. In this case there must exist (possibly multiple) contract-term efficient allocations that dominate $X^{\mathbf{s}}_{f,A}$ through a change of contract-terms. Among those allocations pick the one that has the highest utility for the firm. Since every worker in A weakly prefers this allocation to $X^{\mathbf{s}}_{f,A}$ it must correspond to a salary vector $\mathbf{s}' \in S^A$ with $\mathbf{s} \leq \mathbf{s}'$ such that the dominating allocation is $X^{\mathbf{s}'}_{f,A}$. We then define $v_f(A, \mathbf{s}) = u_f(X^{\mathbf{s}'}_{f,A}) = v_f(A, \mathbf{s}')$.[7]

To completely describe preferences, we also have to specify utilities for the high salaries that do not correspond to any contract. If for some firm f and worker w, the salary $s \in S$ does not correspond to any contract, i.e. $s > |X_{fw}|$ we let $v_w(f, s) > v_w(f, |X_{fw}|)$. Furthermore, for any set of workers $A \subseteq W \setminus \{w\}$ and salaries $\mathbf{s}_{-w} \in S^A$ we let

$$v_f(A, \mathbf{s}_{-w}) > v_f(A \cup \{w\}, (\mathbf{s}_{-w}, \bar{s})).$$

One can check, as we will do in the proof of the following theorem, that the embedding leaves the set of stable allocations (in fact, also the set of Pareto efficient and individually rational allocations) invariant. More interestingly, specifying utility functions in this way yield a very particular form for the demand functions of firms in the market with salaries. To describe the demand function we introduce the following notation. For a contract $x \in X$ let us define the **upper contour set of the worker** x_W **at contract** x by $\mathcal{U}(x) := \{y \in X_{x_W} : u_{x_W}(y) \geq u_{x_W}(x)\}$ and for an allocation Y define $\mathcal{U}(Y) := \bigcup_{y \in Y} \mathcal{U}(y)$. With this notation, we will see that the demand is given by Equation 1 in Theorem 1. In particular, we will later show in Theorem 2 that the demand given by this equation is gross substitutable whenever the choice function in the market with contracts is unilaterally substitutable. Furthermore (see Theorem 3), we will see that the demand satisfies a weak form of gross substitutability whenever the choice functions in the markets with contracts are bilaterally substitutable.

Theorem 1. *Let $(X, (u_i)_{i \in F \cup W})$ be a matching market with contracts. Then there exist a matching market with salaries $(S, (v_i)_{i \in F \cup W})$ and a one-to-one mapping $g : X \to F \times W \times S$ such that*

1. *every worker's utility function is strictly increasing in salaries,*

[7] We specify utility functions such that f is indifferent between $(A, \mathbf{s})$ and $(A, \mathbf{s}')$. Alternatively, we could also define utility functions without indifferences such that $v_f(A, \mathbf{s}') > v_f(A, \mathbf{s})$ but for $(B, \mathbf{s}'')$ with $B \neq A$ we have $v_f(A, \mathbf{s}) > v_f(B, \mathbf{s}'')$ if and only if $v_f(A, \mathbf{s}') > v_f(B, \mathbf{s}'')$ and $v_f(A, \mathbf{s}) < v_f(B, \mathbf{s}'')$ if and only if $v_f(A, \mathbf{s}') < v_f(B, \mathbf{s}'')$.

2. *every firm's utility function is strictly decreasing in salaries when restricted to allocations that correspond to contract-term efficient allocations,*

3. *for every firm f and salaries $\mathbf{s} = (s_w)_{w \in W} \in S^W$ the demand in the market with salaries is given by*

$$D_f(\mathbf{s}) = \left(C_f\left(\mathcal{U}\left(X_f^{\mathbf{s}}\right)\right)\right)_W, \tag{1}$$

where $X_f^{\mathbf{s}} := g^{-1}(\{(f, w, s_w) : w \in W\})$,

4. *every allocation $(\mu, \mathbf{s})$ that does not correspond to an allocation in the market with contract is not individually rational*

5. *for every allocation $(\mu, \mathbf{s})$ that corresponds to an allocation $X^{\mu,\mathbf{s}}$,*

   - *$(\mu, \mathbf{s})$ is efficient if and only $X^{\mu,\mathbf{s}}$ is efficient, i.e. $(\mu, \mathbf{s}) \in \mathcal{P}(S, (v_i)_{i \in F \cup W})$ if and only if $X^{\mu,\mathbf{s}} \in \mathcal{P}(X, (u_i)_{i \in F \cup W})$*
   - *if $(\mu, \mathbf{s})$ is efficient then $v_i(\mu, \mathbf{s}) = u_i(X_i^{\mu,\mathbf{s}})$ for every $i \in F \cup W$, in particular the set of stable allocation remains invariant, i.e. $(\mu, \mathbf{s}) \in \mathcal{S}(S, (v_i)_{i \in F \cup W})$ if and only if $X^{\mu,\mathbf{s}} \in \mathcal{S}(X, (u_i)_{i \in F \cup W})$.*

Proof. The first part of the theorem follows by the construction of the embedding and workers' utility functions in the market with salaries.

For the second part, consider a set of workers $A \subseteq W$ and salaries $\mathbf{s} = (s_w)_{w \in A}, \mathbf{s}' = (s'_w)_{w \in A} \in S^A$ such that $\mathbf{s} < \mathbf{s}'$. Consider the corresponding allocations $X_{f,A}^{\mathbf{s}}, X_{f,A}^{\mathbf{s}'} \subseteq X_f$ in the market with contracts. Assume that both allocations are contract-term efficient. Recall that workers' utility functions are increasing in salaries and thus every worker in A (weakly) prefers $X_{f,A}^{\mathbf{s}'}$ to $X_{f,A}^{\mathbf{s}}$. Thus, we must have $u_f(X_{f,A}^{\mathbf{s}}) > u_f(X_{f,A}^{\mathbf{s}'})$ as otherwise $X_{f,A}^{\mathbf{s}'}$ would dominate $X_{f,A}^{\mathbf{s}}$ through a change of contract-terms. This in turn implies that

$$v_f(A, \mathbf{s}) = u_f(X_{f,A}^{\mathbf{s}}) > u_f(X_{f,A}^{\mathbf{s}'}) = v_f(A, \mathbf{s}').$$

For the third part it is helpful to rephrase the definition of the utility functions in the market with salaries. Consider workers $A \subseteq W$ and salaries $\mathbf{s} = (s_w)_{w \in A} \in S^A$ with $s_w \leq |X_{fw}|$ for every $w \in A$ and the corresponding allocation $X_{f,A}^{\mathbf{s}}$ in the market with contracts. If $X_{f,A}^{\mathbf{s}}$ is contract-term efficient then we must have $u_f(X_{f,A}^{\mathbf{s}}) > u_f(X_{f,A}^{\mathbf{s}'})$ for every $\mathbf{s}' \in S^A$ with $\mathbf{s}' > \mathbf{s}$. If, however, $X_{f,A}^{\mathbf{s}}$ is not contract-term efficient then it is dominated through a change of contract-terms by a contract-term efficient allocation that may be represented by $X^{\mathbf{s}'}_{f,A}$ for salaries $\mathbf{s}' \in S^A$ with $\mathbf{s}' > \mathbf{s}$. Thus we may equivalently define a firm's utility of hiring A under salaries $\mathbf{s} \in S^A$ by

$$v_f(A, \mathbf{s}) = \max_{\mathbf{s}' \geq \mathbf{s}} u_f\left(X^{\mathbf{s}'}_{f,A}\right).$$

With this definition, we immediately obtain

$$D_f(\mathbf{s}) = \arg\max_{A \subseteq W} \max_{\mathbf{s}' \geq \mathbf{s}} u_f\left(X^{\mathbf{s}'}_{f,A}\right) = \left(\arg\max_{Y \subseteq \mathcal{U}(X^{\mathbf{s}'}_f)} u_f(Y)\right)_W = \left(C_f\left(\mathcal{U}\left(X^{\mathbf{s}}_f\right)\right)\right)_W.$$

For the fourth part, notice that any allocation $(\mu, \mathbf{s})$ such that $s_w > |X_{\mu(w)w}|$ for some worker w is not individually rational for the firm $\mu(w)$.

For the fifth part, let $(\mu, \mathbf{s})$ be an allocation in $(S, (v_i)_{i \in F \cup W})$ that corresponds to an allocation $X^{\mu,\mathbf{s}}$ in $(X, (u_i)_{i \in F \cup W})$. Note that we had defined utility functions in the market with contracts such that $v_w(\mu, \mathbf{s}) = v_w(X^{\mu,\mathbf{s}}_w)$ for every worker w. Furthermore, we have $v_f(\mu, \mathbf{s}) \neq u_f(X^{\mu,\mathbf{s}}_f)$ for some firm f if and only if there exists salaries $\mathbf{s}' > \mathbf{s}$ such that $u_f(X^{\mu,\mathbf{s}'}_f) > u_f(X^{\mu,\mathbf{s}}_f)$. In this case $(\mu, \mathbf{s}')$ dominates $(\mu, \mathbf{s})$ and $X^{\mu,\mathbf{s}'}$ dominates $X^{\mu,\mathbf{s}}$. So whenever utility functions in the market with contracts and in the market with salaries disagree for some allocation then neither is it efficient in the market with contracts nor in the market with salaries. But this implies the fifth part of the theorem. □

We make three remarks about the theorem that we will further elaborate on in the subsequent sections of the paper.

First, note that whenever the allocation $X^{\mathbf{s}}_{f,A}$ is contract-term efficient then we have $C_f(\mathcal{U}(X^{\mathbf{s}}_{f,A})) = C_f(X^{\mathbf{s}}_{f,A})$. In particular, whenever preferences are Pareto separable the demand is specified in the same way as in the paper by Echenique (2012) who defines the demand of a firm f at salaries $\mathbf{s} \in S^W$ corresponding to *efficient contracts* to be $D_f(\mathbf{s}) = (C_f(X^{\mathbf{s}}_f))_W$. So the result of Echenique is a special case of our result.

The embedding only differs in the way it treats contract-term inefficient allocation. A salary s offered by f to w now corresponds to the upper contour set of the contract x^s_{fw} rather than the contract x^s_{fw} itself. So we may say that behind the embedding result lies an implicit change of the contract set. A contract x is replaced by a new contract that allows the firm in f to choose any of the original contracts in X that the worker involved in x likes as least as much as x. A firm's new preferences are such that it chooses a worker under some salary whenever it would choose a contract from the upper contour corresponding to the salary in the original market.

With this interpretation, the relation between the embedding result and the cumulative offer mechanism becomes clear. Recall that in the cumulative offer mechanism (compare Hatfield and Kojima (2010)), in each round a worker makes an offer of his favorite contract among contracts that have not been previously rejected. The firms cumulate offers in the sense that a firm may choose among all contracts that have been offered to it including contracts that it has previously rejected. But clearly, if a firm is offered some contract x by worker w but the firm prefers the contract y with the same worker and it has previously rejected y then y must lie in the upper contour set of x for the worker w. Thus, the cumulative offer mechanism may be interpreted as a normal worker-proposing deferred-acceptance algorithm (i.e. an algorithm in which each round firms only consider current offers but not previously rejected ones) but applied to a slightly modified version of the original market. For unilaterally substitutable choice functions even more can be said. In this case, not only does the cumulative offer mechanism correspond to a worker-proposing deferred-acceptance mechanism for the modified market. Also, a firm-proposing deferred acceptance mechanism applied to the modified market corresponds to (a class) of firm-proposing algorithm for markets with unilaterally substitutable choice functions that have not been discussed in the literature so far. We will further develop these points in Section 2.3.

Finally, we note that since we do not require the utility functions to be quasi-linear in salaries the demand function does not entirely pin down the preferences of a firm. The demand function only contains information about the firm's preferences among different sets of workers for a given salary vector. On the other hand, it does not tell us anything about the firm's preferences between hiring the same set of workers under some salary vector or some other salary vector. We allowed for firm utility functions that were strictly decreasing for salary vectors that correspond to contract-term efficient allocation but only weakly-decreasing for general salary vectors. This allowed us to have a demand of the form as in Equation 1 while leaving the set of stable allocation invariant under the embedding. In Section 2.3, we will see what will happen if we require that, for the same demand function, utility functions are strictly monotone in salaries even if an allocation corresponds to a contract-term inefficient allocation. In this case we will in general change the set of stable allocations such that it is no longer isomorphic in the two markets. However, the set of stable allocations will change in a very particular way. Each allocation that is stable in the original market will also be stable in the new market and the only stable allocations that are added were previously contract-term dominated by a stable allocation.

2.1 Unilateral Substitutes

Next we show that unilateral substitutability of firms' choice functions in the market with contracts implies gross substitutability of the demand in the corresponding market with salaries. Thus, we generalize the result of Echenique (2012) to the larger domain of unilaterally substitutable preferences and in particular to the cadet-to-branch matching framework of Sönmez and Switzer (2013).

Theorem 2. *Let $(X,(u_i)_{i\in F\cup W})$ be a market with contracts and $(S,(v_i)_{i\in F\cup W})$ the corresponding market with salaries. If firms' choice functions in $(X,(u_i)_{i\in F\cup W})$ are unilaterally substitutable, then firms' demand functions in $(S,(v_i)_{i\in F\cup W})$ are gross substitutable.*

Proof. By Theorem 1, the demand of a firm f in the market with salaries is given by:

$$D_f(\mathbf{s}) = \left(C_f\left(\mathcal{U}(X_f^{\mathbf{s}})\right)\right)_W$$

Thus, we have to show that for salaries $\mathbf{s},\mathbf{s}'\in S^W$ with $\mathbf{s}\leq\mathbf{s}'$ the following holds: Whenever for some worker $\bar{w}\in W$ we have $s_{\bar{w}}=s'_{\bar{w}}$ and there exists an $x\in C_f(\mathcal{U}(X_f^{\mathbf{s}}))$ with $x_W=\bar{w}$, then there also exist a $x'\in C_f(\mathcal{U}(X_f^{\mathbf{s}'}))$ with $x'_W=\bar{w}$. Defining the sets

$$Y := \mathcal{U}\left(X^{\mathbf{s}'}_{f,W\setminus\bar{w}}\right),\quad Z_1 := \mathcal{U}\left(X^{\mathbf{s}}_{f,W\setminus\bar{w}}\right)\setminus Y,\quad Z_2 := \mathcal{U}\left(x^{s_{\bar{w}}}_{f,\bar{w}}\right)$$

this can be reformulated as follows: If there is a $x\in Z_2$ such that $x\in C_f(Y\cup Z_1\cup Z_2)$, then there is a $x'\in Z_2$ such that $x'\in C_f(Y\cup Z_2)$.

Let $x\in Z_2$ be such that $x\in C_f(Y\cup Z_1\cup Z_2)$. Since Z_2 contains only contracts with $\bar{w}$ and a firm can sign at most one contract with a worker, we must have (by a revealed preference argument): $x\in C_f(Y\cup Z_1\cup\{x\})$. As $x_W=\bar{w}\notin(Y\cup Z_1)_W$, unilateral substitutability implies that $x\in C_f(Y\cup\{x\})$. Again by a revealed preference argument, $C_f(Y\cup Z_2)\cap Z_2=\emptyset$ would imply $x\notin C_f(Y\cup\{x\})$ contradicting $x\in C_f(Y\cup\{x\})$. Thus there exists a $x'\in Z_2$ such that $x'\in C_f(Y\cup Z_2)$. □

Next we show that unilateral substitutability is not only sufficient but also necessary for the embedding result.

Proposition 1. *Consider a market with contracts and its corresponding market with salaries. If a firm's choice function in the market with contracts is not unilaterally substitutable then there exist utility functions for the workers such that in the corresponding market with salaries the demand of the firm is not gross substitutable.*

Proof. Assume that there exists a firm f whose choice function violates unilateral substitutability. Then there exist contracts $x, z \in X_f$ and $Y \subseteq X_f$ with $x_W \notin Y_W$ such that $x \in C_f(Y \cup \{x, z\})$ but $x \notin C_f(Y \cup \{x\})$. Let $A := C_f(Y \cup \{x, z\})$ and $B := C_f(Y \cup \{x\})$. For every $y \in A \cup B$ we let $u_{y_W}(y) > u_{y_W}(\emptyset)$ and for every $y \notin A \cup B$ we let $u_{y_W}(y) < u_{y_W}(\emptyset)$. Furthermore, for every $a \in A$ and $b \in B$ such that $a \neq b$ and $a_W = b_W$ we let $u_{a_W}(a) > u_{a_W}(b)$. We embed the market into a market with salaries $(S, (v_i)_{i \in F \cup W})$ as in Theorem 1.

Let us now consider the salary vectors $\mathbf{s} = (s_w)_{w \in A_W}$ and $\mathbf{s}' = (s'_w)_{w \in B_W}$ that corresponds to the allocations A and B, i.e. $X^{\mathbf{s}}_{f,A_W} = A$ and $X^{\mathbf{s}'}_{f,B_W} = B$. In the case that $A_W \neq W$ we extend $\mathbf{s}$ to a salary vector in S^W by specifying salaries for workers in $W \setminus A_W$ as follows: If $w \notin A_W \cup B_W$ then we define $s_w = \bar{s}$ (recall that $\bar{s}$ is the highest salary in S and does not correspond to any contract) and whenever $w \in B_W \setminus A_W$ we define $s_w = s'_w$. We extend the salary vector $\mathbf{s}'$ to a salary vector in S^W by specifying for every $w \in W \setminus B_W$, $s'_w = \bar{s}$.

By construction, we have $s_w \leq s'_w$ for every $w \in W$. Note furthermore, that $C_f(Y \cup \{x, z\}) = A \subseteq \mathcal{U}(X^{\mathbf{s}}_f) \subseteq Y \cup \{x, z\}$. By a revealed preference argument, this implies $C_f(\mathcal{U}(X^{\mathbf{s}}_f)) = C_f(Y \cup \{x, z\})$. Similarly, we have $C_f(Y \cup \{x\}) = B \subseteq \mathcal{U}(X^{\mathbf{s}'}_f) \subseteq Y \cup \{x\}$. By a revealed preference argument, this implies $C_f(\mathcal{U}(X^{\mathbf{s}'}_f)) = C_f(Y \cup \{x\})$. Thus, we have $x_W \in D_f(\mathbf{s}) = (C_f(Y \cup \{x, z\}))_W$ and $x_W \notin D_f(\mathbf{s}') = (C_f(Y \cup \{x\}))_W$. Thus, we have a violation of gross substitutes. □

Salary Adjustment Processes

Kelso and Crawford (1982) give a constructive proof for the existence of stable allocations under gross substitutability by means of the so called salary adjustment process. This procedure can be interpreted as an ascending auction that starts at workers' reservation salaries and raises in each round the salaries of workers who have rejected an offer of a firm in the previous round (see the original paper for details). In Theorem 1, we only assumed non-increasing utilities in salaries and the terminal allocation of a salary-adjustment process is not necessarily stable because it can fail to be efficient. But the salary adjustment process can be augmented by a final step where after the termination of the process each worker's salary is raised to the highest salary among the salaries that give a firm the same utility. Depending on which worker's salary is raised first, second, third etc. this might lead to different allocations. But each of these allocations is stable.

For the model with contracts, this result yields a new class of firm-proposing algorithms that generalize the firm-proposing deferred-acceptance algorithm of Hatfield and Milgrom (2005) to markets with unilaterally substitutable choice functions. We discuss this issue in more detail in Section 2.3.

Symmetrically, a stable allocation can be found by a descending auction. Under the assumption of gross substitutability a descending auction will terminate in a stable allocation. In contrast to the ascending process, an additional final step is unnecessary and the resulting allocation is the worker-optimal stable allocation. In the next section we will discuss the convergence of a descending salary adjustment process under weaker assumptions than gross substitutability.

2.2 Bilateral Substitutes

Hatfield and Kojima (2010) have pointed out that in a market with contracts weaker assumptions than unilateral substitutability guarantee that a worker-proposing algorithm terminates in a stable allocation. Next we translate this insight into the framework with salaries. For this purpose we introduce a new condition on firms' demands in markets with salaries that we call *weak gross substitutability* and show that under this condition a descending auction terminates in a stable allocation.[8] We then show that, although a market with contracts where choice functions are bilaterally substitutable cannot be embedded into a market with salaries such that firms' demands become gross substitutable (as already pointed out by Echenique (2012)), it can be embedded such that the weak gross substitutes condition holds in the market with salaries.

In the following we consider a **descending salary adjustment process**. For notational convenience, we augment the salary set S by a low salary 0 such that no worker will ever work for this salary, i.e. $v_w(f,0) < v_w(\emptyset)$ for every firm f and worker w. It will not be important how we specify firms' utilities for salary vectors that contain a 0 salary.

1. Start with salaries $s_{fw} := \bar{s}$ for every $(f,w) \in F \times W$.

2. Every worker makes an offer to his favorite firm under current salaries respectively stays alone if he finds no firm acceptable under current

[8] Note that this result does not contradict Gul and Stacchetti (1999) who show that gross substitutability is a necessary condition in the maximal domain sense for the existence of stable allocations in the Kelso-Crawford model. In contrast to the model considered by Gul and Stacchetti (1999) we only consider markets with finite sets of permissible salaries and do not require that utility functions are quasi-linear.

salaries.

3. For every worker w who has made an offer to a firm f but whose offer is rejected by the firm under current salaries, we reduce s_{fw} to the next smallest salary in $S \cup \{0\}$ and repeat step 2. If there is no such worker we go to step 4.

4. We match each worker to the firm he makes an offer to in the last round, respectively to himself if he does not make an offer in the last round.

5. In an arbitrary order, for every matched worker-firm pair the salary is raised to the highest salary among the salaries that gives the firm the same utility.

As for the ascending salary adjustment process, the last step is only necessary because we allow firm utility functions to be weakly instead of strictly decreasing in salaries. The following condition on preferences is sufficient for the descending salary adjustment process to converge to a stable allocation. It relaxes gross substitutable in the way that the condition is now only required to hold if the workers whose salaries are raised are not demanded after the salary raise.

Definition 7. D_f satisfies **weak gross substitutability** if for every worker $\bar{w}$ and salaries $\mathbf{s} = (s_w)_{w \in W}, \mathbf{s}' = (s'_w)_{w \in W} \in S^W$ such that $s_{\bar{w}} < s'_{\bar{w}}$ and for every other worker $w \neq \bar{w}$, $s_w = s'_w$, it holds that for every other worker $w \neq \bar{w}$ we have

$$w \in D_f(\mathbf{s}), \bar{w} \notin D_f(\mathbf{s}') \Rightarrow w \in D_f(\mathbf{s}').$$

Proposition 2. *If firms' demands in a market with salaries satisfy weak gross substitutability, then the descending salary adjustment process converges to a stable allocation.*

Proof. Because $S \cup \{0\}$ is finite the algorithm terminates in a finite number of rounds. The terminal allocation is feasible because each worker makes at most one offer per round. Next we check that the terminal allocation is stable. Let μ be the terminal matching. For every firm f and worker w let s_{fw} be the lowest salary at which w made an offer to f (i.e. we raise the final salary for unmatched pairs by one unit). Note that if the salary between a firm f and worker w is changed in some round of the process, then w's offer was not accepted by f. Thus, by the weak gross substitutes condition, decreasing the salary between f and w will not make any worker in $W \setminus \{w\}$

that was not accepted before by f accepted afterwards. Thus in the terminal allocation we must have $\mu^{-1}(f) = D_f((s_{fw})_{w\in W})$. The terminal allocation can only be blocked by a firm f, workers $A \subseteq W$ and salaries $(s'_{fw})_{w\in A}$ such that $s'_{fw} \geq s_{fw}$ for every $w \in A$. But then

$$v_f(\mu, \mathbf{s}) = \max_{A\subseteq W} v_f(A, (s_{fw})_{w\in W}) \geq v_f(A, (s_{fw})_{w\in A}) \geq v_f(A, (s'_{fw})_{w\in A}),$$

where the first equality follows by the definition of the demand and the last inequality follows as the utility function is weakly increasing in salaries. Thus, unless $A = \mu^{-1}(f)$ and f is indifferent between hiring the workers under lower or higher salaries, f and A will not block. But in the fifth step of the algorithm we adjust salaries such that they are efficient. □

With a very similar proof as the one for Theorem 2 it can be shown that bilaterally substitutable choice functions in a market with contracts induce weakly gross substitutable demand functions in the corresponding market with salaries. The proof can be found in the appendix.

Theorem 3. *Let $(X, (u_i)_{i\in F\cup W})$ be a market with contracts and $(S, (v_i)_{i\in F\cup W})$ the corresponding market with salaries. If firms' choice functions in $(X, (u_i)_{i\in F\cup W})$ are bilaterally substitutable, then firms' demands in $(S, (v_i)_{i\in F\cup W})$ are weakly gross substitutable.*

As it was the case for unilateral substitutability (see Proposition 1), we can prove a converse of Theorem 3. The proof can be found in the appendix.

Proposition 3. *Consider a market with contracts and its corresponding market with salaries. If a firm's choice function in the market with contracts is not bilaterally substitutable then there exist utility functions for the workers such that in the corresponding market with salaries the demand of the firm is not weakly gross substitutable.*

2.3 Monotonicity and Firm-Proposing Algorithms under Unilateral Substitutes

Earlier in Section 2, we had remarked that in order to have an embedding that leaves the set of stable allocations invariant and yields a demand of the form given by Equation 1 we had to give up strict monotonicity of utility functions. A firm's utility function was strictly decreasing in salaries only for salaries corresponding to contract-term efficient allocations. Next we discuss what will happen if additionally to requiring that the demand takes the particular form given in Equation 1 we also let the utility function be strictly monotonic everywhere. So instead of the utility functions $(v_i)_{i \in F \cup W}$ from Theorem 1 we look at utility functions $(\hat{v}_i)_{i \in F \cup W}$ that satisfy the following conditions. Let $A, B \subseteq W$ be sets of workers and $\mathbf{s} \in S^A$ and $\mathbf{s}' \in S^B$ be vectors of salaries. Whenever $A \neq B$ we let preferences be as before,

$$\hat{v}_f \left(A, \mathbf{s}'\right) > \hat{v}_f \left(B, \mathbf{s}\right) \Leftrightarrow v_f \left(A, \mathbf{s}'\right) > v_f \left(B, \mathbf{s}\right).$$

Now in the case that $A = B$ and only the salary vectors differ we let the firm always prefer lower salaries,

$$\mathbf{s}' > \mathbf{s} \Rightarrow \hat{v}_f \left(A, \mathbf{s}\right) > \hat{v}_f \left(A, \mathbf{s}'\right).$$

It does not really matter (for stability) how we specify preferences among hiring the same set of workers under two different salary vectors if neither of them has weakly higher salaries for every worker (neither $\mathbf{s}' > \mathbf{s}$ nor $\mathbf{s}' < \mathbf{s}$) as long as it is consistent with the above two conditions. For example, we may let the firm be indifferent if the preference between the two allocations are not already defined by the above two condition. Up to a monotonic transformation the conditions specify a utility function for each firm. A worker's utility function is defined as before by $\hat{v}_w(f, s) = v_w(f, s) = u_w(x^s_{fw})$.

As a firm's preferences over allocations that match different workers to the firm are defined as before in Theorem 1, the utility function induces the same demand as before given by Equation 1. Moreover, for every stable allocation in the market with contracts there still exists a corresponding stable allocation in the market with salaries.[9] However, it now might be the case that the set of stable allocations in the market with salaries is larger than in the market with contracts. Whenever an allocation Y in the market with contracts is dominated through a change of contract-terms by a stable allocation Z then under the new utility functions $(\hat{v}_i)_{i \in F \cup W}$ in the market with salaries the allocation corresponding to Y becomes stable whereas it is not stable according to the utility functions $(v_i)_{i \in F \cup W}$.

One way to understand this extension of the set of stable allocations is from the point of view of lattice theory. It is well-known that the existence of stable allocations in the Kelso-Crawford model also follows from lattice-theoretic consideration (Blair, 1988, see also Fleiner, 2003; Echenique and Oviedo, 2004). More precisely, one can show that stable allocations can be found by iterating a mapping on the complete lattice of salary vectors

[9]Note that we only have modified firm preferences over different salaries. But since the new utility functions are strictly increasing in salaries for workers and strictly decreasing for firms it is never the case that an allocation is blocked (according to the new utility functions) solely by a change of salaries while keeping the matching fixed.

where gross substitutability guarantees that the iterated mapping is monotone. In particular, by Tarski's fixpoint theorem the set of stable allocations itself forms a complete lattice. Similarly, under the assumption of substitutability the set of stable allocations in the Hatfield-Milgrom model forms a complete lattice (Hatfield and Milgrom, 2005).[10] On the other hand, unilateral substitutability does not yield a lattice structure on the set of stable allocations (Hatfield and Kojima, 2010). Theorems 1 and 2 seem to contradict these findings. Given that the gross substitutes condition is satisfied in the market with salaries it seems to be the case that the set of stable allocations should form a complete lattice in the market with salaries but not in the corresponding market with contracts contradicting the existence of a one-to-one correspondence of stable allocations in the two markets. But, as discussed before, by allowing firms to be indifferent between different salary schedules if they do not correspond to contract-term efficient allocations certain allocations that otherwise would be stable are not stable. Now if we require strict monotonicity everywhere then all these allocations become stable and the set of stable allocations is completed to a lattice.

The lattice-completion result, can also be understood as a result about the model of contracts rather than a result about a related market with salaries and in the following theorem we phrase it in this way. If we understand the utility functions above as modified utility functions in the market with contracts ($\hat{u}_f(X^{\mathbf{s}}_{f,A}) := \hat{v}_f(A, \mathbf{s})$) rather than utility functions in a related market with salaries the main insight is the following. The modified utility functions induce modified choice functions that are Pareto separable. Moreover, unilateral substitutability is preserved by the modification. But, as unilateral substitutability and Pareto separability imply substitutability, the modified choice functions are substitutable whenever the original choice functions are unilaterally substitutable. Thus the set of stable allocations in the market with modified choice functions forms a complete lattice. The lattice is a completion of the semi-lattice of stable market under the original unilaterally substitutable choice functions. Moreover, for every newly formed stable matching Y there exists a stable allocation Z that matches the same workers and firms but dominates Y through a change of contract-terms under the original preferences. This result gives us a sense in which every unilateral substitutable choice function is "almost" substitutable.[11]

[10]The set of stable allocations is a complete lattice with respect to both *aggregate preferences of firms* $\preceq_F$ defined by $Y \preceq_F Z \Leftrightarrow u_f(Y) \leq u_f(Z) \quad \text{for all } f \in F$ and *aggregate preferences of workers* $\preceq_W$ defined by $Y \preceq_W Z \Leftrightarrow u_w(Y) \leq u_w(Z) \quad \text{for all } w \in W$.

[11]This result seems to be very related to the notion of substitutable completability

$\succeq_f$	$\succeq_{w_1}$	$\succeq_{w_2}$	$\widetilde{\succeq}_f$	$\widehat{\succeq}_f$
$\{x', z\}$	x	z	$\{x', z\}$	$\{x', z'\}$
$\{x, z'\}$	x'	z'	$\sim \{x', z'\}$	$\{x', z\}$
$\{x, z\}$	$\emptyset$	$\emptyset$	$\{x, z'\}$	$\{x, z'\}$
$\{x', z'\}$			$\{x, z\}$	$\{x, z\}$
$\{x'\}$			$\{x'\}$	$\{x'\}$
$\{x\}$			$\{x\}$	$\{x\}$
$\{z'\}$			$\{z'\}$	$\{z'\}$
$\{z\}$			$\{z\}$	$\{z\}$
$\emptyset$			$\emptyset$	$\emptyset$

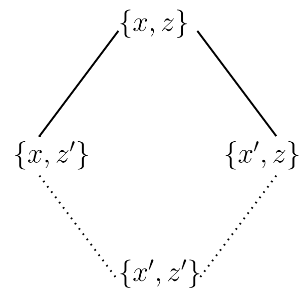

Figure 1: We consider a market with two workers and one firm where the firm's preferences induce unilaterally substitutable choices. $\widetilde{\succeq}_f$ represents the modified preferences of Theorem 1 that leave the set of stable allocations unchanged. The stable semi-lattice is represented by solid lines in the right figure. $\widehat{\succeq}_f$ represents the modified preferences of the firm that make choices substitutable. The allocation $\{x', z'\}$ becomes stable and the set of stable allocations forms a complete lattice.

An illustrative example is given in Figure 1. The proof of the theorem is given in the appendix.

Theorem 4. *Let $(X, (u_i)_{i \in F \cup W})$ be a market with contracts. Then there exist modified utility functions $(\hat{u}_i)_{i \in F \cup W}$ such that*

1. *the choice functions in $(X, (\hat{u}_i)_{i \in F \cup W})$ are Pareto separable,*
2. *$\mathcal{S}(X, (u_i)_{i \in F \cup W}) \subseteq \mathcal{S}\left(X, (\hat{u}_i)_{i \in F \cup W}\right)$, and*
3. *for every $Y \in \mathcal{S}\left(X, (\hat{u}_i)_{i \in F \cup W}\right) \setminus \mathcal{S}(X, (u_i)_{i \in F \cup W})$ there exists a $Z \in S(X, (u_i)_{i \in F \cup W})$ that dominates Y by a change of contract-terms.*
4. *If choice functions in $(X, (u_i)_{i \in F \cup W})$ are unilaterally substitutable, then choice functions in $(X, (\hat{u}_i)_{i \in F \cup W})$ are substitutable.*

The following result, originally proved by Hatfield and Kojima (2010), is an immediate consequence of the theorem.

Corollary 1. *If firms' choice functions in a market with contracts are unilaterally substitutable, then there exists a unique worker optimal stable allocation.*

introduced by Hatfield and Kominers (2014). Indeed, it seems to be the case that Theorem 4 is true for the same reasons that unilaterally substitutable preferences can always be substitutable completed, as recently shown by Kadam (2014).

Hatfield and Milgrom (2005) define two algorithms that find extremal stable matchings in their model. Under the assumption that firms' choice functions are substitutable, the worker-proposing deferred-acceptance algorithm finds the unique worker-optimal stable allocation whereas the firm-proposing algorithm finds the unique firm-optimal stable allocation.

Under unilateral substitutes a firm-optimal stable allocation does not need to exist. But Theorem 4 allows us to define a class of firm-proposing algorithms even when firms' choice functions are unilaterally substitutable but not substitutable. By the theorem, there exist modified versions of the unilaterally substitutable choice functions that are substitutable. We can apply the firm-proposing deferred acceptance algorithm of Hatfield and Milgrom to these modified choice functions. The algorithm terminates in an allocation Y that is stable under the modified choice functions but not necessarily stable under the original choice functions. However there exists - in general multiple - allocations that are stable under the original choice functions and match the same agents as Y but under different contract-terms. We can then augment the algorithm by a final round in which contract-terms are changed such that for each worker we pick his most preferred contract with the same firm as in Y among all contracts under which the firm chooses the same set of workers as in Y. The final allocation will depend on the order in which we change the contract-terms. Thus we have defined a class of algorithms rather than a single algorithm.

As an example, consider the market in Figure 1. Here, the firm proposing deferred-acceptance algorithm applied to the modified choice functions terminates in the unstable allocation $\{x', z'\}$. In the final round, we can either first adjust the contract-terms between f and w_1 or the contract-terms between f and w_2. In the first case the final stable outcome is $\{x, z'\}$ and in the second case it is $\{x', z\}$.

Theorem 4 also allows for a natural theoretical interpretation of the cumulative-offer algorithm. It simply corresponds to the worker-proposing deferred algorithm but applied to the modified choice functions.

3 Discussion

Choice Functions versus Strict Utility Functions

Instead of using strict preferences for firms, as in the original model of Hatfield and Milgrom (2005), Aygün and Sönmez (2013) propose to work directly with firms' choice functions as the primitive of the model and impose a version of the weak axiom of revealed preferences that they call the "Irrelevance of Rejected Contracts (IRC)"[12] on firms' choices. Sönmez and Switzer (2013) use a market with choice functions that satisfy unilateral substitutes and the IRC condition to model the matching between cadets and branches at the United States Military Academy. In the following, we sketch how the results of this paper can be rephrased in the setting proposed by Aygün and Sönmez (2013).

Assume that we are given a market with contracts where each worker w has a strict utility function $u_w : X_w \cup \{\emptyset\} \to \mathbb{R}$ and each firm f has a choice function $C_f : 2^{X_f} \to 2^{X_f}$ that satisfies IRC. We define the embedding and workers' utility functions in the market with salaries in the exact same way as before. Instead of defining utility functions for firms we now specify for every firm f a choice function $\tilde{C}_f : 2^{F\times W\times S} \to 2^{F\times W\times S}$ for the market with salaries as follows. For every set $\{(f, w, s_w) : w \in A \subseteq W\} \subseteq F \times W \times S$ that contains at most one salary per worker we let (with the notation from Section 2)

$$\bar{C}_f(\{(f, w, s_w) : w \in A\}) := \left\{(f, w, s_w) : w \in C_f(\mathcal{U}(X^{\mathbf{s}}_{f,A}))_W\right\}.$$

This implies that the demand for a salary vector $\mathbf{s} = (s_w)_{w\in W}$ is, as before, given by

$$D_f(\mathbf{s}) = \left(\bar{C}_f(\{(f, w, s_w) : w \in W\})\right)_W = \left(C_f(\mathcal{U}(X^{\mathbf{s}}_{f,A}))\right)_W.$$

Now if a set $Y \subseteq F \times W \times S$ contains multiple salaries with some of the workers, we let $(f, w, s) \in \bar{C}_f(Y)$ if and only if for some contract $y \in X_{fw}$ we have $y \in C_f(g^{-1}(Y))$ *and* s is the highest salary with $(f, w, s) \in Y$ such that y lies in the corresponding upper contour set $\mathcal{U}(x^s_{fw})$, i.e. such that $u_w(y) \geq u_w(x^s_{fw})$.

Stability for the market with salaries is now defined in the same way as for the market with contracts interpreting the market with salaries as a market with contracts where the contract set is $F\times W\times S$, a firm's choice functions is $\bar{C}_f$ and a worker's utility function is $\tilde{u}_w(f, w, s_{fw}) \equiv v_w(f, s_{fw})$. One can check that the embedding leaves the set of stable allocations invariant. With the same argument as in the proof of Theorem 2 (note that IRC guarantees that revealed preference arguments are valid) one can show that whenever C_f is unilaterally substitutable then D_f is gross substitutable. Similarly, one can obtain a parallel result to Theorem 3.

[12] The choice function C_f satisfies **irrelevance of rejected contracts (IRC)** (Aygün and Sönmez, 2013) if for all $Y \subseteq X$, $z \in X \setminus Y$,

$$z \notin C_f(Y \cup \{z\}) \Rightarrow C_f(Y) = C_f(Y \cup \{z\}).$$

Many-to-Many Matching

Assuming that preferences on one side but not necessarily on the other side of the market are Pareto-separable the construction proposed in this paper generalizes to the many-to-many case as studied in Kominers (2012). The assumption of Pareto-separability cannot be easily dropped for both sides of the market as the proposed embedding method depends crucially on the fact that each worker can rank his contracts with a given firm independently of his contracts with other firms. Thus Pareto-separability of workers' preferences seems to be necessary for an embedding. In particular, it appears to be necessary that the model is *unitary* (Kominers, 2012), i. e. that workers want to sign at most contract with any firm.

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A Proof of Theorem 3

Proof. By Theorem 1 the demand of a firm f in the market with salaries is given by:

$$D_f(\mathbf{s}) = \left(C_f\left(\mathcal{U}\left(X_f^{\mathbf{s}}\right)\right)\right)_W$$

Consider a worker w_1 and salaries $\mathbf{s}, \mathbf{s}' \in S^W$ such that we have $s_{w_1} < s'_{w_1}$ and $s_w = s'_w$ for every other worker $w \neq w_1$. We have to show the following: If there exists a $x \in C_f(\mathcal{U}(X_f^{\mathbf{s}}))$ with $x_W \neq w_1$ and $w_1 \notin (C_f(\mathcal{U}(X_f^{\mathbf{s}'})))_W$ then there exist a $x' \in C_f(\mathcal{U}(X_f^{\mathbf{s}'}))$ with $x'_W = x_W$. Now let us assume that $x \in C_f(\mathcal{U}(X_f^{\mathbf{s}}))$ with $x_W \neq w_1$ exists and let $w_2 := x_W$. We define the sets

$$Y := \mathcal{U}\left(X^{\mathbf{s}}_{f,W\setminus\{w_1,w_2\}}\right), \quad Z_1 := \mathcal{U}\left(x^{s_{w_1}}_{f,w_1}\right), \quad Z_2 := \mathcal{U}\left(x^{s_{w_2}}_{f,w_2}\right).$$

Note that assuming $w_1 \notin (C_f(\mathcal{U}(X_f^{\mathbf{s}'})))_W$ implies, by a revealed preference argument, $C_f(\mathcal{U}(X_f^{\mathbf{s}'})) = C_f(Y \cup Z_2)$. Thus we need to show that $x \in C_f(Y \cup Z_1 \cup Z_2)$ implies that there exists a $x' \in Z_2$ such that $x' \in C_f(Y \cup Z_2)$.

Since Z_2 contains only contracts with w_2 and a firm can sign at most one contract with a worker, we must have (by a revealed preference argument): $x \in C_f(Y \cup Z_1 \cup \{x\})$. By the same argument, since Z_1 contains only contracts with w_1, we have either $x \in C_f(Y \cup \{x\})$ (in case that $C_f(Y \cup Z_1 \cup \{x\})$ contains no contract with w_1) or $x \in C_f(Y \cup \{x, z\})$ for some $z \in Z_1$. In the latter case, as $w_1, w_2 \notin Y_W$, bilateral substitutability implies that $x \in C_f(Y \cup \{x\})$. Again by a revealed preference argument, $C_f(Y \cup Z_2) \cap Z_2 = \emptyset$ would imply $x \notin C_f(Y \cup \{x\})$ contradicting $x \in C_f(Y \cup \{x\})$. Thus, there exists a $x' \in Z_2$ such that $x' \in C_f(Y \cup Z_2)$. □

B Proof of Proposition 3

Proof. Assume that there exists a firm f whose choice function violates bilateral substitutability. Then there exist contracts $x, z \in X_f$ and $Y \subseteq X_f$ with $x_W, z_W \notin Y_W$ such that $x \in C_f(Y \cup \{x, z\})$ but $x \notin C_f(Y \cup \{x\})$. Let $A := C_f(Y \cup \{x, z\})$ and $B := C_f(Y \cup \{x\})$. For every $y \in A \cup B$ we let $u_{y_W}(y) > u_{y_W}(\emptyset)$ and for every $y \notin A \cup B$ we let $u_{x_W}(y) < u_{x_W}(\emptyset)$. Furthermore, for every $a \in A$ and $b \in B$ such that $a \neq b$ and $a_W = b_W$ we let $u_{a_W}(a) > u_{a_W}(b)$. We embed the market into a market with salaries $(S, (v_i)_{i \in F \cup W})$ as in Theorem 1.

Let us now consider the salary vector $\mathbf{s} = (s_w)_{w \in A_W}$ that corresponds to the allocations A, i.e. $X_f^{\mathbf{s}} = A$. In the case that $A_W \neq W$ we extend $\mathbf{s}$ to a salary vector in S^W by specifying salaries for workers in $W \setminus A_W$ as follows: If $w \notin A_W \cup B_W$ then we define $s_w = \bar{s}$ (recall that $\bar{s}$ is the highest salary in S and does not correspond to any contract). Whenever $w \in B_W \setminus A_W$ we let s_w be the salary corresponding to the respective contract in B, i.e. such that $x_{fw}^{s_w} \in B \setminus A$. Furthermore we define a salary vector $\mathbf{s}' \in S^W$ by $s'_{z_W} = \bar{s}$ and $s_w = s'_w$ for $w \neq z_W$.

By construction, we have $s_w \leq s'_w$ for every $w \in W$. Note furthermore, that $C_f(Y \cup \{x, z\}) = A \subseteq \mathcal{U}(X_f^{\mathbf{s}}) \subseteq Y \cup \{x, z\}$. By a revealed preference argument, this implies $C_f(\mathcal{U}(X_f^{\mathbf{s}})) = C_f(Y \cup \{x, z\})$. Similarly, we have $C_f(Y \cup \{x\}) = B \subseteq \mathcal{U}(X_f^{\mathbf{s}'}) \subseteq Y \cup \{x\}$. By a revealed preference argument, this implies $C_f(\mathcal{U}(X_f^{\mathbf{s}'})) = C_f(Y \cup \{x\})$. Thus, we have $x_W \in D_f(\mathbf{s}) = (C_f(Y \cup \{x, z\}))_W$ and $x_W \notin D_f(\mathbf{s}') = (C_f(Y \cup \{x\}))_W$. Furthermore, by construction, $z_W \notin D_f(\mathbf{s}') = (C_f(Y \cup \{x\}))_W$. Thus, we have a violation of weak gross substitutes. □

C Proof of Theorem 4

Proof. We consider the utility functions defined by $\hat{u}_f\left(X_{f,A}^{\mathbf{s}}\right) = \hat{v}_f(A, \mathbf{s})$ for every $f \in F$, $X_{f,A}^{\mathbf{s}} \subseteq X_f$ and $\hat{u}_w \equiv u_w$ for every $w \in W$. Alternatively, we can define the utility functions without referring to an associated market with salaries: Firm f's utility function is defined up to a monotonic transformation by requiring that for allocations $Y, Z \subseteq X_f$ with $Y_W \neq Z_W$ we have

$$\hat{u}_f(Y) > u_f(Z) \Leftrightarrow \max_{\tilde{Y} \subseteq \mathcal{U}(Y), \tilde{Y}_f = Y_f} u_f\left(\tilde{Y}\right) > \max_{\tilde{Z} \subseteq \mathcal{U}(Z), \tilde{Z}_f = Z_f} u_f\left(\tilde{Z}\right)$$

and for allocation with $Y_W = Z_W$ we have

$$\hat{u}_f(Y) > \hat{u}_f(Z) \Leftrightarrow \mathcal{U}(Y) \supsetneq \mathcal{U}(Z).$$

The modified choices $\hat{C}_f(Y) = \arg\max_{Z \subseteq Y} \hat{u}_f(Z)$ are Pareto separable by construction. Indeed, for any contracts $x, x' \in X_f$ with $x_W = x'_W$ we have $\hat{u}_f(Y \cup \{x\}) > \hat{u}_f(Y \cup \{x'\})$ if and only if $u_{x_W}(x) < u_{x_W}(x')$ independently of the choice of $Y \subseteq X_f$.

The second and third part follow by the previous discussion of stability in the market with salaries and modified utility functions $(S, (\hat{v}_i)_{i \in F \cup W})$, since an allocation is stable in $(X, (\hat{u}_i)_{i \in F \cup W})$ if and only if the corresponding allocation in $(S, (\hat{v}_i)_{i \in F \cup W})$ is stable.

Finally, we show that under the new utilities firms' choices are substitutable. As remarked earlier, substitutability is satisfied if and only if unilateral substitutability and Pareto separability hold (for a proof, see Theorem 3 of Hatfield and Kojima (2010)). Thus it suffices to show that $\hat{C}_f$ satisfies unilateral substitutability. Consider contracts $Y \subseteq X_f$, $x, z \in X_f$ with $x_W \notin Y_W$ and assume that $x \in \hat{C}_f(Y \cup \{x, z\})$. We want to show that $x \in \hat{C}_f(Y \cup \{x\})$. If $x_W = z_W$ then $\hat{C}_f(Y \cup \{x, z\}) = C_f(Y \cup \{x\})$ and trivially $x \in \hat{C}_f(Y \cup \{x\})$. Thus it remains to consider the case that $x_W \neq z_W$. If $x \in \hat{C}_f(Y \cup \{x, z\})$ then, by the construction of the modified utility functions, there exists an $x' \in \mathcal{U}(x)$ with $x' \in C_f(\mathcal{U}(Y \cup \{x, z\})) = C_f(\mathcal{U}(Y) \cup \mathcal{U}(x) \cup \mathcal{U}(z))$. By an argument parallel to the proof of Theorem 2 (now $\mathcal{U}(Y)$ plays the role of the set Y, $Z_1 = \mathcal{U}(z)$ and $Z_2 = \mathcal{U}(x)$) there exists some $x'' \in \mathcal{U}(x)$ with $x'' \in C_f(\mathcal{U}(Y) \cup \mathcal{U}(x)) = C_f(\mathcal{U}(Y \cup \{x\}))$. But as $x_W \notin Y_W$ this implies that $x \in \hat{C}_f(Y \cup \{x\})$. $\square$