Public Investment, Time to Build, and the Zero Lower Bound*

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First draft: January 2014
This version: October 2014

Abstract

We study the effectiveness of public investment in stimulating an economy stuck in a liquidity trap. We do so in the context of a tractable new-Keynesian economy in which a fraction of government spending increases the stock of public capital subject to a time-to-build constraint. Public investment projects typically entail significant time-to-build delays, which often span several years from approval to completion. We show that this feature implies that the spending multiplier associated with public investment can be substantially large—nearly twice as large as the multiplier associated with public consumption—in a liquidity trap. Intuitively, when the time to build is sufficiently long, and to the extent that public capital raises the marginal productivity of private inputs, the resulting deflationary effect will occur after the economy has escaped from the liquidity trap. At the same time, the increase in households's expected wealth amplifies aggregate demand while the economy is still in the liquidity trap. Using a medium-scale model extended to allow for the accumulation of public capital, we quantify the multiplier associated with the spending component of the 2009's ARRA, which allocated roughly 40% of the authorized funds to public investment. We find a peak multiplier of 2.31. Our results also indicate that failing to account for the composition of the stimulus by overlooking its investment component would lead one to underestimate the spending multiplier by about 50%.

JEL classification: E4, E52, E62, H54
Key words: Public spending, Public investment, Time to build, Multiplier, Zero lower bound.

*Financial support from SSHRC and FRQSC is gratefully acknowledged. We thank seminar participants at the University of Ottawa, Université de Lyon 2, the ENSAI, the Norges Bank, HEC Lausanne, the 2014 T2M Conference, and the 2014 Congress of the European Economic Association. We are also grateful to Philippe Bacchetta, Matteo Cacciatore, Larry Christiano, Giancarlo Corsetti, Patrick Fève, Sylvain Leduc, Ferhat Mihoubi, Gernot Müller, Céline Poilly, Federico Ravenna, and Víctor Ríos Rull for helpful comments and discussions. Most of the work on this paper by Jordan Roulleau-Pasdeloup has been done at the Centre de Recherche en Économie et Statistique (CREST) during his PhD and he would like to thank them for their kind hospitality and support.

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"*We can create some room to invest in things that make America stronger, like rebuilding America's infrastructure.*"

Timothy F. Geithner, The New York Times, December 4, 2012

"*The U.S. must address its infrastructure, education, and training needs. Moreover, it must support aggregate demand to repair the damage caused by the Great Recession.*"

Barry Eichengreen, Secular Stagnation: Facts, Causes and Cures, 2014

1 Introduction

One of the most widely debated questions since the onset of the latest global recession has been the effectiveness of public spending as a tool to stimulate the economy. This effectiveness is commonly judged by the size the spending multiplier, that is, the dollar change in aggregate output that results from a dollar increase in public expenditures. From an empirical standpoint, estimates of the spending multiplier range from roughly 0 to well above 1, depending on the sample period and the identifying assumptions.[1] Most theoretical models, on the other hand, yield a spending multiplier in the neighborhood of 1.[2] The main assumption underlying the latter prediction, one that is also implicit in the empirical literature, is that the economy is in "normal times".

Recent theoretical research by Christiano et al. (2011), Eggertsson (2011), and Woodford (2011), however, shows that during sharp recessions that drive nominal interest rates down to their lower bound of zero—rendering conventional monetary policy useless—an increase in public spending can be very effective in stimulating economic activity. In this situation, often referred to as a liquidity trap, the spending multiplier is 2 to 3 times larger than in normal times, under plausible parameter values. Intuitively, by raising aggregate demand, government spending does what monetary policy cannot do: generate inflation. Since the nominal interest rate is stuck at zero, the real interest rate falls, thus further boosting aggregate demand. While very few economies had been plagued by liquidity traps before the Great Recession, there is an increasing fear among academics and policy-makers that such episodes will become recurrent events in the near future.[3] A series of recent papers collected in Baldwin & Teulings (2014) indeed discuss the specter of secular stagnation, i.e., a prolonged period of lackluster growth, that would keep interest rates close to their lower bound.

[1] See, for example, Ramey & Shapiro (1998), Ramey (2011), Fatás & Mihov (2001), Blanchard & Perotti (2002), Perotti (2005), Galí et al. (2007), Mountford & Uhlig (2009), Barro & Redlick (2011), Auerbach & Gorodnichenko (2012) and Bouakez et al. (2014) among others.

[2] See Perotti (2008) and Hall (2009) for comprehensive surveys of the theoretical literature.

[3] For instance, in his 2013 speech at the IMF Economic Forum, Lawrence Summers stated that "*We may well need, in the years ahead, to think about how we manage an economy in which the zero nominal interest rate is a chronic and systemic inhibitor of economic activity, holding our economies back below their potential*".

Perhaps surprisingly, existing studies of the effects of public spending when the zero lower bound (ZLB) on nominal interest rates is binding almost invariably assume that government expenditures are entirely wasteful and have no direct effect on the marginal productivity of private inputs. More specifically, those studies abstract from public investment despite the fact that it represents a non-negligible fraction of total public spending, averaging roughly 23 percent in the U.S., for example. More importantly, Bachmann & Sims (2012) show that, conditional on a positive government spending shock, the ratio of public investment to public consumption tends to rise more during recessions than during expansions. The largest fiscal stimulus plan in U.S. history—the American Recovery and Reinvestment Act (ARRA) of 2009—allocated roughly 40% of non-transfer spending to public investment.

The purpose of this paper is to evaluate the effectiveness of public investment as a fiscal stimulus both in normal times and when the ZLB binds. We start by presenting an analytically tractable new-Keynesian model in which a fraction of government spending is in the form of public investment. The latter increases the stock of public capital, which is an external input in the production technology. In order to account for the implementation delays associated with the completion of investment projects, we assume that multiple periods are required to build new productive (public) capital.[4] Monetary policy sets the nominal interest rate according to a Taylor-type rule subject to a non-negativity constraint. As in Christiano et al. (2011), Eggertsson (2011), and Woodford (2011), we assume that states of the world in which the ZLB binds are the result of a large negative shock to the discount rate, which increases agents' desire to save.[5]

In addition to its direct effect on current aggregate demand, an increase in public investment has similar effects to those of an anticipated technology shock. On the one hand, it exerts downward pressure on real marginal cost and inflation once public capital becomes productive; we call this the supply-side effect.[6] On the other hand, by increasing households' expected wealth, it raises aggregate demand, real marginal cost, and inflation even before public capital becomes productive. The effectiveness of public investment in stimulating output therefore depends on the relative importance of these two forces at different time horizons, which in turn crucially depends on the time-to-build delays.

[4] The assumption of time to build was first introduced by Kydland & Prescott (1982) in the context of a real business cycle model.

[5] Alternatively, Mertens & Ravn (2010) assume that the economy is plunged in a liquidity trap by a sudden change in agents' beliefs that is not motivated by fundamentals.

[6] There is by now ample empirical evidence, dating back to Aschauer (1989) and surveyed in Bom & Ligthart (2013), that public investment raises the return on private inputs.

In order to illustrate the importance of these delays, we consider a simplified version of our model in which public investment becomes immediately productive (i.e., without time-to-build delay), and public capital fully depreciates after one period. Since this version of the model has no (endogenous) state variable, the spending multiplier can be easily characterized. We show that, in normal times, the spending multiplier increases linearly with the fraction of public investment in a stimulus plan, whereas the opposite result holds when the ZLB binds. Intuitively, in this version of the model, an increase in public spending has only contemporaneous effects on real marginal cost: a positive effect due to the direct increase in aggregate demand and a negative effect due to the externality associated with the productive component of the stimulus. The larger this component, the lower current and expected inflation. In normal times, this translates into a smaller response of the real interest rates and thus a larger output response. When the ZLB binds, on the other hand, this implies a larger response of the real interest rate and a smaller output response.

We then show that the main insights from the simplified version of the model carry over to a more general case in which the time-to-build delays are short and public capital depreciates at an empirically plausible rate. Under these assumptions, the spending multiplier continues to be an increasing function of the fraction of public investment in normal times and a decreasing function of this fraction when the ZLB binds. Intuitively, when the time to build is relatively short, the deflationary effect associated with the increase in labor productivity comes about rapidly, thus attenuating the increase in inflation resulting from higher aggregate demand and from the increase in households' expected wealth. In normal times, this implies a smaller initial increase in the long-term real interest rate, and therefore a larger spending multiplier, as the fraction of investment becomes larger. When the ZLB binds, the deflationary effect persists for a prolonged period of time while the economy is in the liquidity trap, which mitigates the decline in the long-term real interest rate and diminishes the multiplier to an extent that depends positively on the fraction of public investment.

When the time to build is relatively long, these results are reversed: the multiplier decreases with the fraction of public investment in normal times but increases with it when the ZLB binds. Intuitively, with a long time-to-build delay, the supply-side effect of public investment comes into play far in the future so that the inflationary pressure lasts for a long period of time. In normal times, this amplifies the increase in the long-term real interest rate and reduces the multiplier to an extent that depends positively on the fraction of public investment. When the ZLB binds, the further in time capital becomes productive, the more likely the resulting deflationary pressure will occur after the economy has escaped from the liquidity trap. At the same time, the positive wealth effect associated with the expected increase in labor productivity adds to inflation while the economy is still in the liquidity trap. This tends to amplify the decline in the long-term real interest rate and, thus the multiplier, to an extent that increases with the fraction of public investment. In this sense, our paper is related to the work of Fernández-Villaverde et al. (2011) and Eggertsson et al. (2014)), who show that supply-side policies like structural reforms and market deregulation can stimulate an economy trapped at the ZLB, precisely because they increase future output, which generates a wealth effect that boosts current aggregate demand.

Importantly, we find that the spending multiplier can be substantially large at the ZLB when a significant fraction of public spending is invested in public capital and when the time-to-build delay is relatively long. With a time to build of 16 quarters, the multiplier associated with public investment is roughly five times larger than in normal times and nearly twice as large as the multiplier associated with public consumption. We also show that this result remains remarkably robust when we (*i*) focus on the present-value rather than the impact multiplier, (*ii*) take into account the gradual nature of public investment outlays, and (*iii*) allow for the possibility of endogenous exit from the liquidity trap. Given that public investment projects entail significant time-to-build delays—often spanning several years, our results suggest that public investment can be a highly effective tool to stimulate an economy stuck in a liquidity trap.

In the last part of the paper, we extend the medium-scale model developed by Christiano et al. (2013) to allow for the accumulation of public capital and for time-to-build delays, and we use the extended model to gauge the output effects of the spending component of the ARRA. To this end, we calibrate the model based on the actual composition of the stimulus, the nature of public investment projects that it comprised, and their financing scheme. We find that the spending component of the ARRA had a peak multiplier of 2.31. Our results also indicate that counterfactually assuming that the stimulus was exclusively composed of public consumption spending would lead one to underestimate the multiplier by roughly 50%.

In addition to the literature cited above, this paper is closely related to those by Baxter & King (1993), Linnemann & Schabert (2006), Leeper et al. (2010), Leduc & Wilson (2013), Drautzburg & Uhlig (2013), Albertini et al. (2014), and Bom & Ligthart (2014), who study the business-cycle implications of public investment in the context of general-equilibrium models. Baxter & King (1993), Leeper et al. (2010), and Bom & Ligthart (2014) consider a neoclassical framework, whereas the remaining studies consider models with nominal rigidities. Among all these contributions, only those by Drautzburg & Uhlig (2013) and Albertini et al. (2014) consider the case of a binding ZLB, but none of these two papers takes into account the time-to-build delay associated with public investment projects, a feature which, as we show in this paper, has crucial implications for the results.

The rest of this paper is organized as follows. Section 2 presents a simple new-Keynesian economy with public capital and time to build. Section 3 derives the spending multiplier in a special version of the model without time to build and with full depreciation of the stock of public capital. Section 4 studies the relationship between the spending multiplier, the fraction of public investment in a stimulus, and the time to build. It also performs a sensitivity analysis and studies the robustness of the results to an alternative measure of the multiplier and to alternative modelling assumptions. Section 5 quantifies the effects of the spending component of the ARRA in a medium-scale new-Keynesian model. Section 6 concludes.

2 A Simple Model with Public Investment and Time to Build

We consider a simple new-Keynesian economy without private capital. The economy is composed of infinitely lived households, firms, a government, and a monetary authority. The key feature of the model is that a fraction of government spending is invested in public capital subject to a time-to-build requirement. The stock of public capital enters as an external input in the production of intermediate goods, which are used to produce an homogenous final good. The latter is used for consumption and investment purposes. Intermediate-good producers are monopolistically competitive and set their prices à la Calvo, whereas final-good producers are perfectly competitive. The monetary authority sets the nominal interest rate according to a Taylor-type rule subject to a non-negativity constraint.

2.1 Households

The economy is populated by a large number of identical households who have the following lifetime utility function:[7]

$$E_t \sum_{s=0}^{\infty} d_{t+s} U\left(C_{t+s}, N_{t+s}\right), \tag{1}$$

[7] The equilibrium dynamics of this economy are identical to those of an economy in which public spending affects preferences in an additively separable manner (as in Christiano et al. (2011)). Since the focus of this paper is not on the normative implications of fiscal policy, we simply assume that public spending does not enter the utility function.

where C_t is consumption, N_t denotes hours worked, and d_t is a time-varying discount factor defined by

$$d_{t+s} = \begin{cases} \beta_{t+1}\beta_{t+2}...\beta_{t+s} & s \geq 1 \\ 1 & s = 0, \end{cases}$$

where $0 < \beta_{t+s} < 1$ is known in $t+s-1$ with certainty.

The representative household enters period t with $\mathcal{B}_{t-1}$ units of one-period riskless nominal bonds. During the period, it receives a wage payment, $\mathcal{W}_t N_t$, and dividends, $\mathcal{D}_t$, from the monopolistically competitive firms. This income is used to pay a lump-sum tax, $\mathcal{T}_t$, to the government, to consumption, and to the purchase of new bonds. The household's budget constraint is therefore

$$P_t C_t + \mathcal{T}_t + \frac{\mathcal{B}_t}{1+i_t} \leq \mathcal{W}_t N_t + \mathcal{D}_t + \mathcal{B}_{t-1}, \tag{2}$$

where P_t is the price of the final good, $\mathcal{W}_t$ is nominal wage rate, and $\frac{1}{1+i_t}$ is the price of a nominal bond purchased at time t, i_t being the nominal interest rate. The household maximizes (1) subject to (2) and to no-Ponzi-game condition. The first order conditions for this problem are given by

$$W_t = -\frac{U_N(C_t, N_t)}{U_C(C_t, N_t)}, \tag{3}$$

$$\frac{1}{1+i_t} = \beta_{t+1} E_t \left(\frac{U_C(C_{t+1}, N_{t+1})}{U_C(C_t, N_t)} \frac{P_t}{P_{t+1}} \right), \tag{4}$$

where $W_t = \frac{\mathcal{W}_t}{P_t}$ is the real wage rate and $U_X(C_t, N_t) = \partial U(C_t, N_t) / \partial X_t$. Note that the conditional-expectation operator, E_t, is not applied to β_{t+1} because the latter is known in period t.

2.2 Firms

The final good is produced by perfectly competitive firms using the following constant-elasticity-of-substitution technology:

$$Y_t = \left[\int_0^1 X_t(z)^{1-1/\theta} dz \right]^{\frac{\theta}{\theta-1}}, \tag{5}$$

where $X_t(z)$ is the quantity of intermediate good z and $\theta \geq 1$ is the elasticity of substitution between intermediate goods. Denoting by $P_t(z)$ the price of intermediate good z, demand for z is given by

$$X_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\theta} Y_t. \tag{6}$$

Firms in the intermediate-good sector are monopolistically competitive, each producing a differentiated good using labor as a direct input and public capital as an external input

$$X_t(z) = F(N_t(z), K_t) = J(N_t(z))H(K_t), \tag{7}$$

where the function $J(.)$ is such that $J(U_tV_t) = J(U_t)J(V_t)$. Intermediate-good producers set their prices à la Calvo. That is, in each period, a given firm resets its price with probability $1-\phi$. Denoting by P_t^o the optimal price chosen in period t, the firm's problem is

$$\max_{P_t^o} E_t \sum_{s=0}^{\infty} \phi^s Q_{t,t+s} \left\{ P_t^o X_{t,t+s} - (1-\tau)\mathcal{W}_{t+s} N_{t,t+s} \right\},$$

subject to

$$X_{t,t+s} = F(N_{t,t+s}, K_{t+s}),$$

and

$$X_{t,t+s} = \left(\frac{P_t^o}{P_{t+s}}\right)^{-\theta} Y_{t+s},$$

where $Q_{t,t+s} = d_{t+s}U_C(C_{t+s}, N_{t+s}) / d_tU_C(C_t, N_t)$ is the stochastic discount factor; $X_{t,t+s}$ and $N_{t,t+s}$ are, respectively, the quantity of intermediate good produced and labor demand in period $t+s$ if the price set at time t is still in effect; there is a subsidy $\tau = 1/\theta$ that corrects the steady-state distortion stemming from monopolistic competition in the goods market.

The first order condition for this program is given by

$$E_t \sum_{s=0}^{\infty} \phi^s Q_{t,t+s} X_{t,t+s} [P_t^o - \mu MC_{t,t+s}] = 0, \tag{8}$$

where $MC_{t,t+s} = \frac{(1-\tau)\mathcal{W}_{t+s}}{F_N(N_{t,t+s}, K_{t+s})}$ is the marginal cost of producing an additional unit of output in period $t+s$ if the price set at time t is still in effect, and $\mu = \theta/(\theta-1)$ is the desired steady-state markup over marginal cost.

Given the price setting mechanism just described, the price of the final good evolves according to

$$P_t^{1-\theta} = (1-\phi)(P_t^o)^{1-\theta} + \phi P_{t-1}^{1-\theta}. \tag{9}$$

2.3 Fiscal and monetary authorities

The government levies lump-sum taxes to finance its expenditures and the subsidy given to firms in the intermediate-good sector. Its budget constraint is given by

$$P_t G_t + (1-\tau)\mathcal{W}_t N_t = \mathcal{T}_t, \tag{10}$$

where G_t is government spending, which is composed of two parts, public consumption and public investment

$$G_t = G_t^c + G_t^i. \tag{11}$$

Public investment increases the stock of public capital according to the following accumulation equation:

$$K_t = (1-\delta)\,K_{t-1} + G_{t-T}^i, \tag{12}$$

where $T \geq 0$. This specification allows for the possibility that several periods may be required to build new productive capital, i.e., time to build (see Kydland & Prescott (1982)). This feature reflects the implementation delays typically associated with the different stages of public investment projects (planning, bidding, contracting, construction, etc.).[8]

In normal times, public spending is determined by the following process:

$$G_t = (1-\rho)G + \rho G_{t-1} + \epsilon_t, \tag{13}$$

where $0 \leq \rho < 1$, G is the steady-state level of public spending, and ϵ_t is a zero-mean serially uncorrelated disturbance. The dynamics of public spending when the ZLB binds will be described in Section 3.2. Moreover, we assume that public investment is determined by the following policy rule:

$$G_t^i = G^i + \alpha\,(G_t - G)\,, \tag{14}$$

where $0 \leq \alpha \leq 1$ and G^i is the steady-state level of public investment. The policy parameter α measures the fraction of public investment in a spending-based stimulus plan. Note that this fraction need not be equal to the steady-state share of public investment in total public expenditures.

The monetary policy is described by a simple Taylor-type rule in which the inflation target is

[8] In general, the outlays of an investment project occur gradually over time after the budget has been authorized. To keep the model analytically tractable, however, we abstract for now from this issue and simply assume that investment expenditures are outlaid in full at the time of the authorization. We will relax this assumption in sections 4.4.2 and 5.

0 and which is subject to a non-negativity constraint on the nominal interest rate:

$$i_t = \max\left(0; r_\beta + \phi_\pi \pi_t\right), \tag{15}$$

where $r_\beta = \beta^{-1} - 1$ and $\pi_t = P_t/P_{t-1} - 1$ is the inflation rate between $t-1$ and t.

2.4 Market clearing and equilibrium

Market clearing for each intermediate good z requires that

$$F\left(N_t\left(z\right), K_t\right) = J(N_t\left(z\right))H(K_t) = \left(\frac{P_t(z)}{P_t}\right)^{-\theta} Y_t,$$

which implies

$$N_t(z) = J^{-1}\left[\frac{Y_t}{H(K_t)}\left(\frac{P_t(z)}{P_t}\right)^{-\theta}\right].$$

Aggregating across all intermediate-good producers and imposing labor market equilibrium, we obtain

$$N_t = J^{-1}\left(\frac{Y_t}{H(K_t)}\right)\int J^{-1}\left[\left(\frac{P_t(z)}{P_t}\right)^{-\theta}\right]dz.$$

This yields

$$Y_t = \frac{F\left(N_t, K_t\right)}{\Delta_t},$$

where $\Delta_t = J\left(\int J^{-1}\left[\left(\frac{P_t(z)}{P_t}\right)^{-\theta}\right]dz\right)$ is a measure of dispersion of relative prices.

Since households are identical, the net supply of bonds must be zero in equilibrium ($\mathcal{B}_t = 0$). Finally, the resource constraint is

$$Y_t = C_t + G_t. \tag{16}$$

A competitive intertemporal equilibrium for this economy is a sequence of prices $\{P_t(z), P_t, W_t, i_t\}_{t=0}^{\infty}$ and quantities $\{X_t(z), N_t(z), N_t, Y_t, C_t, K_t, G_{t-T}^i\}_{t=0}^{\infty}$ such that, for a given sequence of exogenous variables $\{\beta_t, G_t\}_{t=0}^{\infty}$, households and firms solve their respective optimization problems, the accumulation equation of public capital holds, the spending and monetary rules hold, and all markets clear. The model equations are listed in Appendix A.1.

2.5 Linearized model

The model is solved by linearizing the equilibrium conditions around a deterministic zero-inflation steady state. In what follows, variables without a time subscript denote steady-state values and variables in lowercase denote percentage deviations from steady state ($z_t = \left(Z_t - Z\right)/Z$), except for $g_t = (G_t - G)/Y$ and $g_t^i = (G_t^i - G^i)/Y$. Defining $r_{\beta,t} = \beta_t^{-1} - 1$, $\bar{g} = G/Y$ and $\bar{\alpha} = G^i/G$, the linearized model is given by (see Appendices A.2 and A.3 for details)

$$c_t = E_t c_{t+1} - \Phi_r (i_t - E_t \pi_{t+1} - r_{\beta,t}) + \Phi_g (g_t - E_t g_{t+1}) - \Phi_k (k_t - E_t k_{t+1}), \tag{17}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa (\Theta_c c_t + \Theta_g g_t - \Theta_k k_t), \tag{18}$$

$$k_t = (1-\delta) k_{t-1} + \alpha \tilde{\delta} g_{t-T}, \tag{19}$$

$$i_t = \max(0; r_\beta + \phi_\pi \pi_t), \tag{20}$$

where

$$\Phi_r = \frac{-\frac{U_C}{U_{CC}C}}{1 + (1-\bar{g}) \frac{U_{CN}N}{U_{CC}C} \frac{F}{F_N N}} \quad \Phi_g = \frac{F}{F_N N} \frac{U_{CN}N}{U_C} \Phi_r, \quad \Phi_k = \frac{F_K K}{F} \Phi_g,$$

$$\Theta_c = \left(\frac{U_{CN}C}{U_N} - \frac{U_{CC}C}{U_C} \right) + (1-\bar{g}) \Theta_g,$$

$$\Theta_g = \left(\frac{U_{NN}N}{U_N} - \frac{U_{CN}N}{U_C} - \frac{F_{NN}N}{F_N} \right) \frac{F}{F_N N}, \quad \Theta_k = \frac{F_{NK}K}{F_N} + \frac{F_K K}{F} \Theta_g,$$

and

$$\kappa = \frac{(1-\phi)(1-\beta\phi)}{\phi} \frac{\frac{F_N N}{F}}{\frac{F_N N}{F} - \theta \frac{F_{NN}N}{F_N}} > 0,$$

$$\tilde{\delta} = \frac{\delta}{\bar{\alpha}\bar{g}} \geq 0.$$

Model (17)–(20) nests the framework considered by Christiano et al. (2011), Eggertsson (2011) and Woodford (2011), in which public spending plays no productive role. This case can be recovered either by assuming that $F_K = 0$ (which implies that $\Phi_k = \Theta_k = 0$) or by setting $\alpha = 0$. In the former case, public capital does not affect the marginal productivity of private inputs. In the latter, the fraction of total public spending devoted to investment is nil, implying that the stock of public capital remains constant at its steady-state level.

3 A Special Case

Consider the case where (*i*) public investment increases public capital contemporaneously, i.e., with no time-to-build delay ($T = 0$), (*ii*) public capital depreciates fully at the end of each period ($\delta = 1$), and (*iii*) the utility function is additively separable in consumption and leisure ($U_{CN} = 0$). The first two conditions imply that the model has no (endogenous) state variable. The third condition implies that $\Phi_g = \Phi_k = 0$ and that Φ_r, Θ_c, Θ_g, and Θ_k are all positive. In this case, model (17)–(20)

collapses to

$$c_t = E_t c_{t+1} - \Phi_r \left(i_t - E_t \pi_{t+1} - r_{\beta,t} \right), \tag{21}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \Big(\Theta_c c_t + (\Theta_g - \alpha \tilde{\delta} \Theta_k) g_t \Big), \tag{22}$$

$$i_t = \max \left(0; r_\beta + \phi_\pi \pi_t \right). \tag{23}$$

This version of the model will allow us to obtain a tractable analytical characterization of the effects of government spending in normal times and when the ZLB binds.

3.1 The spending multiplier in normal times

In normal times, the nominal interest rate is strictly positive and is determined by

$$i_t = r_\beta + \phi_\pi \pi_t.$$

For simplicity, we also assume that the discount factor is constant (i.e., $r_{\beta,t} = r_\beta$). Under these assumptions, system (21)–(23) becomes

$$c_t = E_t c_{t+1} - \Phi_r \left(\phi_\pi \pi_t - E_t \pi_{t+1} \right), \tag{24}$$

$$\pi_t = \beta E_t \pi_{t+1} + \kappa \Big(\Theta_c c_t + (\Theta_g - \alpha \tilde{\delta} \Theta_k) g_t \Big). \tag{25}$$

Under the assumption that $\phi_\pi > 1$, and given the process (13), the unique linear rational expectation solution of the system above is given by

$$c_t = \vartheta_g g_t, \tag{26}$$

$$\pi_t = \zeta_g g_t, \tag{27}$$

where

$$\vartheta_g = -\frac{\kappa \left(\phi_\pi - \rho \right) \left(\Theta_g - \alpha \tilde{\delta} \Theta_k \right) \Phi_r}{\left(1 - \rho \right) \left(1 - \beta \rho \right) + \kappa \left(\phi_\pi - \rho \right) \Phi_r \Theta_c},$$

$$\zeta_g = \frac{\kappa \left(1 - \rho \right) \left(\Theta_g - \alpha \tilde{\delta} \Theta_k \right)}{\left(1 - \rho \right) \left(1 - \beta \rho \right) + \kappa \left(\phi_\pi - \rho \right) \Phi_r \Theta_c}.$$

Using the linearized resource constraint, $y_t = \left(1 - \bar{g} \right) c_t + g_t$, we obtain the following expression for the spending multiplier, $m \equiv dY_t / dG_t = y_t / g_t$:

$$m = 1 + \left(1 - g \right) \vartheta_g.$$

With little algebra, and using the fact that $\Theta_c = \Phi_r^{-1} + (1-\bar{g})\,\Theta_g$ when $U_{CN} = 0$, the multiplier can be rewritten as

$$m = \frac{1+\psi+\alpha\,(1-\bar{g})\,\Phi_r\tilde{\delta}\Theta_k}{1+\psi+(1-\bar{g})\,\Phi_r\Theta_g}, \tag{28}$$

where $\psi = \frac{(1-\rho)(1-\beta\rho)}{\kappa(\phi_\pi-\rho)} \geq 0$ is a parameter that captures the Keynesian aspect of this economy.

Denote by m^n the neoclassical multiplier, that is, the multiplier obtained under fully flexible prices ($\kappa \to \infty$ or, equivalently, $\psi = 0$), and by m_u the multiplier corresponding to the case where all public spending is unproductive ($\alpha = 0$). These quantities are given by

$$m^n = \frac{1+\alpha\,(1-\bar{g})\,\Phi_r\tilde{\delta}\Theta_k}{1+(1-\bar{g})\,\Phi_r\Theta_g}, \tag{29}$$

$$m_u = \frac{1+\psi}{1+\psi+(1-\bar{g})\,\Phi_r\Theta_g}. \tag{30}$$

The following proposition summarizes the main properties of the spending multiplier in normal times.

Proposition 1 *In normal times, the spending multiplier, m, is linearly increasing in the share of public investment, α, and is larger than* 1 *if and only if*

$$\alpha \geq \frac{\Theta_g}{\tilde{\delta}\Theta_k}.$$

In this case, we have

$$m_u \leq 1 \leq m \leq m^n.$$

In order to get some intuition for the results stated in Proposition 1, it is useful to use equations (26), (27), and the fact that $E_t g_{t+1} = \rho g_t$, to express $E_t\pi_{t+1}$ and $E_t c_{t+1}$ as functions of π_t and c_t

$$E_t c_{t+1} = \rho c_t,$$

$$E_t \pi_{t+1} = \rho \pi_t.$$

Inserting these expressions in (24) and (25), we obtain

$$c_t = -\frac{(\phi_\pi-\rho)\Phi_r}{1-\rho}\pi_t, \tag{31}$$

$$\pi_t = \frac{\kappa\Theta_c}{1-\beta\rho}c_t + \frac{\kappa(\Theta_g-\alpha\tilde{\delta}\Theta_k)}{1-\beta\rho}g_t. \tag{32}$$

The upper panels of Figure 1 provide a graphical representation of these two equations in the (c_t, π_t)

plan. The curve labeled "*Euler*" corresponds to (31) and is downward sloping. The curve labeled "*NKPC*" corresponds to (32) and its positive slope implies a finite value of κ.

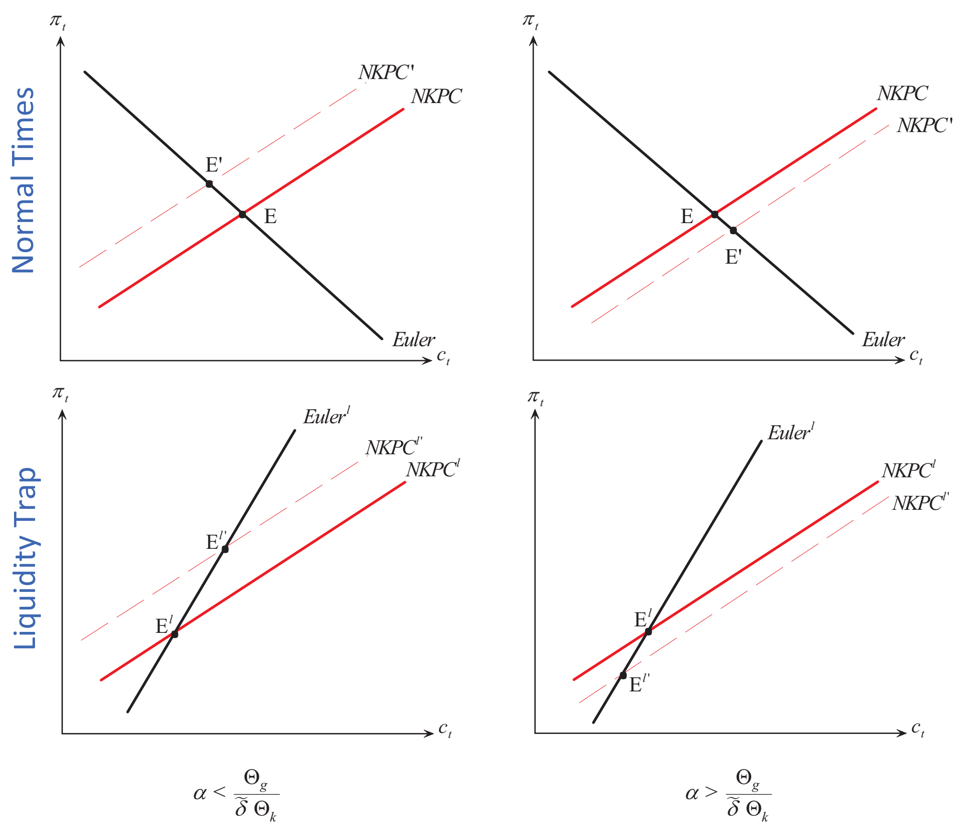

Figure 1: Graphical representation of the equilibrium in normal times and when the ZLB binds in a model with no time-to-build and full depreciation of public capital.

An increase in government spending leaves the *Euler* curve unchanged but shifts the *NKPC*. In the standard case where public spending is unproductive, an increase in government expenditures raises aggregate demand, which in turn raises real marginal cost by a factor of Θ_g (see equation 25). As a result, the *NKPC* shifts to the left, leading to an increase in inflation and a fall in consumption in equilibrium. This in turn yields a multiplier that is smaller than 1. Productive public spending, on the other hand, acts like a technology shock, raising aggregate supply and lowering real marginal cost by a factor of $\alpha\tilde{\delta}\Theta_k$. When $\alpha < \Theta_g/(\tilde{\delta}\Theta_k)$, the latter effect is not strong enough to induce a fall in real marginal cost and the *NKPC* still shifts to the left. In contrast, when $\alpha > \Theta_g/(\tilde{\delta}\Theta_k)$, the supply-side effect dominates and, as a result, the *NKPC* shifts to the right. This yields an increase in consumption and a multiplier larger than 1. The magnitude of the rightward shift in the *NKPC* depends positively on the term $\alpha\tilde{\delta}\Theta_k - \Theta_g$, which increases monotonically with α. Moreover, as prices become less rigid (that is, as κ increases), the *NKPC* becomes steeper and shifts by a larger amount, thus implying a larger multiplier. The latter reaches its maximum value, m^n, when prices are fully flexible, i.e., when the slope of the *NKPC* is infinite.

3.2 The spending multiplier in a liquidity trap

We now study the effects of a government spending shock under the assumption that the ZLB on the nominal interest rate binds. The ZLB becomes binding as a result of shock that raises the discount factor. As in Christiano et al. (2011), Eggertsson (2011) and Woodford (2011), we assume that the discount factor can take only two possible values, β and $\beta^l > \beta$, and evolves according to the following process:

$$\begin{aligned} Pr\left[\beta_{t+1} = \beta^l | \beta_t = \beta^l\right] &= p, \\ Pr\left[\beta_{t+1} = \beta^l | \beta_t = \beta\right] &= 0, \end{aligned} \tag{33}$$

where p is the probability that the discount factor remains high. For simplicity, we also follow this literature by assuming that $g_t = g^l$ as long as the ZLB is binding and $g_t = 0$ otherwise. Substituting $i_t = 0$ in equation (17) and denoting by a superscript l the values taken by the variables when the economy is in a liquidity trap, we obtain the following system:

$$c_t^l = E_t c_{t+1} + \Phi_r (E_t \pi_{t+1} + r_\beta^l), \tag{34}$$

$$\pi_t^l = \beta E_t \pi_{t+1} + \kappa \left(\Theta_c c_t^l + (\Theta_g - \alpha\tilde{\delta}\Theta_k) g^l\right). \tag{35}$$

The absence of an endogenous state variable in this system means that whenever the economy leaves the ZLB state, it jumps immediately to its steady state. Thus, $E_t c_{t+1} = p c_t^l$ and $E_t \pi_{t+1} = p \pi_t^l$. Solving for c_t^l and π_t^l yields

$$\begin{aligned} c^l &= \frac{p\kappa(\Theta_g - \alpha\tilde{\delta}\Theta_k)\Phi_r}{(1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c} g^l + \frac{\kappa(1-\beta p)\Phi_r}{(1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c} r_\beta^l, \\ \pi^l &= \frac{\kappa(1-p)(\Theta_g - \alpha\tilde{\delta}\Theta_k)}{(1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c} g^l + \frac{\kappa\Phi_r\Theta_c}{(1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c} r_\beta^l, \end{aligned}$$

where $(1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c > 0$ is a necessary condition for determinacy.

The spending multiplier associated with the ZLB state, m^l, is given by

$$m^l = \frac{\psi^l - 1 - \alpha\left(1-g\right)\Phi_r\tilde{\delta}\Theta_k}{\psi^l - 1 - \left(1-\bar{g}\right)\Phi_r\Theta_g}, \tag{36}$$

where $\psi^l = \frac{(1-p)(1-\beta p)}{p\kappa} > 0$.[9] In the case where public spending is entirely unproductive ($\alpha = 0$), the multiplier, denoted by m_u^l, is given by

$$m_u^l = \frac{\psi^l - 1}{\psi^l - 1 - \left(1-\bar{g}\right)\Phi_r\Theta_g}. \tag{37}$$

The following proposition summarizes the main properties of the multiplier in a liquidity trap.

Proposition 2 *When the ZLB binds, the spending multiplier, m^l, is linearly decreasing in the share of public investment, α, and is larger than* 1 *if and only if*

$$\alpha \leq \frac{\Theta_g}{\tilde{\delta}\Theta_k}.$$

In this case, we have

$$m^n < m \leq 1 \leq m^l \leq m_u^l.$$

Again, the intuition for these results can be understood by examining how the *Euler* and *NKPC* equations are affected by the increase in public spending. Substituting the conditional expectations of c_{t+1} and π_{t+1} in (34) and in (35), we obtain

$$c_t^l = \frac{p\Phi_r}{1-p}\pi_t^l, \tag{38}$$

$$\pi_t^l = \frac{\kappa\Theta_c}{1-\beta p}c_t^l + \frac{\kappa(\Theta_g - \alpha\tilde{\delta}\Theta_k)}{1-\beta p}g_t^l. \tag{39}$$

These two curves are represented graphically in the bottom panels of Figure 1, and are labeled *Euler*^{l} and *NKPC*^{l}, respectively. Note that the former is upward sloping when the ZLB binds. Intuitively, higher current inflation leads to a rise in expected inflation and since the nominal interest rate is constant (at zero), the real interest rate must fall, thus causing an increase in consumption, *ceteris paribus*. In this case, an increase in public spending will raise consumption in equilibrium only to the extent that it shifts *NKPC*^{l} to the left, which requires that the demand-side effect of public spending dominates its supply-side effect, i.e., $\Theta_g \geq \alpha\tilde{\delta}\Theta_k$ (see the bottom left panel of Figure 1). The largest increase in consumption—and therefore the largest multiplier—is obtained when the

[9]Note that the determinacy condition stated above implies that $\psi^l > 1 + (1-\bar{g})\,\Phi_r\Theta_g$.

fraction of public investment, α, is nil. On the other hand, the larger this fraction, the further $NKPC^l$ shifts to the right and the smaller the value of the multiplier.

3.3 A first quantitative exercise

So far, the discussion about the size of the spending multiplier has remained mostly qualitative. The purpose of this section is to quantify, within the special case just described, the spending multiplier and its dependence on the share of public investment both in normal times and when the ZLB binds. To this end, we need to specify functional forms for the utility and production functions and to assign values to the model parameters. For ease of comparison with the results obtained by Christiano et al. (2011), we adopt the same functional form for preferences and consider a production function that nests their specification.

The utility function is given by

$$U\left(C_t, N_t\right) \begin{array}{ll} = \dfrac{\left(C_t^{\gamma}(1-N_t)^{1-\gamma}\right)^{1-\sigma}-1}{1-\sigma} & \text{if } \sigma \neq 1 \\ = \gamma \ln C_t + (1-\gamma)\ln(1-N_t) & \text{if } \sigma = 1, \end{array} \tag{40}$$

where $\sigma > 0$ and $0 < \gamma \leq 1$.

The production function is given by

$$F\left(N_t, K_t\right) = N_t^a K_t^b, \tag{41}$$

where $0 \leq a, b \leq 1$. This specification nests the linear technology assumed by Christiano et al. (2011) as a special case in which $a = 1$ and $b = 0$. The expressions of the composite parameters Φ_r, Φ_g, Φ_k, Θ_g, Θ_k, and Θ_c implied by the functional forms assumed above are summarized in Table 1 (see Appendix A.4 for the derivation).

Table 1: Expressions of the composite parameters

Φ_r	Φ_g	Φ_k	Θ_c	Θ_g	Θ_k
1	$\frac{\gamma(\sigma-1)}{1-\bar{g}}$	$\frac{b\gamma(\sigma-1)}{1-\bar{g}}$	$\frac{1}{1-\gamma}$	$\frac{1}{1-\bar{g}}\frac{\gamma}{1-\gamma}$	$b\left(1+\frac{1}{1-\bar{g}}\frac{\gamma}{1-\gamma}\right)$

We closely follow Christiano et al. (2011)'s calibration. In particular, we use their values for β, γ, a, κ, ϕ_π, and ρ, as well as for the steady-state ratio of government spending to output, $\bar{g}$. However, we set $\sigma = 1$ to ensure that $U_{CN}(.) = 0$, as assumed in the simple case discussed above.[10] We also

[10] Christiano et al. (2011) set $\sigma = 2$ in their benchmark calibration. See Bilbiie (2011) and Monacelli & Perotti

need to assign values to two additional parameters that are absent from Christiano et al. (2011)'s model: the steady-state share of public investment in total public spending, $\bar{\alpha}$, and the elasticity of output with respect to public capital, b. The former can be approximated by the historical average ratio of public investment to total public spending, which is roughly 0.23 in the U.S., so we set $\bar{\alpha}$ accordingly. The latter is less straightforward to parameterize as available empirical estimates of b vary considerably depending on the methodology and sample period considered. Using meta-regression analysis of existing empirical studies, Bom & Ligthart (2013) reach an average estimate of 0.08. This estimate lies between the two values considered by Leeper et al. (2010): 0.05 and 0.1. Therefore, we set $b = 0.08$ in our benchmark calibration. The chosen parameter values are summarized in Table 2.

Table 2: Benchmark parameter values

Discount factor	$\beta = 0.99$
Preference parameter	$\sigma = 1$
Preference parameter	$\gamma = 0.29$
Elasticity of output w.r.t labor	$a = 1$
Elasticity of output w.r.t public capital	$b = 0.08$
Elasticity of inflation w.r.t real marginal cost	$\kappa = 0.03$
Inflation feedback parameter	$\phi_\pi = 1.5$
Autocorrelation of the public spending shock	$\rho = 0.8$
Conditional probability that the discount factor remains high	$p = 0.8$
Steady-state ratio of public spending to output	$\bar{g} = 0.2$
Steady-state ratio of public investment to total public spending	$\bar{\alpha} = 0.23$

Using the values reported in Table 2, we compute the spending multiplier as a function of α in normal times (m) and when the ZLB binds (m^l). The results are shown in Figure 2. The figure also reports the neoclassical multiplier, m^n, defined in (29). Figure 2 confirms that, in normal times, the spending multiplier increases linearly with α, from 0.88 when $\alpha = 0$ to 1.5 when $\alpha = 1$. When the ZLB binds, on the other hand, the multiplier declines linearly from 2.26 when $\alpha = 0$ to -4.2 when $\alpha = 1$. When α exceeds 35 percent, the spending multiplier exceeds 1 in normal times and is negative when the ZLB binds. Interestingly, the value of α for which $m = m^l = m^n = 1$ is equal to 19.4 percent,[11] which is even lower than the steady-state share of public investment in total public expenditures ($\bar{\alpha}$). This suggests that the supply-side effects of public spending can be substantial

(2008) for an analysis of the effects of public spending under the assumption of non-separability between consumption and leisure.

[11]This is the value of α such that the aggregate supply and aggregate demand effects of public spending exactly offset each other ($\Theta_g = \alpha\tilde{\delta}\Theta_k$), leaving private spending and inflation unchanged.

even when a modest fraction of productive public spending affects production contemporaneously.

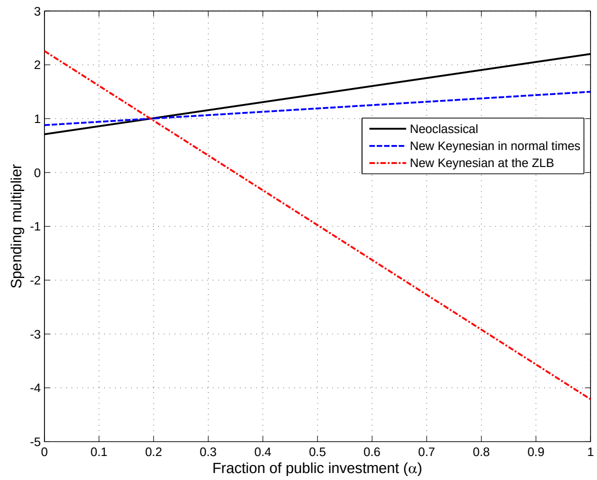

Figure 2: Spending multiplier as a function of the fraction of public investment in a model with no time-to-build and full depreciation of public capital.

It is important to emphasize that the assumption of additive separability is only made for analytical tractability and is not essential for the results outlined so far. With $\sigma = 2$, we obtain very similar patterns for m, m^l and m^n to those depicted in Figure 2. Of course, the numerical values of the multiplier will be slightly different,[12] but the general message remains.[13]

4 Public Investment, Time to Build, and the Spending Multiplier

The simplifying assumptions considered in the previous section, namely that public investment becomes immediately productive and public capital fully depreciates after one period, albeit unrealistic, are nonetheless insightful about the role of investment in this economy. Under these assumptions, the supply-side effect of public spending is largest on impact (i.e., at the time of the shock), thus implying that the spending multiplier at the ZLB may become small and even negative when the fraction of public investment in a stimulus plan is sufficiently large.

In this section, we relax these two assumptions and consider a more general version of the model with time to build ($T \geq 1$) and an empirically plausible value of the depreciation rate of public capital, δ. As we did in section 3, we first study the effects of public capital in normal times before turning to those occurring when the ZLB binds.

[12] In particular, the values of m and m^l obtained when $\alpha = 0$ (i.e., m_u and m_u^l, respectively) are identical to those reported by Christiano et al. (2011) (i.e., 1.05 and 3.7, respectively.)

[13] These results are not reported but are available upon request.

4.1 The spending multiplier in normal times

In normal times, the model is given by equation (17)–(20), with $i_t = r_\beta + \phi_\pi \pi_t$. Under the assumption that $\phi_\pi > 1$, and given the process (13), the unique linear rational expectation solution of the system above is given by

$$c_t = \vartheta_k k_{t-1} + \sum_{\tau=0}^{T} \vartheta_g^\tau g_{t-\tau}, \tag{42}$$

$$\pi_t = \zeta_k k_{t-1} + \sum_{\tau=0}^{T} \zeta_g^\tau g_{t-\tau}, \tag{43}$$

where the coefficients ϑ_k, ζ_k, ϑ_g^τ, and ζ_g^τ $(\tau = 0, ..., T)$ are given in Appendix A.5.1 . The impact multiplier, m^0, is then given by

$$m^0 = 1 + (1 - \bar{g})\,\vartheta_g^0.$$

In order to study the way in which m^0 varies with the share of public investment, α, and the time-to-build delay, T, it is easier to consider again the case with additively separable preferences ($U_{CN}(.) = 0$). Figure 3 shows m^0 as a function of α for $T = 1$, 4, 16. The reported values for m^0 are computed using the parameter values presented in Table 2 and a value of 0.02 for the depreciation rate of public capital, δ (as in Leeper et al. (2010)). The case $T = 1$ corresponds to the one-quarter delay assumed in Baxter & King (1993) and Drautzburg & Uhlig (2013). The case $T = 4$ corresponds to quick-to-implement investment projects like minor maintenance of public infrastructure. Finally, the case $T = 16$ corresponds to investment projects that require a long time to complete, like major repairs or construction of new buildings and highways.

The first message that Figure 3 conveys is that the spending multiplier is monotonically increasing in α when the time to build is short or moderate, and monotonically decreasing in α when the time to build is long. To understand the intuition behind this result, it is useful to recall that the size of the multiplier depends on the response of consumption, which in our sticky-price economy depends on the deviation of the long-term ex ante real interest rate from its steady-state value, $\hat{r}_t^{LT} \equiv E_t \sum_{\tau=0}^{\infty} (i_{t+\tau} - \pi_{t+1+\tau} - r_\beta)$.[14] The latter is in turn determined by the entire path of (expected) inflation and ultimately by the dynamic response of real marginal cost (or, alternatively,

[14] Iterating equation (21) forward yields

$$c_t = \lim_{j\to\infty} E_t c_{t+j} - \Phi_r \hat{r}_t^{LT}.$$

In response to a transitory shock, $\lim_{j\to\infty} E_t c_{t+j} = 0$ and current consumption moves in an opposite direction to $\hat{r}_t^{LT}$ (since $\Phi_r > 0$).

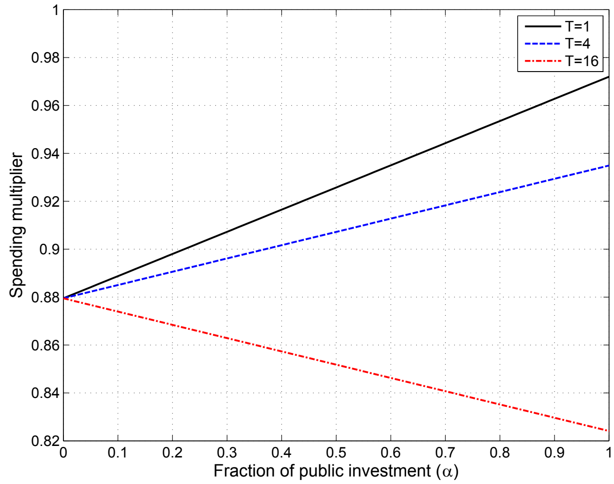

Figure 3: Spending multiplier as a function of the fraction of public investment and the time-to-build delay in normal times.

the output gap).

In addition to its direct effect on current and future aggregate demand (since the shock is serially correlated), an increase in public investment has similar effects to those of an anticipated technology shock. On the one hand, it exerts downward pressure on real marginal cost and inflation once public capital becomes productive (supply-side effect). On the other hand, by increasing households' expected wealth, it raises aggregate demand, real marginal cost, and inflation even before public capital becomes productive.

Figure 4 depicts the dynamic effects of a positive government spending shock for $\alpha = 0$, $\{\alpha = 1,\ T = 4\}$, and $\{\alpha = 1,\ T = 16\}$. Owing to the wealth effect just described, inflation rises more in the short run in response to an increase in public investment ($\alpha = 1$) than in response to an increase in public consumption ($\alpha = 0$). With a time-to-build delay of 4 quarters, the deflationary effect associated with the increase in labor productivity occurs rapidly so that inflation is short-lived. As a result, the initial response of the long-term real interest rate is smaller—implying a larger consumption response and thus a larger spending multiplier—than when $\alpha = 0$. In turn, this explains why the multiplier is increasing in α when the time to build is short. When the time to build is 16 quarters, the supply-side effect of public investment comes into play far in the future so that the inflationary pressure lasts for a long period of time. In response to the public spending shock, the long-term real interest rate thus rises more and consumption falls more—implying a smaller multiplier—than when $\alpha = 0$. This also explains why the multiplier decreases with α in this case.[15]

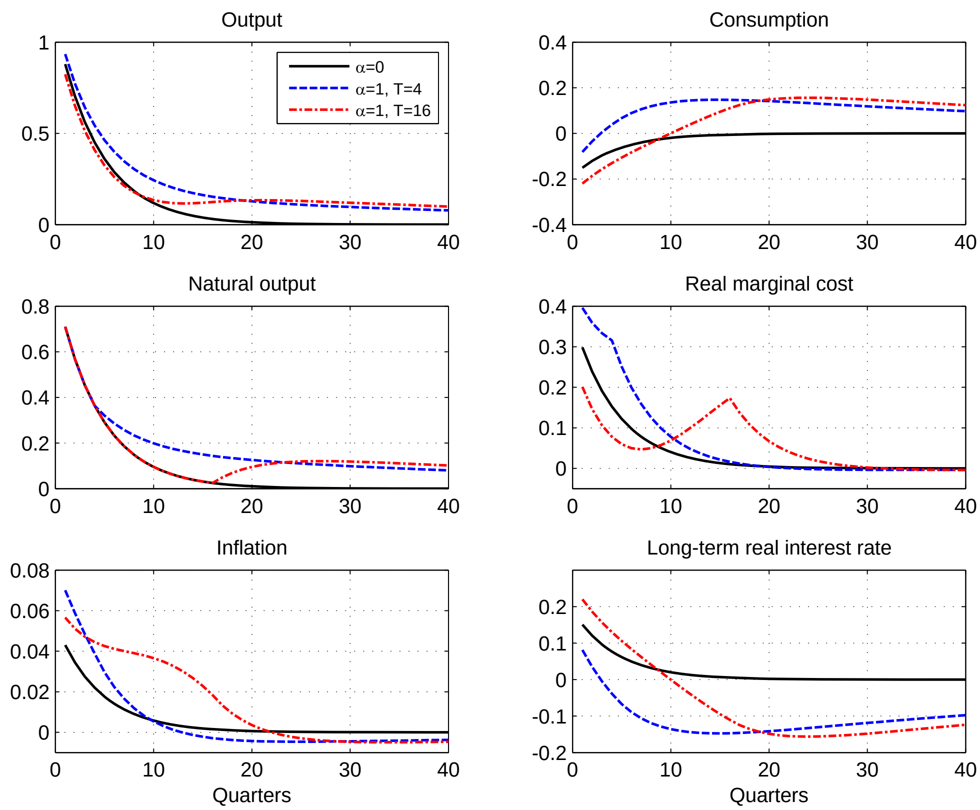

Figure 4: Dynamic effects of an increase in public spending equal to 1 percent of steady-state output in normal times.

Note: All responses are expressed as percentage variations from steady state, except the responses of inflation and the long-term real interest, which are expressed as percentage points deviations from steady state.

The second observation that emerges from Figure 3 is that the spending multiplier is always lower than unity. Thus, under plausible parameter values, the spending multiplier in normal times remains numerically close to that predicted by a standard model in which all public spending is unproductive.

[15] Barsky & Sims (2011) provide evidence that a favorable news shock about future technology raises private consumption and lowers inflation today. To the extent that anticipated technological improvements are expected to occur in the near future, this evidence is consistent with our results regarding the supply-side effects of quick-to-implement public investment projects. For a recent survey of the literature on news-driven business cycles see Beaudry & Portier (2014).

4.2 The spending multiplier in a liquidity trap

When the ZLB binds, the model is given by equation (17)–(20), with $i_t = 0$. As before, we assume that the discount factor remains high, at β^l, with probability p and returns permanently to its steady-state value, β, with probability $1-p$. We also assume that $g_{t+1} = g_t > 0$ if the ZLB is still binding in $t+1$ and $g_{t+1} = 0$ otherwise. Finally, we assume that the ZLB will bind as long as the discount factor is high. Note that the economy does not immediately return to steady state when the ZLB ceases to bind, as the stock of public capital continues to adjust and converges to its steady state only gradually. Conditional on the economy being at the ZLB, the expected value of a given variable in $t+1$ is an average of its possible values in $t+1$ inside and outside the liquidity trap, weighted by the respective probabilities of being in these two states. The linear rational expectation solution of the model when the ZLB binds is given by

$$c_t = \vartheta_i^l r_\beta^l + \vartheta_k^l k_{t-1} + \sum_{\tau=0}^{T} \vartheta_g^{l,\tau} g_{t-\tau}, \tag{44}$$

$$\pi_t = \zeta_i^l r_\beta^l + \zeta_k^l k_{t-1} + \sum_{\tau=0}^{T} \zeta_g^{l,\tau} g_{t-\tau}, \tag{45}$$

where the coefficients ϑ_k^l, ζ_k^l, $\vartheta_g^{l,\tau}$, and $\zeta_g^{l,\tau}$ $(\tau = 0, ..., T)$ are given in Appendix A.5.2 . The impact spending multiplier when the ZLB binds, $m^{l,0}$, is

$$m^{l,0} = 1 + (1-\bar{g})\,\vartheta_g^{l,0}.$$

Focusing again on the case with additively separable preferences ($U_{CN} = 0$), we compute $m^{l,0}$ as a function of α for $T = 1$, 4, 16 using the parameter values in Table 2 and $\delta = 0.02$. The results, depicted in Figure 5, indicate that the spending multiplier is decreasing in α when the time to build is very short (1 quarter), but increases with α when the time to build is moderate or long.

Intuitively, since the nominal interest rate is stuck at zero, the inflationary effect of an increase in public spending translates into a fall in the real interest rate, which further increases aggregate demand, thus creating a virtuous circle that results in a large increase in output. When the time to build is sufficiently short, the deflationary (supply-side) effect associated with the increase in public capital comes about quickly after the initial shock and hence remains in effect for a prolonged period of time while the ZLB is still binding. This effect greatly attenuates the inflationary pressure stemming from the increase in aggregate demand and from the increase in households' expected wealth. As a result, the long-term real interest rate falls less compared with the case $\alpha = 0$, which

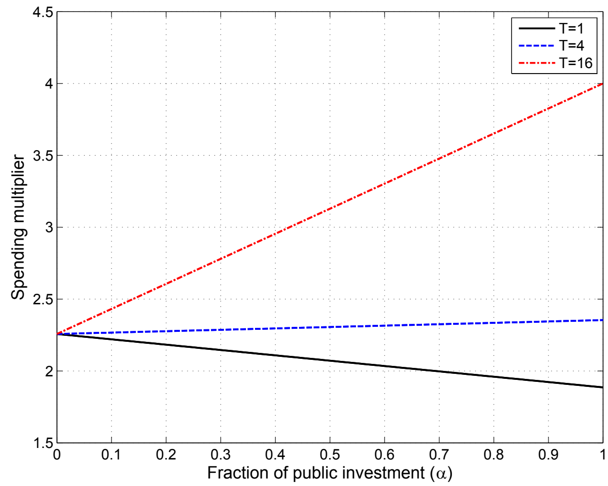

Figure 5: Spending multiplier as a function of the fraction of public investment and the time-to-build delay in a liquidity trap.

in turn implies that the multiplier is a decreasing function of α.

As the time to build increases, the deflationary effect brought about by the increase in public capital is further delayed, becoming increasingly more likely to occur after the economy has escaped from the liquidity trap. At the same time, the positive wealth effect associated with the expected increase in labor productivity amplifies inflation while the economy is still in the liquidity trap. When the time to build is sufficiently long, this implies a larger fall in the long-term real interest rate compared with the case $\alpha = 0$, which also means that the multiplier is increasing in α.

Figure 5 also shows that the spending multiplier can be substantially large at the ZLB when a large fraction of public spending is invested in public capital and when the time-to-build delay is relatively long. With a time to build of 16 quarters, the multiplier associated with public investment is five times larger than in normal times and nearly twice as large as the multiplier associated with public consumption.

4.3 Sensitivity analysis

In this section, we study the sensitivity of our results to perturbations to the benchmark calibration. To save space, we focus on the case where the ZLB is binding, and we restrict our attention to three key parameters: the substitutability between consumption and leisure in utility, the degree of price rigidity, and the probability that the discount factor remains high. The results are shown in Figure 6.

Consider first the substitutability between consumption and leisure in preferences. A (strictly positive) value of σ that is different from 1 implies that preferences are non separable in consumption and leisure (i.e., $U_{CN} \neq 0$). We study two alternative cases: $\sigma = 0.5$ and $\sigma = 2$. Consumption and leisure are complements in the former case and substitutes in the latter.[16] Under substitutability, the increase in labor triggered by higher public spending reduces leisure and lowers the marginal utility of consumption, *ceteris paribus*. This channel reinforces the negative wealth effect associated with the increase in taxes needed to finance the increase in government spending. Thus, consumption falls more in normal times and rises less when the ZLB binds for any given value of α and T (see the upper left panel of Figure 6). The opposite result holds when consumption and leisure are complements (see the upper right panel of Figure 6). Because substitutability/complementarity acts merely as shifter in the Euler equation (see equation 17), it implies that the shape of the relationship between the multiplier and α and T is preserved when one considers alternative values of σ.

Consider next the degree of price rigidity. As prices become more rigid, the slope of the NKPC, κ, becomes smaller. We consider the following two values: $\kappa = 0.02$ and $\kappa = 0.035$, which are, respectively, lower and higher than the benchmark value of 0.03. Price rigidity affects the spending multiplier in two different ways when the economy hits the ZLB. On the one hand, as prices become more rigid, inflation rises less in response to the increase in public spending. This tends to lower the multiplier for any given value of α and T, as can be seen in the middle panels of Figure 6. This result echoes the general findings of Bhattarai et al. (2014), who show that if monetary policy is not responsive enough to inflation, higher price flexibility will exacerbate the response of output and inflation to demand shocks, even if the economy is not at the ZLB.[17,18] On the other hand, price rigidity tends to increase the relative importance of the inflationary effect of public investment. Consequently, for sufficiently rigid prices, the spending multiplier becomes an increasing function of α even for a time-to-build delay that is extremely short (see the middle left panel of Figure 6). Conversely, when price rigidity is sufficiently small, the multiplier becomes a decreasing function of α even when the time to build is moderately long (see the middle right panel of Figure 6).

[16] Recall that $U_{CN} = (1-\gamma)(\sigma-1)\frac{U_C}{1-N}$.

[17] Bhattarai et al. (2014) consider a general Taylor rule whereby the nominal interest rate responds to the output gap in addition to deviations of inflation from its target. In their case, the lack of responsiveness of monetary policy is captured by the condition $\phi_\pi - \rho < 0$, which still yields a determinate solution if monetary policy is sufficiently responsive to the output gap. In our simple model, however, this condition leads to indeterminacy.

[18] It may seem that price flexibility and the time-to-build delay affect the multiplier in a similar way. In fact, price flexibility amplifies the response of inflation while lowering its persistence, whereas the time to build determines the relative durations of the inflationary and deflationary episodes in response to a public investment shock. As we will see in section 4.4.2, the relationship between the multiplier and the time-to-build delay may become non-monotonic.

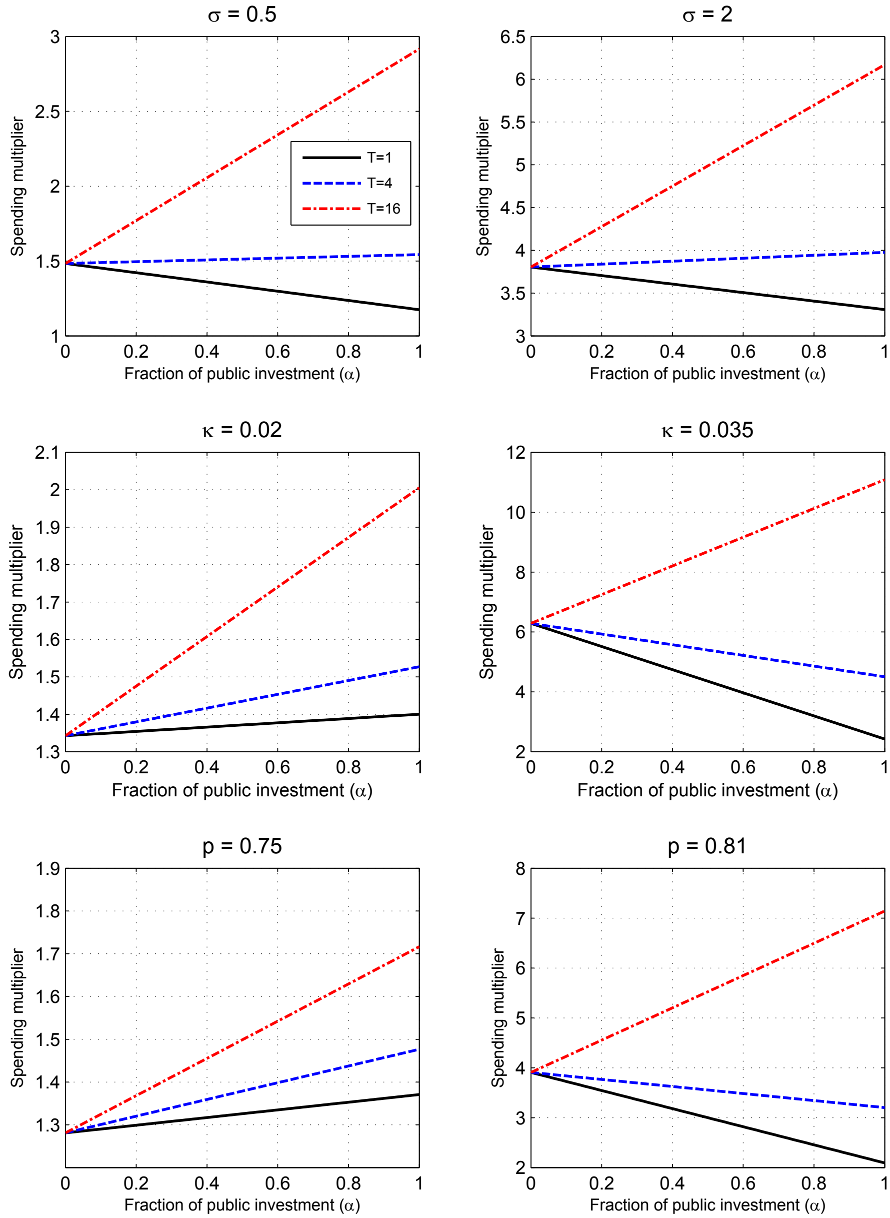

Figure 6: Spending multiplier as a function of the fraction of public investment and the time-to-build delay in a liquidity trap: Sensitivity analysis.

Finally, consider the conditional probability that the stochastic factor remains high, p. The bottom panels of Figure 6 show that the multiplier at the ZLB increases with p for any given value of α and T, thus corroborating the well-established result that the expected duration of the liquidity trap affects positively the size of the multiplier (see, for example, Christiano et al. (2011)). But our results also indicate that when p is sufficiently small, the multiplier becomes an increasing function of α even when the time to build is as short as 1 quarter. Conversely, for sufficiently large values of p, the multiplier falls with α even when the time to build is 4 quarters. Intuitively, when the expected duration of the ZLB state is relatively short, the deflationary effects will predominantly occur when the ZLB is no longer binding. In this case, the multiplier associated with public investment will be larger than that associated with public consumption even if the time to build is extremely short; hence the positive relationship between the multiplier and α. The opposite scenario holds when the expected duration of the ZLB episode is relatively long.

4.4 Robustness analysis

The results discussed so far indicate that when the economy is at the ZLB, the spending multiplier increases with the fraction of public investment in a stimulus plan when the time-to-build is relatively long. Under our benchmark calibration and a time to build of 16 quarters, the multiplier associated with public investment is roughly five times larger at the ZLB than in normal times and nearly twice as large as the multiplier associated with public consumption. Recall, however, that these results are obtained under the specific assumptions that (*i*) the shock that makes the ZLB bind follows a two-state Markov process given by (33), (*ii*) public spending is higher than in steady state as long as the ZLB is binding, and (*iii*) the ZLB binds as long as the shock to the discount factor is in effect.

While these assumptions allow us to obtain an analytical solution of the (linearized) model, they are however restrictive in the following ways. First, the random duration of the liquidity trap only allows one to compute the impact multiplier but not a dynamic or a cumulative multiplier. The latter two variants are particularly relevant in the context of our model given the inherent persistence brought about by the time-to-build requirement. Second, assumption (*ii*) is restrictive in that it prevents us from considering the realistic possibility that spending outlays occur gradually during the course of an investment project, which may exceed the expected duration of the liquidity trap. Finally, assumption (*iii*) precludes endogenous exit from the liquidity trap, that is, an outcome in which the ZLB ceases to bind as a result of higher public spending despite the fact that the shock that caused the ZLB to bind is still in effect. This outcome is more likely to occur the larger the fraction of public investment and the longer the time-to-build delays.

To study these issues, we relax the assumption that the discount factor and public spending follow a two-state Markov process and assume instead that both variables are governed by an AR(1) process. To solve the model under this alternative assumption, we use the piecewise linear algorithm developed by Guerrieri & Iacoviello (2014), which allows to solve dynamic stochastic general-equilibrium (DSGE) models with occasionally binding constraints by computing two linear decision rules, one for the regime in which the constraint binds, and one for the unconstrained regime.

4.4.1 Impact versus present-value multiplier

In addition to being subject to potentially significant time-to-build delays, public capital is a slow-moving variable that has a long lasting effect on economic activity. For this reason, the impact multiplier—on which we have focused so far—may not be a sufficient statistic to gauge the effects of public spending shocks on output. An alternative measure that better captures the dynamic nature of these effects is the present-value multiplier defined as

$$pvm = \frac{\sum_{t=0}^{h}(1+r_\beta)^{-1}\Delta y_t}{\sum_{t=0}^{h}(1+r_\beta)^{-1}\Delta g_t}, \tag{46}$$

where Δx_t is the difference between the response of variable x when the economy is hit both by a preference and a government spending shock and the response of x when there is only a preference shock. For simplicity, we assume that the dynamic responses of output and government spending are discounted at the constant rate r_β.

Figure 7 reports both the impact and the present-value multipliers at the ZLB as functions of α for T = 1, 4, 16. In order to maintain comparability with the results discussed in Section 4.2 regarding the role of public investment and the time to build, we choose the autocorrelation coefficients of the AR(1) processes and the size of the preference shock such that the impact multiplier associated with public consumption (i.e., $\alpha = 0$) at the ZLB is equal to that obtained in the analytical solution (that is, 2.3). We consider a small public spending shock (1 percent of steady-state output) such that there is no endogenous exit from the ZLB state for any given value of α and T. In computing the present-value multiplier, we set $h = 1000$.

The first observation that can be drawn from the left panel of Figure 7 is that the main findings based on the two-state Markov process for the shocks carry over to the specification with AR(1)

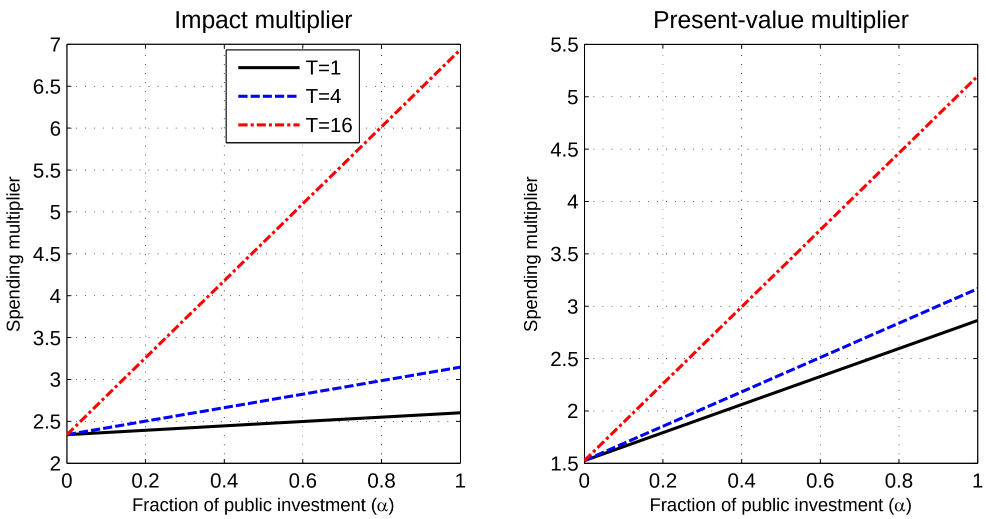

Figure 7: Impact versus present-value spending multiplier in a liquidity trap.

processes: The multiplier increases monotonically with α when the time to build is relatively long, and becomes substantially large when a significant fraction of public spending is allocated to investment. Qualitatively, the only difference with respect to the results shown in Figure 5 is that the spending multiplier becomes increasing in α even when the time to build is 1 quarter. Quantitatively, the impact multiplier is larger under the autoregressive structure of the shocks than under the Markov process for any given value of α and T.

The right panel of Figure 7 shows that the present-value multiplier at the ZLB displays the same pattern as the impact multiplier, taking substantially larger values when all public spending is in the form of investment and the time to build is long than when public spending is entirely wasteful. For instance, when $T = 16$, the present-value multiplier associated with public investment is more than 3 times larger than the multiplier associated with public consumption.

4.4.2 Timing of public investment outlays

Typically, public investment outlays occur gradually over time after the spending has been authorized.[19] To capture this feature of the data, we amend the model presented in Section 2 as follows. Let A_t be the amount authorized in period t and assume, as before, that a fraction α of that amount is allocated to investment projects while the remaining fraction is devoted to public consumption.

[19] See Leeper et al. (2010) and Leduc & Wilson (2013) for a detailed discussion of the legislative process governing public investment decisions in the U.S.

Assume further that only investment outlays occur gradually, and denote by ω_j, $j = 0, 1..., T-1$, the spend-out rates, i.e., the rates at which the authorizations are converted into outlays.

In the linearized version of this extended model, the accumulation equation of public capital becomes

$$k_t = (1-\delta)\, k_{t-1} + \alpha\tilde{\delta} a_{t-T},$$

where a_t (the deviation of A_t from its steady-state value, as a percentage of steady-state output) is exogenously given and follows the AR(1) process

$$a_t = \rho_a a_{t-1} + \epsilon_t^a.$$

For national accounting purposes, government spending is now defined as

$$g_t = (1-\alpha)\, a_t + \alpha \sum_{j=0}^{T-1} \omega_j a_{t-j},$$

where $\sum_{j=0}^{T-1} \omega_j = 1$ and where the second expression on the right hand side of the equation indicates that public investment is the sum of current and past outlays.

In this model, the relevant (and economically meaningful) multiplier is the dollar change in output that results from a dollar change in the authorization, i.e., $dY_t/dA_t = y_t/a_t$.[20] To compute the spending multiplier, we set ρ_a to the same value as ρ_g in the version of the model with outright outlays, and calibrate the spend-out rates as follows. For $T = 1$, we set $\omega_0 = 1$; for $T = 4$, we set $\omega_0 = 0$ and $\omega_1 = \omega_1 = \omega_3 = 1/3$; and for $T = 16$, we set $\omega_0 = 0$, $\omega_1 = \omega_2 = \omega_3 = 0.25/3$, and $\omega_4 = ... = \omega_{15} = 0.75/12$. In other words, when $T = 4$ or $T = 16$, authorizations start to show up as outlays with a lag of one quarter (as in Leeper et al. (2010)). When $T = 16$, the authorized amount is spread equally over the 4 years. Note that this version of the model nests the one presented in Section 2 as a special case in which $\omega_0 = 1$ and $\omega_1 = ... = \omega_{15} = 0$.

Figure 8 shows the impact and present-value multipliers when allowing for gradual investment outlays. For $T = 1$, the results are obviously identical to those depicted in Figure 7. For $T = 4$ and $T = 16$, on the other hand, the impact multiplier is lower than that depicted in Figure 7 for any strictly positive value of α. Intuitively, when the outlays occur gradually, the inflationary effect of public spending is spread over time, implying a lower impact multiplier at the ZLB compared with a scenario in which the funds are disbursed immediately after being authorized. When $T = 4$, because

[20] Indeed, assuming $\alpha = 1$ and $\omega_0 = 0$ implies that the impact multiplier with respect to current public spending is infinite.

public capital becomes productive rather rapidly, the gradual nature of outlays plays a relatively large role in mitigating short-term inflation, actually to the point that the impact multiplier becomes a decreasing function of α. When $T = 16$, on the other hand, the impact multiplier continues to be an increasing function of α and to take substantially large values when α is large.

The right panel of Figure 8 depicts the results for the present-value multiplier. These results are virtually identical to those shown in the corresponding panel of Figure 7. This similarity is not surprising given that we define the multiplier with respect to authorizations rather than to current spending.

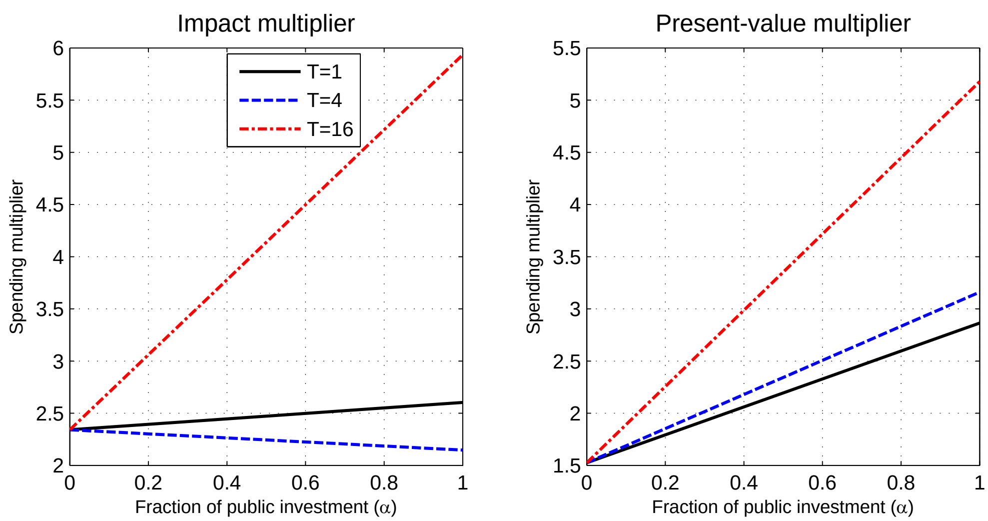

Figure 8: Spending multiplier with gradual investment outlays in a liquidity trap.

4.4.3 Endogenous exit

So far, the magnitude of the public spending shock that we have considered was small enough not to affect the duration of the ZLB spell, that is, we did not allow for endogenous exit from the liquidity trap. An important result established by the literature on the effects of public spending at the ZLB, however, is that fiscal stimuli that lower the duration of the liquidity trap will be characterized by relatively small multipliers (e.g., Fernández-Villaverde et al. (2012) and Erceg & Lindé (2014)). This is because the portion of the stimulus that takes place outside the ZLB generates inflation once the ZLB ceases to bind, which mitigates the fall in the long-term real interest rate and leads to a smaller increase in private spending while the economy is still at the ZLB.

To allow for the possibility of endogenous exit, we consider public spending shocks of larger magnitudes: 2% and 4% of steady-state output. None of these magnitudes generates endogenous exit when the time to build is 1 or 4 quarters. Therefore, we only report the results for $T = 16$, which are shown in Figure 9. Starting with the impact multiplier, the left panel of the figure reveals that exit occurs at decreasing values of α as the shock becomes larger. When the increase in public spending is equal to 2% of steady-state output, the economy leaves the liquidity trap one quarter earlier for $\alpha \geq 0.4$. When the shock is twice as large, the economy exits the ZLB one quarter earlier for $0.05 \leq \alpha < 0.75$ and two quarters earlier for $\alpha \geq 0.75$. In conformity with the result discussed above, endogenous exit reduces the value of the impact multiplier, *ceteris paribus*. Nonetheless, even with a shock that is abnormally large (4% of steady-state output), the reduction in the size of the spending multiplier is limited and the latter remains quantitatively large relative to the case of wasteful spending and relative to normal times when a significant fraction of public spending is in the form of investment. The same observation applies to the present-value multiplier, shown in the right panel of Figure 9.

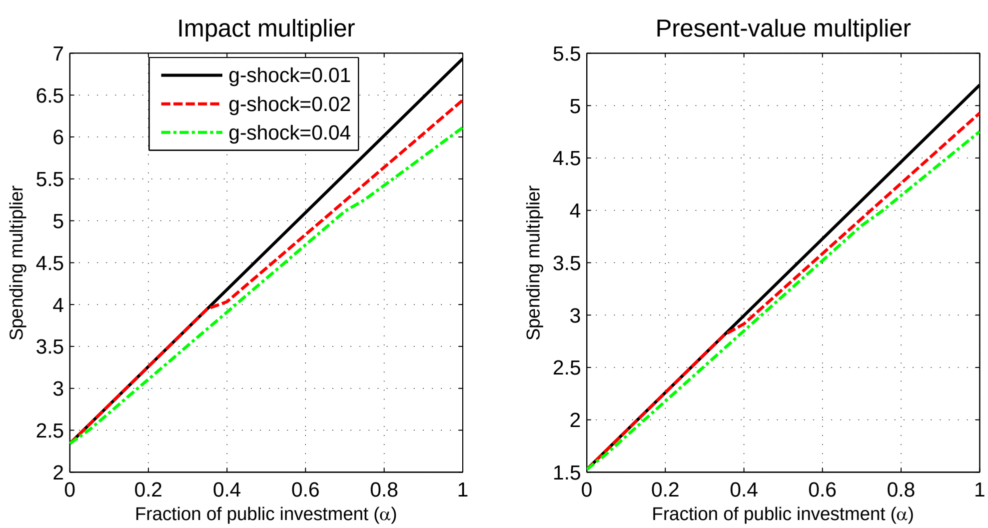

Figure 9: Spending multiplier when there is endogenous exit from the liquidity trap ($T = 16$).

5 Public Investment and Time to Build in a Medium-Scale DSGE Model

Owing to its simplicity, the model presented in Section 2 has been useful to convey the main insights about the macroeconomic effects of public investment both in normal times and when the ZLB binds. A serious quantitative evaluation of these effects, however, requires a richer model that incorporates empirically relevant features that can help account for business-cycle fluctuations, such as the accumulation of private capital, variable capacity of utilization, and labor-market frictions. There are by now a number of DSGE models that perform remarkably well in accounting for aggregate fluctuations. We choose to work with the model recently developed by Christiano et al. (2013), which we extend to allow for the accumulation of public capital and for time-to-build delays. We use the model to evaluate the multiplier associated with the spending component of the 2009's ARRA.

5.1 Overview of the model

We briefly summarize the model here and refer the reader to Christiano et al. (2013) for a complete exposition. The economy is populated by a representative household whose preferences exhibit habit formation with respect to consumption. The household accumulates capital subject to investment-adjustment costs and chooses the utilization rate when renting capital to firms. Working members of the household earn a real wage while the remaining members receive unemployment benefits.

The final good, which is used for consumption and investment purposes, is produced by a competitive representative firm using differentiated inputs produced by monopolistically competitive retailers. Retailers set their prices à la Calvo. Their production technology depends on private capital, public capital, and an intermediate good produced by wholesalers using labor. To hire new workers, wholesale producers post vacancies at zero cost. At the beginning of the period, each worker engages in bilateral bargaining with a representative of the firm, and the equilibrium real wage is the outcome of an alternating offer bargaining process along the lines of Hall & Milgrom (2008). At the end of each period, a constant fraction of employed workers lose their jobs.

The government levies lump-sum taxes to finance its purchases and the unemployment benefits paid to unemployed workers. As in the simple model presented in Section 2, we assume that a fraction α of government spending on goods and services is allocated to investment goods, that public investment projects are subject to time-to-build delays, and that public investment outlays occur gradually over time. Finally, the monetary authority follows a Taylor rule whereby deviations of the nominal interest rate from its steady-state value respond to deviations of inflation, output, and the lagged nominal interest rate from their respective steady-state values. Whenever this rule calls for a negative nominal interest rate, the latter is set to zero.

5.2 Calibration

To assign values to the model parameters, Christiano et al. (2013) fix a subset of them à priori and estimate the rest via a Bayesian minimum-distance strategy using U.S. data. We use their calibrated/estimated values. For the new parameters that we add to the model, namely, the elasticity of output with respect to public capital, b, the fraction of public investment in government spending, α, the length of time-to-build delay, T, and the spend-out rates, ω_i, we proceed as follows. For b, we choose the same value as in Section 3.3, that is, $b = 0.08$. For α, T, and ω_i, we turn to the composition of the spending component of the ARRA, the nature of public investment projects that it comprised, and their financing scheme.

The ARRA allocated roughly \$350 billion to non-transfer spending, i.e., spending on goods and services. Drautzburg & Uhlig (2013) estimate that roughly 40% of this amount was devoted to public investment. Accordingly, we set $\alpha = 0.4$. The bulk of public investment projects financed by the ARRA were infrastructure projects, including efficient and renewable energy projects (\$57 billion), highway construction (\$27.5 billion), transportation infrastructure (\$18 billion), and building government facilities (\$5.5 billion). Infrastructure projects typically involve long time-to-build delays. For instance, citing the Federal Highway Administration (FHWA), the Government Accountability Office (GAO) reports that most highway construction projects in the U.S. take between 4 and 6 years to complete, though the construction of major new highways may take up to 19 years from planning to completion.[21] Non-infrastructure projects may not entail such a long time to build, but they still require several quarters to complete.[22] Based on these arguments, we set $T = 16$ quarters.[23]

According to the Congressional Budget Office, by the end of 2011, more than 90% of public

[21]Source: http://www.gao.gov/assets/110/109588.pdf

[22]Some studies attempted to estimate the time-to-build lags of private investment projects. For instance, Krainer (1968) finds that investment projects in the U.S. automobile industry take 2 to 3 years from approval to completion. Montgomery (1995) estimates that U.S. private non-residential construction projects have a mean time-to-build delay of 17 months. Koeva (1995) reports that the average construction lead time for new plants is around 2 years in most U.S. industries. While we are not aware of studies that estimate the time-to-build delays associated with public investment projects, it is widely believed that these projects take significantly longer to implement than those in the private sector.

[23]In their study of the effects of public infrastructure, Leduc & Wilson (2013) also assume a time-to-build delay of 4 years.

spending under the ARRA was obligated. Outlays, on the other hand, spanned a longer period of time, as the awarded amounts were paid out to recipients gradually. To capture these features, we approximate the path of spending obligations by an AR(1) process with an autocorrelation coefficient of 0.8, and set the spend-out rates as in section 4.4.2.[24] This calibration of the spend-out rates assumes that obligations start to show up as outlays with a lag of one quarter, and that one fourth of the obligated amount is outlaid in each year.

5.3 Quantifying the spending multiplier of the ARRA

To compute the spending multiplier associated with the non-transfer component of the ARRA, we consider the following experiment. We assume that the economy is initially in steady state. Then, a sequence of positive shocks to the discount factor hits, driving the nominal interest rate down to zero. We choose the size and persistence of the preference shock such that the resulting trough in output is 5.65% below the pre-shock level, exactly matching the observed decline in U.S. real per capita GDP between 2007Q4 (peak) and 2009Q2 (trough).[25] This generates a liquidity trap of 14 quarters. Note that because the model economy involves a number of slow-moving state variables, its dynamic response to the discount-factor shock is such that it reaches the ZLB 3 quarters later. As soon as the ZLB binds, the government authorizes an increase in public spending equal to 1% of steady-state output. The composition and persistence of this stimulus mimic those of the non-transfer component of the ARRA. For simplicity, we henceforth refer to it as the ARRA.

Figure 10 depicts the dynamic response of the economy to the ARRA. The response of output is scaled by the initial shock so that it can be interpreted as a dynamic multiplier. The response of private consumption, private investment, and real marginal cost are expressed as percentage deviations from the benchmark scenario in which only the discount-factor shock is in effect, while the response of inflation and the real interest rate are expressed as percentage point deviations from the benchmark scenario. The figure also shows the results of a counterfactual experiment in which the stimulus is exclusively composed of public consumption (i.e., $\alpha = 0$).

Starting with the latter case, the figure shows that the increase in public consumption generates a hump-shaped output response that reaches its peak 2 quarters after the shock. At the peak, the spending multiplier is equal to 1.21. The fact that the multiplier is larger than 1 echoes the by-now well-established result established in the literature on the effects of public spending when the ZLB

[24]More precisely, we set $\omega_0 = 0$ and $\omega_1 = \omega_2 = \omega_3 = 0.25/3$ for the first year, and $\omega_4 = \cdots = \omega_{15} = 0.75/12$ for the subsequent 3 years.

[25]Source: `http://research.stlouisfed.org/fred2/series/A939RX0Q048SBEA`

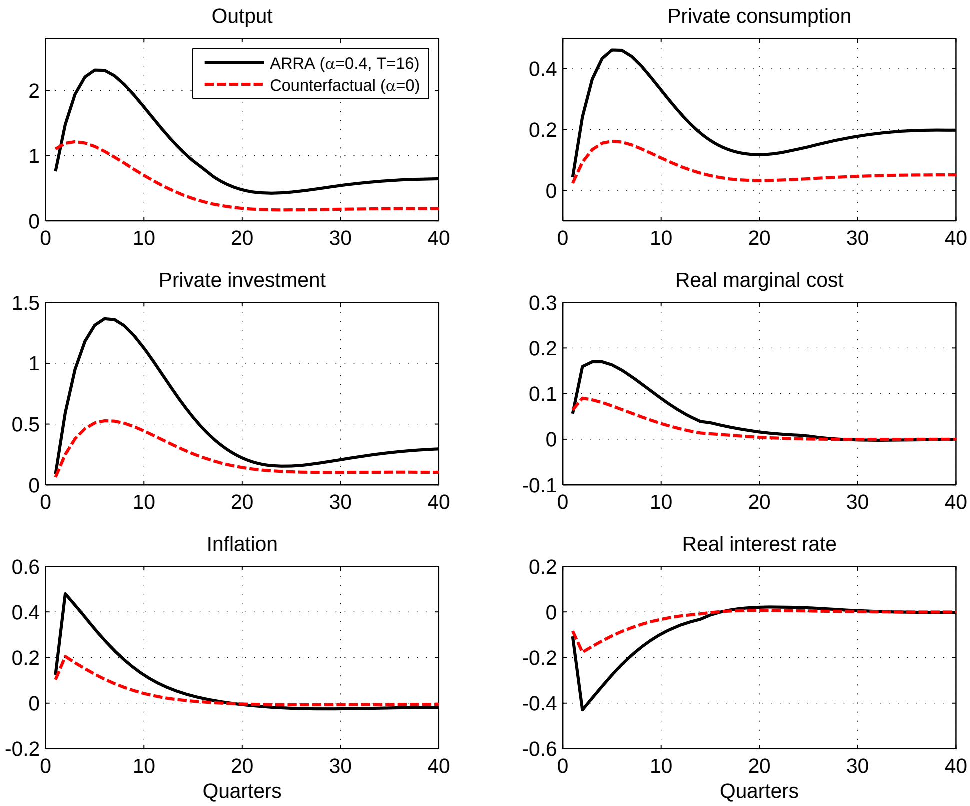

Figure 10: Dynamic effects of the ARRA.

is binding. Because the nominal interest rate cannot adjust, the inflationary effect of higher public spending lowers the real interest rate, which crowds-in of private consumption and investment and further increases aggregate demand. The fact the multiplier is lower than those reported by earlier studies is due to two factors. First, studies that specify an AR(1) process for the spending shock usually assume that the shock is highly persistent—an assumption that is hard to justify in the context of the ARRA. Second, the presence of real wage rigidity arising from the alternating offer bargaining process tends to dampen the response of inflation to demand shocks, thus lowering the size of the spending multiplier.

Next, consider the dynamic effects of the ARRA, which involves 40% of the authorized spending in the form of public investment and where public investment projects take 16 quarters to complete. The output response is larger in magnitude and exhibits a more pronounced hump compared with the case $\alpha = 0$. The peak multiplier is equal to 2.31, that is, nearly twice as large as the maximum multiplier associated with public consumption. The amplification effect of public investment and the time-to-build delay is even more dramatic on the present-value multiplier, which increases from 1.62 when all spending is assumed to be unproductive to 5.41 under the ARRA.[26] As explained in section 4.2, with long time-to-build delays, the deflationary effects of public investment come into play far in the future, while the expected increase in the marginal productivity of private inputs raises households' expected wealth and contributes to increasing aggregate demand an inflation in the short run. Note that under our calibration, this additional inflation occurs while the ZLB is still binding, which amplifies the multiplier.

To summarize, our analysis suggests that the spending component of ARRA had a large multiplier, whether measured at the peak or in present value, and that failing to account for the composition of the stimulus by counterfactually assuming that it was exclusively composed of public consumption spending would lead one to underestimate the multiplier by roughly 50%.

6 Conclusion

The main lesson from the literature on the effects of fiscal policy in a liquidity trap is that policies that stimulate aggregate demand can have substantially larger effects when the ZLB binds than in normal times. Conversely, policies that raise the natural level of output while the economy is in a liquidity trap may depress economic activity even further (Eggertsson (2011)). Yet, Fernández-Villaverde et al. (2011) and Eggertsson et al. (2014) show that supply side policies that take effect after the ZLB has ceased to bind are desirable because they generates a positive wealth effect that stimulates aggregate demand when the ZLB is still binding. In this paper, we have shown that when the time to build the stock of public capital is sufficiently long, public investment will raise both aggregate demand while the economy is in the liquidity trap, and the natural level of output once the economy has exited the liquidity trap, thereby further fueling aggregate demand in the short run. A corollary is that, in states where the ZLB binds, the spending multiplier will be an increasing function of the share of public investment in a stimulus plan. Because the time to build the stock of public capital is typically long—often spanning several years, our analysis suggests that public investment is a highly effective tool to stimulate the economy at the ZLB.[27]

[26] The present-value multiplier is computed for an horizon of 1000 quarters. The peak and present-value multipliers remain virtually unchanged when we consider $T = 20$ or $T = 24$ (results are available upon request).

[27] Wieland (2013) finds little empirical support for the proposition that supply shocks have contractionary effects when the ZLB is binding. Wieland (2013), Kiley (2014) and Bundick (2014) propose different modifications of the standard new-Keynesian model to rationalize this evidence. Unless the time-to-build delays are implausibly short, our paper shows that public investment can stimulate output even if public capital becomes productive before the

Using an extended version of Christiano et al. (2013)'s medium-scale model to quantify the output effects of the 2009's ARRA, we found that the stimulus had a peak multiplier that exceeds 2. Our results also indicate that failing to account for the actual composition of the stimulus by overlooking its investment component would lead one to severely underestimate the associated multiplier.

Given our findings, one might be led to conclude that stimulus packages should be exclusively targeted towards public investment projects. This conclusion is of course unwarranted. A formal analysis of the normative aspects of fiscal policy—including the optimal allocation of public spending in and out of the ZLB—ought to be welfare-based. We leave this task for future research.

economy exits the liquidity trap. From this perspective, our results are consistent with the evidence reported by Wieland (2013).

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A Appendix

A.1 Summary of the model

The model equilibrium conditions are

$$
\begin{aligned}
Y_t &= C_t + G_t, && \text{(A.1)}\\
\Delta_t Y_t &= F\left(N_t, K_t\right), && \text{(A.2)}\\
\frac{1}{1+i_t} &= \beta_{t+1} E_t \left(\frac{U_C\left(C_{t+1}, N_{t+1}\right)}{U_C\left(C_t, N_t\right)} \frac{P_t}{P_{t+1}} \right), && \text{(A.3)}\\
W_t &= -\frac{U_N\left(C_t, N_t\right)}{U_C\left(C_t, N_t\right)}, && \text{(A.4)}\\
0 &= E_t \sum_{s=0}^{\infty} \phi^s Q_{t,t+s} X_{t,t+s} \left(P_t^o - \mu MC_{t,t+s}\right), && \text{(A.5)}\\
P_t^{1-\theta} &= (1-\phi)(P_t^o)^{1-\theta} + \phi P_{t-1}^{1-\theta}, && \text{(A.6)}\\
K_t &= (1-\delta) K_{t-1} + G_{t-T}^i, && \text{(A.7)}\\
i_t &= \max\left(0; \ln \beta^{-1} + \phi_\pi \ln \frac{P_t}{P_{t-1}}\right), && \text{(A.8)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_t &= J\left(\int J^{-1}\left[\left(\frac{P_t(z)}{P_t}\right)^{-\theta}\right] dz\right),\\
Q_{t,t+s} &= d_{t+s} U_C\left(C_{t+s}, N_{t+s}\right) / d_t U_C\left(C_t, N_t\right),\\
MC_{t,t+s} &= \frac{(1-\tau)\mathcal{W}_{t+s}}{F_N\left(N_{t,t+s}, K_{t+s}\right)}.
\end{aligned}
$$

A.2 Steady state

Using equations (A.1) to (A.8) evaluated at steady state, we obtain a system of three equations that uniquely determine private consumption, hours worked and public capital in steady state. These quantities are all we need to compute the log-linearized version of the model.

$$
\begin{aligned}
C &= \left(1-\bar{g}\right) F\left(N, K\right),\\
-\frac{U_N\left(C, 1-N\right)}{U_C\left(C, 1-N\right)} &= \frac{\theta-1}{\left(1-\tau\right)\theta} F_N\left(N, K\right),\\
K &= \frac{\bar{\alpha}}{\delta} \frac{\bar{g}}{1-\bar{g}} C.
\end{aligned}
$$

A.3 The linearized model

We linearize the model around a deterministic zero-inflation steady state. Variables without a time subscript denote steady-state values and variables in lowercase denote percentage deviations from steady state ($z_t = (Z_t - Z)/Z$), except for $g_t = (G_t - G)/Y$ and $g_t^i = (G_t^i - G^i)/Y$. Defining $r_{\beta,t} = \beta_t^{-1} - 1$, $\bar{g} = G/Y$ and $\bar{\alpha} = G^i/G$, the log-linearized model is given by

$$
\begin{aligned}
y_t &= (1-\bar{g})\, c_t + g_t, \\
n_t &= \frac{F}{F_N N} y_t - \frac{F_K K}{F_N N} k_t, \\
c_t &= E_t c_{t+1} + \frac{U_C}{U_{CC} C}\left(i_t - E_t \pi_{t+1} - r_{\beta,t}\right) - \frac{U_{CN} N}{U_{CC} C}\left(n_t - E_t n_{t+1}\right), \\
w_t &= \left(\frac{U_{CN} C}{U_N} - \frac{U_{CC} C}{U_C}\right) c_t + \left(\frac{U_{NN} N}{U_N} - \frac{U_{CN} N}{U_C}\right) n_t, \\
\pi_t &= \beta E_t \pi_{t+1} + \kappa\left(w_t - \frac{F_{NN} N}{F_N} n_t - \frac{F_{NK} K}{F_N} k_t\right), \\
k_t &= (1-\delta)\, k_{t-1} + \alpha \tilde{\delta} g_{t-T}, \\
i_t &= \max\left(0; r_\beta + \phi_\pi \pi_t\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\kappa &= \frac{(1-\phi)(1-\beta\phi)}{\phi} \frac{\frac{F_N N}{F}}{\frac{F_N N}{F} - \theta \frac{F_{NN} N}{F_N}} > 0, \\
\tilde{\delta} &= \frac{\delta}{\bar{\alpha}\bar{g}} \geq 0,
\end{aligned}
$$

and where we have used the fact that $g_t^i = \alpha g_t$.

A.4 Functional forms and the implied composite parameters

We consider the following production and utility functions:

$$
\begin{aligned}
U(C_t, N_t) &= \frac{\left(C_t^\gamma (1-N_t)^{1-\gamma}\right)^{1-\sigma} - 1}{1-\sigma} && \text{if } \sigma \neq 1 \\
&= \gamma \ln C_t + (1-\gamma) \ln(1-N_t) && \text{if } \sigma = 1,
\end{aligned}
$$

and

$$F(N_t, K_t) = N_t^a K_t^b.$$

These specifications imply the following first and second derivatives:

$$\begin{aligned}
F_N &= a\frac{F(N,K)}{N}, \quad F_K = b\frac{F(N,K)}{K}, \\
F_{NN} &= (a-1)\frac{F_N}{N} \quad F_{KK} = (b-1)\frac{F_K}{K}, \quad F_{NK} = a\frac{F_K}{N}, \\
U_C &= \gamma C^{\gamma(1-\sigma)-1}(1-N)^{(1-\gamma)(1-\sigma)}, \\
U_{CC} &= -(1+\gamma(\sigma-1))\frac{U_C}{C}, \\
U_{CN} &= (1-\gamma)(\sigma-1)\frac{U_C}{1-N}, \\
U_N &= -\left[(1-\gamma)C^{\gamma(1-\sigma)}(1-N)^{(1-\gamma)(1-\sigma)-1}\right], \\
U_{NN} &= -\left[(1+(1-\gamma)(\sigma-1))(1-\gamma)C^{\gamma(1-\sigma)}(1-N)^{(1-\gamma)(1-\sigma)-2}\right].
\end{aligned}$$

The steady-state values for C, N, and K are implicitly given by

$$\begin{aligned}
C &= (1-\bar{g})N^a K^b, \\
\frac{\theta-1}{(1-\tau)\theta}\frac{a}{1-\bar{g}} &= \frac{1-\gamma}{\gamma}\frac{N}{1-N}, \\
K &= \frac{\bar{\alpha}}{\delta}\frac{\bar{g}}{1-\bar{g}}C.
\end{aligned}$$

The composite parameters Φ_r, Φ_g, Φ_k, Θ_g, Θ_k, and Θ_c, are given by

$$\begin{aligned}
\Phi_r &= \frac{1}{1-\gamma(1-\sigma)+(1-\gamma)(1-\sigma)\frac{N}{1-N}\frac{1-g}{a}} = 1, \\
\Phi_g &= \frac{1}{a}(1-\gamma)(\sigma-1)\frac{N}{1-N}\Phi_r = \frac{\gamma(\sigma-1)}{1-\bar{g}}, \\
\Phi_k &= b\Phi_g, \\
\Theta_g &= -\frac{1}{a}\left((1-\gamma)(\sigma-1)\frac{N}{1-N}+a-1\right) \\
&\quad + \frac{1}{a}\frac{(1+(1-\gamma)(\sigma-1))(1-\gamma)C^{\gamma(1-\sigma)}(1-N)^{(1-\gamma)(1-\sigma)-1}\frac{N}{1-N}}{(1-\gamma)C^{\gamma(1-\sigma)}(1-N)^{(1-\gamma)(1-\sigma)-1}} \\
&= \frac{1}{1-\bar{g}}\frac{\gamma}{1-\gamma}, \\
\Theta_k &= b(1+\Theta_g), \\
\Theta_c &= 1+(1-\bar{g})\Theta_g = \frac{1}{1-\gamma}.
\end{aligned}$$

A.5 Analytical solution of the model with time to build ($T \geq 1$)

In this subsection, we explain how we solve the model with time to build both in normal times and when the ZLB binds. In both cases, we use the method of undetermined coefficients.

A.5.1 Normal times

Under the assumption that $\phi_\pi > 1$, the unique linear rational expectation solution is given by

$$\begin{aligned} c_t &= \vartheta_k k_{t-1} + \sum_{\tau=0}^{T} \vartheta_g^\tau g_{t-\tau}, \\ \pi_t &= \zeta_k k_{t-1} + \sum_{\tau=0}^{T} \zeta_g^\tau g_{t-\tau}. \end{aligned}$$

Substituting these two expressions into equations (17)-(20) (with $i_t = r_\beta + \phi_\pi \pi_t$).and equating the relevant coefficients, we obtain

$$\begin{aligned} \vartheta_k &= (1-\delta) \frac{\kappa(\phi_\pi - (1-\delta))\Phi_r\Theta_k - \delta(1-\beta(1-\delta))\Phi_k}{\delta(1-\beta(1-\delta)) + \kappa(\phi_\pi - (1-\delta))\Phi_r\Theta_c}, \\ \zeta_k &= -(1-\delta)\frac{\kappa\delta(\Theta_c\Phi_k + \Theta_k)}{\delta(1-\beta(1-\delta)) + \kappa(\phi_\pi - (1-\delta))\Phi_r\Theta_c}, \end{aligned}$$

and

$$\begin{pmatrix} \vartheta_g^0 \\ \zeta_g^0 \end{pmatrix} = \alpha\tilde{\delta}\Lambda\mathbf{A}\begin{pmatrix} \vartheta_g^1 \\ \zeta_g^1 \end{pmatrix} + \begin{pmatrix} (1-\rho\beta)\left[\alpha\tilde{\delta}\Phi_k + (1-\rho)\Phi_g\right] - \kappa(\phi_\pi - \rho)\Phi_r\Theta_g \\ \kappa(1-\rho)\Theta_g + \kappa\left[\alpha\tilde{\delta}\Phi_k + (1-\rho)\Phi_g\right]\Theta_c \end{pmatrix},$$

$$\begin{pmatrix} \vartheta_g^\tau \\ \zeta_g^\tau \end{pmatrix} = \varpi\mathbf{B}\begin{pmatrix} \vartheta_g^{\tau+1} \\ \zeta_g^{\tau+1} \end{pmatrix}, \quad \tau = 1, ..., T-2; \quad \text{for } T > 2,$$

$$\begin{pmatrix} \vartheta_g^{T-1} \\ \zeta_g^{T-1} \end{pmatrix} = \varpi\left[\mathbf{B}\begin{pmatrix} \vartheta_g^T \\ \zeta_g^T \end{pmatrix} + \alpha\tilde{\delta}\begin{pmatrix} \Phi_k \\ \kappa\Theta_c\Phi_k \end{pmatrix}\right], \quad \text{for } T > 1,$$

$$\begin{pmatrix} \vartheta_g^T \\ \zeta_g^T \end{pmatrix} = \alpha\tilde{\delta}\varpi\left[\mathbf{B}\begin{pmatrix} \vartheta_k \\ \zeta_k \end{pmatrix} + \begin{pmatrix} \kappa\phi_\pi\Phi_r\Theta_k - \delta\Phi_k \\ \kappa(\Theta_k + \delta\Theta_c\Phi_k) \end{pmatrix}\right], \quad \text{for } T \geq 1,$$

where

$$\Lambda = \left((1-\rho)(1-\beta\rho) + \kappa(\phi_\pi - \rho)\Phi_r\Theta_c\right)^{-1}, \qquad \varpi = (1 + \kappa\phi_\pi\Phi_r\Theta_c)^{-1},$$

and

$$\mathbf{A} = \begin{pmatrix} 1-\beta\rho & (1-\beta\phi_\pi)\Phi_r \\ \kappa\Theta_c & \kappa\Phi_r\Theta_c + \beta(1-\rho) \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 & (1-\beta\phi_\pi)\Phi_r \\ \kappa\Theta_c & \kappa\Phi_r\Theta_c + \beta \end{pmatrix}.$$

In practice, for any arbitrary T, ϑ_g^τ, and ζ_g^τ ($\tau = 0, ..., T$) are computed backward recursively: Start by computing ϑ_g^T and ζ_g^T, which are only functions of the deep parameters. Once ϑ_g^T and ζ_g^T are known, compute ϑ_g^{T-1} and ζ_g^{T-1}, and so on up to ϑ_g^0 and ζ_g^0.

A.5.2 Liquidity trap

When the ZLB binds ($i_t = 0$), equations (17)-(20) imply

$$\begin{aligned} c_t^l &= E_t c_{t+1} - \Phi_r \left(-E_t \pi_{t+1} - i_{\beta,t}\right) + \Phi_g \left(g_t^l - E_t g_{t+1}\right) - \Phi_k \left(\delta k_t - \alpha \tilde{\delta} E_t g_{t-(T-1)}\right), \\ \pi_t^l &= \beta E_t \pi_{t+1} + \Theta_c c_t^l + \Theta_g g_t^l - \Theta_k k_t. \end{aligned}$$

The unique linear rational expectation solution is given by

$$\begin{aligned} c_t^l &= \vartheta_i^l r_\beta^l + \vartheta_k^l k_{t-1} + \sum_{\tau=0}^{T} \vartheta_g^{l,\tau} g_{t-\tau}, \\ \pi_t^l &= \zeta_i^l r_\beta^l + \zeta_k^l k_{t-1} + \sum_{\tau=0}^{T} \zeta_g^{l,\tau} g_{t-\tau}. \end{aligned}$$

Unlike the simple case discussed in Section 3.2, whenever the ZLB ceases to bind, the economy does not immediately jump to the steady state, as the stock of public capital continues to adjust. Thus, the expected value of a given variable in $t+1$ is an average of its possible values in $t+1$ inside and outside the liquidity trap, weighted by the respective probabilities of being in these two states. Using the equilibrium paths of consumption and inflation inside and outside the ZLB state, we obtain

$$\begin{aligned} E_t c_{t+1} &= p \left(\vartheta_i^l r_\beta^l + \vartheta_k^l k_{t-1} + \vartheta_g^{l,0} g_t + \sum_{\tau=1}^{T} \vartheta_g^{l,\tau} g_{t+1-\tau}\right) + (1-p)\left(\vartheta_k k_{t-1} + \sum_{\tau=1}^{T} \vartheta_g^{\tau} g_{t+1-\tau}\right), \\ E_t \pi_{t+1} &= p \left(\zeta_i^l r_\beta^l + \zeta_k^l k_{t-1} + \zeta_g^{l,0} g_t + \sum_{\tau=1}^{T} \zeta_g^{l,\tau} g_{t+1-\tau}\right) + (1-p)\left(\zeta_k k_{t-1} + \sum_{\tau=1}^{T} \zeta_g^{\tau} g_{t+1-\tau}\right), \end{aligned}$$

where we have used the fact that

$$\begin{aligned} g_{t+1} &= g_t \text{ if the ZLB is still binding in } t+1, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Equating the relevant coefficients yields

$$\begin{aligned} \vartheta_i^l &= \frac{(1-\beta p)\,\Phi_r}{(1-p)(1-\beta p) - p\kappa \Phi_r \Theta_c}, \\ \zeta_i^l &= \frac{\Theta_c \Phi_r}{(1-p)(1-\beta p) - p\kappa \Phi_r \Theta_c}, \end{aligned}$$

$$\vartheta_k^l = \frac{(1-p)(1-\delta)\left[(1-p\beta(1-\delta))\vartheta_k + \Phi_r\zeta_k\right] - p\kappa(1-\delta)^2\Phi_r\Theta_k - (1-\delta)(1-p\beta(1-\delta))\delta\Phi_k}{(1-p(1-\delta))(1-\beta p(1-\delta)) - p\kappa(1-\delta)\Phi_r\Theta_c},$$

$$\zeta_k^l = \frac{(1-p)(1-\delta)\left\{\kappa\Theta_c\vartheta_k + \left[\beta(1-p(1-\delta)) + \kappa\Theta_c\Phi_r\right]\zeta_k\right\} - \kappa(1-\delta)\left[(1-p\beta(1-\delta))\Theta_k + \delta\Theta_c\Phi_k\right]}{(1-p(1-\delta))(1-\beta p(1-\delta)) - p\kappa(1-\delta)\Phi_r\Theta_c},$$

and

$$\begin{aligned}
\begin{pmatrix}\vartheta_g^{l,0}\\ \zeta_g^{l,0}\end{pmatrix} &= \alpha\tilde{\delta}\Lambda^l\mathbf{A}^l\left[p\begin{pmatrix}\vartheta_g^{l,1}\\ \zeta_g^{l,1}\end{pmatrix} + (1-p)\begin{pmatrix}\vartheta_g^{1}\\ \zeta_g^{1}\end{pmatrix}\right] + \Lambda^l\begin{pmatrix}(1-p\beta)(1-p)\Phi_g + p\kappa\Phi_r\Theta_g\\ \kappa(1-p)(\Theta_g + \Phi_g\Theta_c)\end{pmatrix},\\
\begin{pmatrix}\vartheta_g^{l,\tau}\\ \zeta_g^{l,\tau}\end{pmatrix} &= \mathbf{B}^l\left[p\begin{pmatrix}\vartheta_g^{l,\tau+1}\\ \zeta_g^{l,\tau+1}\end{pmatrix} + (1-p)\begin{pmatrix}\vartheta_g^{\tau+1}\\ \zeta_g^{\tau+1}\end{pmatrix}\right], \quad \tau = 1,...,T-2, \quad \text{for } T>2,\\
\begin{pmatrix}\vartheta_g^{l,T-1}\\ \zeta_g^{l,T-1}\end{pmatrix} &= \mathbf{B}^l\left[p\begin{pmatrix}\vartheta_g^{l,T}\\ \zeta_g^{l,T}\end{pmatrix} + (1-p)\begin{pmatrix}\vartheta_g^{T}\\ \zeta_g^{T}\end{pmatrix}\right] + \alpha\tilde{\delta}\begin{pmatrix}\Phi_k\\ \kappa\Theta_c\Phi_k\end{pmatrix}, \quad \text{for } T>1,\\
\begin{pmatrix}\vartheta_g^{T}\\ \zeta_g^{T}\end{pmatrix} &= \alpha\tilde{\delta}\mathbf{B}^l\left[p\begin{pmatrix}\vartheta_k^{l}\\ \zeta_k^{l}\end{pmatrix} + (1-p)\begin{pmatrix}\vartheta_k\\ \zeta_k\end{pmatrix}\right] - \alpha\tilde{\delta}\begin{pmatrix}\delta\Phi_k\\ \kappa(\delta\Theta_c\Phi_k + \Theta_k)\end{pmatrix}, \quad \text{for } T\geq 1,
\end{aligned}$$

where

$$\Lambda^l = ((1-p)(1-\beta p) - p\kappa\Phi_r\Theta_c)^{-1},$$

and

$$\mathbf{A}^l = \begin{pmatrix}1-p\beta & \Phi_r\\ \kappa\Theta_c & \kappa\Phi_r\Theta_c + \beta(1-p)\end{pmatrix}, \quad \mathbf{B}^l = \begin{pmatrix}1 & \Phi_r\\ \kappa\Theta_c & \kappa\Phi_r\Theta_c + \beta\end{pmatrix}.$$

Again, the coefficients $\vartheta_g^{l,\tau}$ and $\zeta_g^{l,\tau}$ $(\tau = 0,...,T)$ are computed recursively starting from period T.