

Object Allocation via Deferred-Acceptance: Strategy-Proofness and Comparative Statics*

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December 2014

Abstract

We study the problem of assigning indivisible and heterogenous objects (*e.g.*, houses, jobs, offices, school or university admissions *etc.*) to agents. Each agent receives at most one object and monetary compensations are not possible. We consider mechanisms satisfying a set of basic properties (*unavailable-type-invariance*, *individual-rationality*, *weak non-wastefulness*, or *truncation-invariance*).

In the house allocation problem, where at most one copy of each object is available, deferred-acceptance (DA)-mechanisms allocate objects based on exogenously fixed objects' priorities over agents and the agent-proposing deferred-acceptance-algorithm. For house allocation we show that DA-mechanisms are characterized by our basic properties and (i) *strategy-proofness* and *population-monotonicity* or (ii) *strategy-proofness* and *resource-monotonicity*.

Once we allow for multiple identical copies of objects, on the one hand the first characterization breaks down and there are unstable mechanisms satisfying our basic properties and (i) *strategy-proofness* and *population-monotonicity*. On the other hand, our basic properties and (ii) *strategy-proofness* and *resource-monotonicity* characterize (the most general) class of DA-mechanisms based on objects' fixed choice functions that are acceptant, monotonic, substitutable, and consistent. These choice functions are used by objects to reject agents in the agent-proposing deferred-acceptance-algorithm. Therefore, in the general model *resource-monotonicity* is the "stronger" comparative statics requirement because it characterizes (together with our basic requirements and *strategy-proofness*) choice-based DA-mechanisms whereas *population-monotonicity* (together with our basic properties and *strategy-proofness*) does not.

JEL Classification: D63, D70.

Keywords: indivisible objects allocation, deferred-acceptance-algorithm, strategy-proofness, resource-monotonicity, population-monotonicity.

*We are grateful to two anonymous referees, Battal Doğan, and Jan Christoph Schlegel for their helpful comments and suggestions. Lars Ehlers acknowledges financial support from the SSHRC (Canada) and the FQRSC (Québec) and Bettina Klaus acknowledges financial support from the Swiss National Science Foundation (SNSF).

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1 Introduction

We study the simple model of assigning indivisible and heterogenous objects (*e.g.*, houses, jobs, offices, school or university admissions *etc.*) to agents. In many real-life applications economists have recommended practitioners to use (variants of) the deferred-acceptance (DA)-algorithm to assign the objects to agents. Since in our model only one side of the market consists of agents and the other of objects, by DA-algorithm we refer to the agent-proposing DA-algorithm. For the DA-algorithm to be well-defined, objects need to be endowed with priorities over agents in order to determine which agents to reject in case of too many applications. When at most one copy of each object is available, priorities are given by strict orders of agents, and a priority structure is a collection of strict orders, one for each object. Once multiple identical copies of an object are available, priorities are given by choice functions, which choose for any set of agents wanting an object some agents to receive one each (no more than the number of available copies). Choice structures are a collection of choice functions, one for each object. Recent papers (Ehlers et al., 2014; Kamada and Kojima, 2014) constructed choice structures taking several important features from applications into account (like controlled school choice constraints or regional caps and target capacities) and then applied the DA-algorithm based on those constructed choice structures.

One key reason for applying the DA-algorithm in real life is that it is *strategy-proof* if the choice structure satisfies certain conditions. In addition, economists have advocated the attractive comparative statics properties of the DA-algorithm (*e.g.*, Crawford, 1991, Hatfield and Milgrom, 2005, and Chambers and Yenmez, 2014): (i) *population-monotonicity*: once some agents leave, all remaining agents weakly benefit from this decrease in competition and (ii) *resource-monotonicity*: once more objects become available, all agents weakly benefit from this increase in resources. Our main results characterize responsive DA-algorithms and choice-based DA-algorithms via a set of basic properties in conjunction with *strategy-proofness* and one of the two comparative statics properties.

We first go back one step and consider the house allocation model¹ where at most one copy of each object is available. We show that in the house allocation model DA-mechanisms with strict priority structures are characterized (in the fixed resources model) by our basic properties, *population-monotonicity*, and *strategy-proofness*. However, this result does not hold in the general model where multiple identical copies of an object may be available and even if we add *truncation-invariance* (the chosen allocation remains unchanged if an agent truncates his preference below his assigned object): there are unstable mechanisms satisfying all our basic properties, *population-monotonicity*, *strategy-proofness*, and *truncation-invariance*. In addition, all these properties minus *strategy-proofness* do not imply stability of the mechanism: the Boston mechanism satisfies all these properties except *strategy-proofness* (but the Boston mechanism is not stable).

¹The search for “good” mechanisms to solve house allocation problems has been the subject of various contributions (among others): Ehlers (2002), Ehlers and Klaus (2004, 2007, 2011), Kesten (2009), and Pápai (2000).

Considering the house allocation model with a fixed population and variable resources, our basic properties, *resource-monotonicity*, and *strategy-proofness* do not characterize responsive DA-mechanisms. However, adding *truncation-invariance* to these properties yields another characterization of responsive DA-mechanism via our basic properties, *resource-monotonicity*, *strategy-proofness*, and *truncation-invariance*. In the general model our main result characterizes choice-based DA-algorithms via a set of basic properties in conjunction with *resource-monotonicity*, *strategy-proofness*, and *truncation-invariance*. In addition, we show that any mechanism satisfying all these properties except *strategy-proofness* must be stable, *i.e.*, we must be able to construct a choice structure such that for any problem the mechanism chooses an allocation that is stable with respect to this choice structure.

On the one hand, in the house allocation model *population-monotonicity* is the “stronger” comparative statics requirement because it characterizes responsive DA-mechanisms with our basic properties and *strategy-proofness* whereas *resource-monotonicity* does not (without adding *truncation-invariance*). On the other hand, in the multi-unit allocation model *resource-monotonicity* is the “stronger” comparative statics requirement because it characterizes choice-based DA-mechanisms with our basic properties, *strategy-proofness* and *truncation-invariance* whereas *population-monotonicity* does not.

Therefore, in the general model, choice-based DA-mechanisms are the unique recommendation once we insist in addition to our basic properties on *resource-monotonicity*, *population-monotonicity*, and *strategy-proofness*. Once we drop one of the key properties, choice-based DA-mechanisms remain no longer the *unique* recommendation. This result advocates the DA-algorithm in applications in a stronger fashion than the usual ones whereby a mechanism is advocated because of its properties. The basic properties we use in our characterizations are satisfied by any real-life mechanism we are aware of, *e.g.*, priority mechanisms, the Boston mechanism, linear programming mechanisms, the top trading cycles mechanism, the objects-proposing DA-algorithm *etc.*

Kojima and Manea (2010) were the first to obtain a characterization of choice-based DA-mechanisms in the general model. They provided two characterizations, (a) one using the two properties of *non-wastefulness* and “individually rational monotonicity” and (b) one using the three properties of *non-wastefulness*, *population-monotonicity*, and “weak Maskin monotonicity”. Whereas *non-wastefulness* is a basic requirement, “individual rational monotonicity” and “weak Maskin monotonicity” are new axioms that are arguably more difficult to explain to school boards in policy debates. Ehlers and Klaus (2014) characterize the smaller class of responsive DA-mechanisms whereby each choice function is based on a strict order and chooses from any set of agents the k most preferred elements. In their main result, “two-agent consistent conflict resolution” plays a key role which says that at maximal conflict situations always the same agent should win the conflict (or receive the object). This property is violated by some choice-based DA-mechanisms. Furthermore, in contrast to these characterizations, we show that our main result minus *strategy-proofness* gives

us stability, *i.e.*, any such mechanism must be stable with respect to a given choice structure. Note that in all these contributions priorities are derived from a mechanism via a set of properties. Other papers take exogenous priorities as given and impose properties on the mechanism using these exogenous priorities. Balinski and Sönmez (1999) and Morrill (2013) are example of this approach and characterize the deferred-acceptance mechanisms based on “responsive” priorities and “substitutable” priorities.

The paper is organized as follows. In Section 2 we introduce our general object allocation model, properties of mechanisms, and the classes of deferred-acceptance-mechanisms that are either based on acceptant and responsive priority structures or on acceptant, monotonic, substitutable, and consistent choice structures over sets of agents. In Section 3 we state our characterization of responsive-deferred-acceptance mechanisms in the house allocation model using *population-monotonicity* and show how some previous and some new results are implied. Section 4 contains our main result (Theorem 3) for the general object allocation model using *resource-monotonicity*, a characterization of the class of choice-deferred-acceptance-mechanisms (where the choice structure satisfies acceptance, monotonicity, substitutability, and consistency). Section 5 concludes. All proofs and the independences of properties in our characterizations can be found in the Appendix.

2 Object Allocation

2.1 The Model and Notation

Let² N denote a finite *set of agents*, $|N| \geq 2$. We define the set of all nonempty subsets of N by $\mathcal{N} \equiv \{S : S \subseteq N \text{ and } S \neq \emptyset\}$. Let O denote a set of potential (real) object types or *types* for short. We assume that O contains at least two elements and that O is finite.³ Not receiving any real object is called “receiving the null object.” Let \emptyset represent the *null object*.

For a given set of agents $S \in \mathcal{N}$, each agent $i \in S$ is equipped with a *preference relation* R_i over all types $O \cup \{\emptyset\}$. The preference relation R_i is strict, *i.e.*, R_i is a linear order over $O \cup \{\emptyset\}$. Given $x, y \in O \cup \{\emptyset\}$, $x P_i y$ means that agent i strictly prefers x to y (and $x \neq y$) and $x R_i y$ means that agent i weakly prefers x to y (*i.e.*, $x P_i y$ or $x = y$). Let \mathcal{R} denote the *set of all preferences* over $O \cup \{\emptyset\}$, and \mathcal{R}^S the *set of all (preference) profiles* $R = (R_i)_{i \in S}$ such that for all $i \in S$, $R_i \in \mathcal{R}$.

Given $R \in \mathcal{R}^S$ and $M \subseteq S$, let R_M denote the profile $(R_i)_{i \in M}$. It is the restriction of R to the set of agents M . We also use the notation $R_{-M} = R_{S \setminus M}$ and $R_{-i} = R_{S \setminus \{i\}}$ (whenever the set of agents S has been fixed and no confusion can arise). Given $O' \subseteq O \cup \{\emptyset\}$, let $R_i|_{O'}$ denote the restriction of R_i to O' and $R|_{O'} = (R_i|_{O'})_{i \in S}$. Given $i \in S$ and $R_i \in \mathcal{R}$, type $x \in O$ is *acceptable under* R_i if $x P_i \emptyset$. Let $A(R_i) = \{x \in O : x P_i \emptyset\}$ denote the *set of acceptable types under* R_i .

²Our allocation model is identical to that described in Ehlers and Klaus (2014).

³Our results remain unchanged when O is infinite. For expositional convenience, finiteness of O is assumed.

For each type $x \in O$, at most $\bar{q}_x \in \mathbb{N}$ copies are available in any economy with $1 \leq \bar{q}_x \leq |N|$.⁴ Let $q_x \in \{0, 1, \dots, \bar{q}_x\}$ denote the number of available objects or the *capacity* of type x . Let $q = (q_x)_{x \in O}$ denote a *capacity vector* and $\mathcal{Q} = \times_{x \in O} \{0, 1, \dots, \bar{q}_x\}$ denote the *set of all capacity vectors*. The null object is always available without scarcity and therefore we set $q_\emptyset = \infty$. Given a capacity vector q , let $O_+(q) = \{x \in O : q_x > 0\}$ denote the *set of available real types* under q . The *set of available types* is $O_+(q) \cup \{\emptyset\}$ and includes the null object

An (*allocation*) *problem (with capacity constraints)* consists of a set of agents $S \in \mathcal{N}$, a preference profile $R \in \mathcal{R}^S$, and a capacity vector q . We denote a problem by (R, q) and the set of all problems by $(\cup_{S \in \mathcal{N}} \mathcal{R}^S) \times \mathcal{Q}$.

Given $S \in \mathcal{N}$ and $q \in \mathcal{Q}$, each agent i is allocated exactly one object of a type in $O \cup \{\emptyset\}$ taking capacity constraints into account. An *allocation (for S and q)* is a list $a = (a_i)_{i \in S}$ such that for all $i \in S$, $a_i \in O \cup \{\emptyset\}$, and any real type $x \in O$ is not assigned more than q_x times, *i.e.*, for all $x \in O$, $|\{i \in S : a_i = x\}| \leq q_x$. Note that \emptyset , the null object, can be assigned to any number of agents and that not all real objects have to be assigned. Let $a(x) = \{i \in S : a_i = x\}$ denote the set of agents who are assigned type x under a . Let $\mathcal{A}(S, q)$ denote the *set of all allocations for S and q* , and $\mathcal{A} = \cup_{S \in \mathcal{N}} \cup_{q \in \mathcal{Q}} \mathcal{A}(S, q)$ the *set of all allocations*.

A *mechanism* is a function $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \times \mathcal{Q} \rightarrow \mathcal{A}$ such that for all $S \in \mathcal{N}$, all $R \in \mathcal{R}^S$ and all $q \in \mathcal{Q}$, $\varphi(R, q) \in \mathcal{A}(S, q)$. Given $i \in S$, we call $\varphi_i(R, q)$ the *allotment* of agent i at $\varphi(R, q)$.

2.2 Properties of Mechanisms

A natural requirement for a mechanism is that the chosen allocation depends only on preferences over the set of available types. Given a capacity vector q , a type x is *unavailable* if $q_x = 0$.

Unavailable-Type-Invariance: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, and all $R' \in \mathcal{R}^N$ such that $R|_{O_+(q) \cup \{\emptyset\}} = R'|_{O_+(q) \cup \{\emptyset\}}$, $\varphi(R, q) = \varphi(R', q)$.⁵

By *individual-rationality* each agent should weakly prefer his allotment to the null object (which may represent an outside option such as off-campus housing in the context of university housing allocation, or private schools or home schooling in the context of student placement in public schools).

Individual-Rationality: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, and all $i \in S$, $\varphi_i(R, q) R_i \emptyset$.

⁴By introducing “global upper bounds” via \bar{q}_x ($x \in O$) we can, for instance, specify the so-called *house allocation model* where at most one object of each type is available, *i.e.*, for all $x \in O$, $\bar{q}_x = 1$.

⁵*Unavailable-type-invariance* is similar to the *independence if irrelevant commodities* (IIC) property for social ordering functions introduced by Fleurbaey and Tadenuma (2007). It is important to note that “IIC is similar to the well known condition proposed by Arrow (1951), *independence of irrelevant alternatives* (IIA), but it turns out to be much weaker and, we believe, much less controversial.” In Ehlers and Klaus (2004, 2011) we refer to *unavailable-type-invariance* as *independence of irrelevant objects*.

Next, we introduce two properties that require a mechanism to not waste any resources. First, *non-wastefulness* (Balinski and Sönmez, 1999) requires that no agent prefers an available object that is not assigned to his allotment.

Non-Wastefulness: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in N$, if $x P_i \varphi_i(R, q)$, then $|\{j \in S : \varphi_j(R, q) = x\}| = q_x$.

Next, we weaken *non-wastefulness* by requiring that no agent receives the null object while he prefers an available object that is not assigned.

Weak Non-Wastefulness: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in S$, if $x P_i \varphi_i(R, q)$ and $\varphi_i(R, q) = \emptyset$, then $|\{j \in S : \varphi_j(R, q) = x\}| = q_x$.

Weak non-wastefulness is a limited efficiency requirement. For example, suppose that a central agency registers all agents who did not receive anything (or are unemployed) and all those agents report all real types (or jobs) which are acceptable to them. Then it should not be the case that some agent who receives nothing prefers one of the available real objects (or available jobs) to the null object.

Of course, no resources are wasted if a mechanism is *(Pareto) efficient*.

Efficiency: For all $S \in \mathcal{N}$ and all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, there exists no feasible allocation $a \in \mathcal{A}(S, q)$ such that for all $i \in S$, $a_i R_i \varphi_i(R, q)$, and for some $j \in S$, $a_j P_j \varphi_j(R, q)$.

Note that *efficiency* implies *individual-rationality* and *(weak) non-wastefulness*.

When the set of objects varies, another natural requirement is *resource-monotonicity*. As already explained in the Introduction, this is a widely used solidarity property introduced by Chun and Thomson (1988) and it describes the effect of a change in the available resources on the welfare of the agents. A mechanism is *resource-monotonic* if the availability of more real objects has a (weakly) positive effect on all agents.

Resource-Monotonicity: For all $S \in \mathcal{N}$, all $R \in \mathcal{R}^S$, and all $q, q' \in \mathcal{Q}$, if $q_x \leq q'_x$ for all $x \in O$, then for all $i \in S$, $\varphi_i(R, q') R_i \varphi_i(R, q)$.

Another solidarity property concerning variations of the population of agents is *population-monotonicity*; it requires that as the set of agents becomes larger, the initially present agents all get (weakly) worse off. This property goes back to Thomson (1983), who also presents a survey of population monotonicity in various economic models (Thomson, 1995).

Population-Monotonicity: For all $T \subseteq S \in \mathcal{N}$ and all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, we have for all $i \in T$, $\varphi_i(R_T, q) R_i \varphi_i(R, q)$.

Many mechanisms that are used in real life ignore agents' preferences below their allotments (*e.g.*, any mechanism based on or equivalent to the famous deferred-acceptance-algorithm or the so-called priority mechanisms, Roth and Sotomayor, 1990, Sections 5.4.1 and 5.5.1). That is, an

allocation does not change if an agent changes his reported preferences below his allotment. We formulate a weaker version of this invariance property by restricting agents' changes below their allotments to truncations.

A truncation strategy is a preference relation that ranks the real types in the same way as the corresponding original preference relation and each real type which is acceptable under the truncation strategy is also acceptable under the original preference relation. Formally, given $S \in \mathcal{N}$, $i \in S$ and $R_i \in \mathcal{R}$, a strategy $\bar{R}_i \in \mathcal{R}$ is a *truncation (strategy) of R_i* if (t1) $\bar{R}_i|_O = R_i|_O$ and (t2) $A(\bar{R}_i) \subseteq A(R_i)$. Loosely speaking, a truncation strategy of R_i is obtained by moving the null object “up.”

If an agent truncates his preference relation in a way such that his allotment remains acceptable under the truncated preference relation, then *truncation-invariance* requires that the allocation is the same under both profiles. The property is quite natural on its own in the sense that the chosen allocations do not depend on where any agent, who receives a real object, ranks the null object below his allotment.

Truncation-Invariance: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, all $i \in S$, and all $\bar{R}_i \in \mathcal{R}_i$, if \bar{R}_i is a truncation of R_i and $\varphi_i(R, q)$ is acceptable under \bar{R}_i (i.e., $\varphi_i(R, q) \in A(\bar{R}_i)$), then $\varphi((\bar{R}_i, R_{-i}), q) = \varphi(R, q)$.

The well-known non-manipulability property *strategy-proofness* requires that no agent can ever benefit from misrepresenting his preferences.

Strategy-Proofness: For all $S \in \mathcal{N}$, all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, all $i \in S$, and all $\bar{R}_i \in \mathcal{R}_i$, $\varphi_i(R, q) R_i \varphi_i((\bar{R}_i, R_{-i}), q)$.

The following strengthening of *strategy-proofness* requires that no group of agents can ever benefit by misrepresenting their preferences.

Group-Strategy-Proofness: For all $S \in \mathcal{N}$, $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, all $M \subseteq S$ ($M \neq \emptyset$), and all $\bar{R}_M \in \mathcal{R}^M$, if for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), q) R_i \varphi_i(R, q)$, then for all $i \in M$, $\varphi_i((\bar{R}_M, R_{-M}), q) = \varphi_i(R, q)$.

2.3 Stability and Deferred Acceptance

2.3.1 “Classic” Stability

In this subsection, let $S \in \mathcal{N}$ be fixed and consider fixed-population *mechanisms* $\varphi : \mathcal{R}^S \times \mathcal{Q} \rightarrow \mathcal{A}$.

Given type $x \in O$, let \succ_x denote a *priority ordering on S* , e.g., $\succ_x: 1 \ 2 \ \dots \ |S|$ means that agent 1 has higher priority for type x than agent 2, who has higher priority for type x than agent 3, etc. Let $\succ \equiv (\succ_x)_{x \in O}$ denote a *priority structure for S* . Then, given a priority structure \succ for S , and a problem $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, we can interpret (R, q, \succ) as a *responsive many-to-one matching market*

(Gale and Shapley, 1962; Roth and Sotomayor, 1990) where the set of agents S corresponds to the set of students, the set of available real types $O_+(q)$ corresponds to the set of colleges, preferences $R|_{O_+(q) \cup \{\emptyset\}}$ correspond to students' preferences over colleges and being unmatched, and the priority structure $(\succ_x)_{x \in O}$ corresponds to colleges' (acceptant) responsive preferences over students in S .⁶ Stability is an important requirement for many real-life matching markets and it will turn out to be essential in our context of allocating indivisible objects to agents as well.

\succ -Stability: Given $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, an allocation $a \in \mathcal{A}(S, q)$ is \succ -stable if

- (s1) it is *individually-rational* and *non-wasteful*, i.e., there exists no agent-type pair $(i, x) \in S \times (O_+(q) \cup \{\emptyset\})$ such that $x P_i a_i$ and $|\{j \in S : a_j = x\}| < q_x$ and
- (s2) there are *no blocking pairs*, i.e., there exists no agent-type pair $(i, x) \in S \times O_+(q)$ such that $x P_i a_i$ and there exists $k \in N$ such that $a_k = x$ and $i \succ_x k$.

Note that \succ -stability implies *individual-rationality* and (*weak*) *non wastefulness*, but it does not imply *efficiency*. Fixed-population mechanism φ is \succ -stable if for all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, $\varphi(R, q)$ is \succ -stable.

\succ -Agent-Proposing Deferred-Acceptance-Algorithm: given priority structure \succ and problem $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, the agent-proposing deferred-acceptance-algorithm (Gale and Shapley, 1962) is defined as follows:

- at the first step of the deferred-acceptance-algorithm, every agent applies to his favorite type in $O_+(q) \cup \{\emptyset\}$. For each available real type $x \in O_+(q)$, the q_x applicants who have highest priority for x are placed on the waiting list of x (all if there are fewer than q_x applicants), and all others are rejected. The null object \emptyset accepts all agents.
- At the r -th step of the deferred-acceptance-algorithm, those applicants who were rejected at step $r - 1$ apply to their next favorite type in $O_+(q) \cup \{\emptyset\}$. For each available real type $x \in O_+(q)$, the q_x applicants among the new applicants and those on the waiting list at the end of Step $r - 1$ who have the highest priority for x are placed on the (updated) waiting list of x (all if there are fewer than q_x applicants), and all others are rejected. The null object \emptyset accepts all agents.

The deferred-acceptance-algorithm terminates when every agent is on a waiting list. Once the algorithm ends, available real objects are assigned to the agents on the available real type waiting lists (all other agents were accepted by and receive the null object) and the resulting allocation is the agent-optimal stable allocation for the responsive many-to-one matching market (R, q, \succ) , denoted by $DA^\succ(R, q)$.

⁶Formally, a priority relation P_x on $\{S' \subseteq S (S' \neq \emptyset) : |S'| \leq q_x\}$ is (acceptant) responsive to \succ_x if the following two conditions hold: (r1) for all $S' \subseteq S (S' \neq \emptyset)$ such that $|S'| < q_x$ and all $i \in S \setminus S'$, $S' \cup \{i\} P_x S'$ and (r2) for all $S' \subseteq S (S' \neq \emptyset)$ such that $|S'| < q_x$ and all $i, j \in S \setminus S'$, $(S' \cup \{i\}) P_x (S' \cup \{j\})$ if and only if $i \succ_x j$.

Responsive-Deferred-Acceptance-Mechanisms: A fixed-population mechanism $\varphi : \mathcal{R}^S \times \mathcal{Q} \rightarrow \mathcal{A}$ is a *responsive-deferred-acceptance-mechanism* if there exists a strict priority structure \succ such that for all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, $\varphi(R, q) = DA^\succ(R, q)$.

2.3.2 Choice Function Stability

In this subsection we return to our variable population setup and instead of using (acceptant) responsive priorities over sets of agents, we use a choice function approach.

For any real type $x \in O$, let $\mathcal{C}_x : \mathcal{N} \times \{1, \dots, \bar{q}_x\} \rightarrow \mathcal{N}$ denote a *choice function* such that for all $S \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, $\mathcal{C}_x(S, l) \subseteq S$ and $\mathcal{C}_x(S, l) \neq \emptyset$. We next list some properties of choice functions.

Acceptance: Choice function \mathcal{C}_x is *acceptant* if for all $S \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, $|\mathcal{C}_x(S, l)| = \min\{|S|, l\}$.

Monotonicity: Choice function \mathcal{C}_x is *monotonic* if for all $S \in \mathcal{N}$ and all $l \in \{2, \dots, \bar{q}_x\}$, $\mathcal{C}_x(S, l-1) \subseteq \mathcal{C}_x(S, l)$.

Substitutability: Choice function \mathcal{C}_x is *substitutable* if for all $S \in \mathcal{N}$, all $l \in \{1, \dots, \bar{q}_x\}$, and all $i, j \in S$ with $i \neq j$, $i \in \mathcal{C}_x(S, l)$ implies $i \in \mathcal{C}_x(S \setminus \{j\}, l)$.

Consistency: Choice function \mathcal{C}_x is *consistent*⁷ if for all $S, S' \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, $\mathcal{C}_x(S, l) \subseteq S' \subseteq S$ implies $\mathcal{C}_x(S, l) = \mathcal{C}_x(S', l)$.

Law of Aggregate Demand (LAD): Choice function \mathcal{C}_x satisfies the *law of aggregate demand (LAD)* if for all $S, S' \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, $S' \subseteq S$ implies $|\mathcal{C}_x(S', l)| \leq |\mathcal{C}_x(S, l)|$.

Path Independence: Choice function \mathcal{C}_x is *path independent* if for all $S, S' \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, $\mathcal{C}_x(S \cup S', l) = \mathcal{C}_x(S \cup \mathcal{C}_x(S', l), l)$.

Lemma 1.

- (a) If choice function \mathcal{C}_x is acceptant, then it satisfies the law of aggregate demand (LAD).
- (b) If choice function \mathcal{C}_x satisfies the law of aggregate demand (LAD) and substitutability, then it satisfies consistency.
- (c) Choice function \mathcal{C}_x is path independent if and only if it is substitutable and consistent.

Proof.

(a) LAD follows immediately from the definition of acceptance.

(b)⁸ Let $S, S' \in \mathcal{N}$ and $l \in \{1, \dots, \bar{q}_x\}$ be such that $\mathcal{C}_x(S, l) \subseteq S' \subseteq S$. By substitutability, $\mathcal{C}_x(S, l) \subseteq \mathcal{C}_x(S', l)$. By LAD, $|\mathcal{C}_x(S', l)| \leq |\mathcal{C}_x(S, l)|$ which implies $\mathcal{C}_x(S, l) = \mathcal{C}_x(S', l)$.

⁷Consistency is referred to as *irrelevance of rejected contracts (IRC)* by Aygün and Sönmez (2013).

⁸Aygün and Sönmez (2013) state “Some of the analysis in Hatfield and Milgrom (2005) assumes the *law of aggregate demand (LAD)* condition in addition to the substitutes condition. These two conditions together imply the IRC condition, ...” (we refer to the IRC condition as consistency).

(c) Chambers and Yenmez (2014), Lemma 1 (they refer to Aizerman and Malishevski, 1981). \square

A choice structure is a profile of choice functions, *i.e.*, $\mathcal{C} = (\mathcal{C}_x)_{x \in O}$. For all $S \in \mathcal{N}$ and all $l \in \{1, \dots, \bar{q}_x\}$, we define $\mathcal{C}_\emptyset(S, l) = S$ (the null object always chooses all agents). We say that choice structure \mathcal{C} is *acceptant / monotonic / substitutable / consistent / path independent* if for all real types $x \in O$, \mathcal{C}_x is acceptant / monotonic / substitutable / consistent / path independent.

\mathcal{C} -Stability: Given $S \in \mathcal{N}$ and $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, an allocation $a \in \mathcal{A}(S, q)$ is \mathcal{C} -stable if

- (s1) it is *individually-rational for both sides of the market*, *i.e.*, for all $i \in S$, $a_i R_i \emptyset$ and for all $x \in O_+(q) \cup \{\emptyset\}$, $\mathcal{C}_x(a(x), q_x) = a(x)$ and
- (s2) there are *no blocking pairs*, *i.e.*, there exists no agent-type pair $(i, x) \in S \times O_+(q)$ such that $x P_i a_i$ and $i \in \mathcal{C}_x(a(x) \cup \{i\}, q_x)$.

Note that \mathcal{C} -stability implies *individual-rationality* and (*weak*) *non wastefulness*, but it does not imply *efficiency*. Mechanism φ is \mathcal{C} -stable if for all $S \in \mathcal{N}$ and $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, $\varphi(R, q)$ is \mathcal{C} -stable.

\mathcal{C} -Agent-Proposing Deferred-Acceptance-Algorithm: Given choice structure \mathcal{C} , a set of agents $S \in \mathcal{N}$, and a problem $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, the agent-proposing deferred-acceptance-algorithm is defined as follows:

- at the first step of the deferred-acceptance-algorithm, every agent applies to his favorite type in $O_+(q) \cup \{\emptyset\}$. For each available real type $x \in O_+(q)$, let $S_{x,1}$ denote the set of agents who applied to real type x , applicants in $\mathcal{C}_x(S_{x,1}, q_x)$ are placed on the waiting list of x , and all others are rejected. The null object \emptyset accepts all agents.
- At the r -th step of the deferred-acceptance-algorithm, those applicants who were rejected at step $r - 1$ apply to their next favorite type in $O_+(q) \cup \{\emptyset\}$. For each available real type $x \in O_+(q)$, let $S_{x,r}$ denote the set of new applicants and those on the waiting list at the end of Step $r - 1$, applicants in $\mathcal{C}_x(S_{x,r}, q_x)$ are placed on the (updated) waiting list of x , and all others are rejected. The null object \emptyset accepts all agents.

The deferred-acceptance-algorithm terminates when every agent is on a waiting list. Once the algorithm ends, available real objects are assigned to the agents on the available real type waiting lists (all other agents were accepted by and receive the null object). Let $DA^{\mathcal{C}}(R, q)$ denote the allocation determined by the choice-based deferred-acceptance-algorithm when applied to (R, q, \mathcal{C}) .

Hatfield and Milgrom (2005) and Aygün and Sönmez (2013) have showed that if \mathcal{C} is acceptant, substitutable, and consistent, then $DA^{\mathcal{C}}(R, q)$ is \mathcal{C} -stable (and mechanism $DA^{\mathcal{C}}$ is *strategy-proof*).⁹

⁹Earlier results for the existence of stable allocations under substitutability are Kelso and Crawford (1982) and Roth (1984).

Choice-Deferred-Acceptance-Mechanisms: A mechanism $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \times \mathcal{Q} \rightarrow \mathcal{A}$ is a *choice-deferred-acceptance-mechanism* if there exists a choice structure \mathcal{C} such that for all $S \in \mathcal{N}$ and all $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$, $\varphi(R, q) = DA^{\mathcal{C}}(R, q)$.

3 Population-Monotonicity

For the so-called house allocation model at most one copy of each real type is available. Hence, for all $x \in O$, $\bar{q}_x = 1$. We denote the set of all house allocation capacity vectors by \mathcal{Q}_1 .

We keep the set of types O fixed and vary the set of agents to be $S \in \mathcal{N}$. Hence, a mechanism is a function $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \rightarrow \mathcal{A}$ (with a fixed set of types O). The definitions of \succ -stability and the \succ -proposing deferred-acceptance-algorithm are as before with the understanding that \succ is a priority structure for N that induces corresponding priority structures for all $S \in \mathcal{N}$, *i.e.*, if agent $i \in N$ has a higher priority for a type x than agent $j \in N$, then for any set $S \in \mathcal{N}$ such that $i, j \in S$, agent i again has a higher priority for type x than agent j . The house allocation model (with fixed resources and a variable population) is identical to that described in Ehlers, Klaus, and Pápai (2002).

Theorem 1. *In the house allocation model (with fixed resources and a variable population), responsive-deferred-acceptance-mechanisms are the only mechanisms satisfying individual-rationality, weak non-wastefulness, population-monotonicity, and strategy-proofness.*

We prove Theorem 1 in Appendix A.1. We prove the independence of the properties in Theorem 1 in Appendix A.2.

Interestingly, in the house allocation model (with fixed resources and a variable population) all properties of Theorem 1 minus *strategy-proofness* do not imply \succ -stability (even if we added *truncation-invariance*): the so-called Boston (or immediate-acceptance) algorithm defines a mechanism that satisfies all properties of Theorem 1 (and *truncation-invariance*) but *strategy-proofness* and it is not \succ -stable for any priority structure \succ (see also our later Remark 1).

We introduce an acyclicity condition which equals Ergin's (2002) acyclicity condition for the general model when adapted to house allocation problems.

Cycles and Acyclicity: Given a priority structure \succ a *cycle* consists of distinct $x, y \in O$ and $i, j, k \in S$ such that $i \succ_x j \succ_x k$ and $k \succ_y i$. A priority structure \succ is *acyclic* if it has no cycles. A responsive-deferred-acceptance-mechanism is *acyclic* if the associated priority structure is acyclic.

Ergin (2002, Theorem 1) shows that the acyclicity of the priority structure \succ is equivalent to *efficiency* or *group-strategy-proofness* of the induced responsive-deferred-acceptance-mechanism DA^\succ . Furthermore, with Kesten's (2009, Theorem 1) result it follows that for house allocation the

class of *efficient* responsive-deferred-acceptance-mechanisms equals the class of restricted endowment inheritance rules in Ehlers, Klaus, and Pápai (2002). Then, Theorem 1 implies the following characterizations of the subclass of acyclic responsive-deferred-acceptance-mechanisms.

Corollary 1.

(a) *In the house allocation model (with fixed resources and a variable population), responsive-deferred-acceptance-mechanisms with acyclic priority structures are the only mechanisms satisfying efficiency, population-monotonicity, and strategy-proofness.*

(b) *In the house allocation model (with fixed resources and a variable population), responsive-deferred-acceptance-mechanisms with acyclic priority structures are the only mechanisms satisfying individual-rationality, weak non-wastefulness, population-monotonicity, and group-strategy-proofness.*

Corollary 1 (a) equals Ehlers, Klaus, and Pápai's (2002) Theorem 1. We prove Corollary 1 in Appendix A.3.

The following example demonstrates that Theorem 1 does not hold for our more general model where more than one copy of each object might be available. This is even true if we added *truncation-invariance* to the properties in Theorem 1 because the mechanism below satisfies *truncation-invariance*. In addition we show that the example below is not \mathcal{C} -stable with an *acceptant* choice structure \mathcal{C} , *i.e.* even the more general approach does not yield a positive result.

Example 1. Let $N = \{1, 2, 3, 4\}$, $O = \{x, y\}$, and there are two copies of x and one copy of y available, *i.e.*, $q_x = 2$, $q_y = 1$ and $q = (q_x, q_y)$. We then denote a problem by $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$). The following mechanism f satisfies *individual-rationality*, *weak non-wastefulness*, *population-monotonicity*, and *strategy-proofness*.

Let $\succ_x: 1\ 2\ 3\ 4$, $\succ'_x: 1\ 2\ 4\ 3$, and $\succ_y: 1\ 2\ 3\ 4$. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Given R_i , let $\text{top}(R_i)$ denote agent i 's favorite object in $O \cup \{\emptyset\}$. Then, for each problem $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$),

$$f(R) = \begin{cases} DA^{\succ'}(R, q) & \text{if } \text{top}(R_1) = \text{top}(R_2) = y \text{ and} \\ DA^{\succ}(R, q) & \text{otherwise.} \end{cases}$$

Note that f is not \mathcal{C} -stable with an *acceptant* choice structure \mathcal{C} (and therefore, f is not a responsive-deferred-acceptance-mechanism): suppose that \mathcal{C} is an *acceptant* choice structure and f is \mathcal{C} -stable; if $R \in \mathcal{R}^N$ is such that $R_1 : y \emptyset x$ and $R_2 = R_3 = R_4 : x \emptyset y$, then $f(R) = DA^{\succ}(R, q)$, *i.e.*, agents 2 and 3 receive x and agent 4 receives \emptyset (not the desired x), implying $4 \notin \mathcal{C}_x(\{2, 3, 4\}, 2)$; if $R'_2 : y\ x\ \emptyset$ and $R' = (R'_2, R_{-2})$, then $f(R') = DA^{\succ'}(R', q)$, *i.e.*, agents 2 and 4 receive x and agent 3 receives \emptyset (not the desired x), implying $3 \notin \mathcal{C}_x(\{2, 3, 4\}, 2)$; but now $3, 4 \notin \mathcal{C}_x(\{2, 3, 4\}, 2)$ and \mathcal{C}_x is not *acceptant*, a contradiction. Note that this argument also establishes that there does not exist a collection of *acceptant* choice structures $(\hat{\mathcal{C}}^S)_{S \in \mathcal{N}}$ (one for each $S \in \mathcal{N}$) such that for any $R \in \mathcal{R}^S$

($S \in \mathcal{N}$), $f(R)$ is stable with respect to $\hat{\mathcal{C}}^S$: we would have to have both $3, 4 \notin \hat{\mathcal{C}}_x^N(\{2, 3, 4\}, 2)$ and $\hat{\mathcal{C}}_x^N$ is not *acceptant*, a contradiction.

It is easy to see that f satisfies *individual-rationality*, *weak non-wastefulness*, and *strategy-proofness* (and *truncation-invariance*).

For *population-monotonicity*, let $R \in \mathcal{R}^N$ and $T, S \in \mathcal{N}$ such that $T \subsetneq S$. It suffices to check for possible violations of *population-monotonicity* when f uses a different priority structure for R_S and R_T . Hence, $1, 2 \in S$, $\text{top}(R_1) = \text{top}(R_2) = y$, and $1 \notin T$ or $2 \notin T$. Thus, $f_1(R_S) = y$.

If $R_2 : y \emptyset x$, then $f_2(R_S) = \emptyset$ and *population-monotonicity* is satisfied if 1 or 2 leaves problem R_S (because both copies of x are available for the remaining agent(s)). Let $R_2 : y x \emptyset$. Then $f_2(R_S) = x$. If 1 leaves problem R_S and $2 \in T$, then $f_2(R_T) = y$ and again *population-monotonicity* is satisfied (because both copies of x are available for the remaining agent(s)). The case where 1 leaves problem R_S and $2 \notin T$ is trivial. If 2 leaves problem R_S and $1 \in T$, then $f_1(R_T) = y$ and again *population-monotonicity* is satisfied (because both copies of x are available for the remaining agent(s)). The case where 2 leaves problem R_S and $1 \notin T$ is trivial. \diamond

4 Resource-Monotonicity

Instead of *population-monotonicity* and unit capacities, we now consider *resource-monotonicity* and non-unit capacities with a fixed population and variable resources. Let N be fixed and consider fixed-population mechanisms $\varphi : \mathcal{R}^N \times \mathcal{Q} \rightarrow \mathcal{A}$.

We first show that *individual-rationality*, *weak non-wastefulness*, *resource-monotonicity*, and *truncation-invariance* imply stability with respect to some *acceptant*, *monotonic*, and *path-independent* choice structure.

Theorem 2. *If a mechanism φ satisfies unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, and truncation-invariance, then there exists a choice structure \mathcal{C} satisfying acceptance, monotonicity, and path-independence such that φ is \mathcal{C} -stable.*

The main characterization for our general allocation model is as follows.

Theorem 3. *Choice-deferred-acceptance-mechanisms (where the choice structure satisfies acceptance, monotonicity, and path-independence) are the only mechanisms satisfying unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.*

We prove Theorems 2 and 3 in Appendix A.4. We prove the independence of properties in Theorem 3 in Appendix A.5.

Note that the object-proposing choice-deferred-acceptance-mechanisms (where the choice structure satisfies acceptance, monotonicity, and path-independence) satisfy all properties of Theorem 2 and are \mathcal{C} -stable. However, they are not *strategy-proof*.

In the general setup Kumano (2009) introduced an acyclicity condition on choice structures which is necessary and sufficient for the induced choice-deferred-acceptance mechanisms to satisfy *efficiency* or *group-strategy-proofness*.

***-Cycles and *-Acyclicity:** Given a choice structure \mathcal{C} , a *-cycle consists of distinct $x, y \in O$, $l \in \{1, \dots, \bar{q}_x\}$, $m \in \{1, \dots, \bar{q}_y\}$, distinct $i, j, k \in N$ and disjoint $S_x, S_y \subseteq N \setminus \{i, j, k\}$ such that (i) $j \notin \mathcal{C}_x(S_x \cup \{i, j\}, l)$, $k \notin \mathcal{C}_x(S_x \cup \{j, k\}, l)$, and $i \notin \mathcal{C}_y(S_y \cup \{k, i\}, m)$ and (ii) both $|S_x| = l - 1$ and $|S_y| = m - 1$. A choice structure \mathcal{C} is *-acyclic if it has no *-cycles.

A choice-deferred-acceptance mechanism is *-acyclic if the associated choice structure is *-acyclic.

The result below follows from Kumano (2009, Theorem 1) and Kojima and Manea (2010, Proposition 1).

Corollary 2.

- (a) **-Acyclic choice-deferred-acceptance-mechanisms are the only mechanisms satisfying unavailable-type-invariance, efficiency, resource-monotonicity, and truncation-invariance.*
- (b) **-Acyclic choice-deferred-acceptance-mechanisms are the only mechanisms satisfying individual-rationality, weak non-wastefulness, resource-monotonicity, and group-strategy-proofness.*

The proof of Corollary 2 is in Appendix A.6.

5 Conclusion

5.1 Unit Capacities

First, note that for the house allocation model with fixed population and variable resources the following are immediate corollaries from Theorem 2 and Theorem 3.

Corollary 3. *In the house allocation model (with a fixed population and variable resources), if a mechanism satisfies unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, and truncation-invariance, then there exists a priority structure \succ such that the mechanism is \succ -stable.*

Note that in contrast to Theorem 1 we need *truncation-invariance* below.

Corollary 4. *In the house allocation model (with a fixed population and variable resources), responsive-deferred-acceptance-mechanisms are the only mechanisms satisfying unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, truncation-invariance, and strategy-proofness.*

We show the independence of the properties in Corollary 4 in Appendix A.7. Corollaries 3 and 4 were the main results of Ehlers and Klaus (2013).

Next, we briefly discuss two important real-life classes of mechanisms (for house allocation as well as for our more general model) and state their properties.

Remark 1. *The Boston or Immediate-Acceptance-Mechanism*

A mechanism that has also been playing an important role in market design is the so-called Boston or immediate-acceptance mechanism. Any immediate-acceptance mechanism is based on a priority structure and satisfies *individual-rationality*, *weak non-wastefulness*, *population-monotonicity*, and *truncation-invariance*. However, it is easy to check that the immediate-acceptance mechanism violates *strategy-proofness*, and *stability*. Whether an immediate-acceptance mechanism satisfies *unavailable-type-invariance* or *resource-monotonicity* depends on how available objects are treated in the first round of the immediate-acceptance-algorithm (if in round one agents have to apply to objects that are not available, then the resulting immediate-acceptance mechanism satisfies *resource-monotonicity* but not *unavailable-type-invariance*; if in round one agents do not apply to objects that are not available, then the resulting immediate-acceptance mechanism satisfies *unavailable-type-invariance* but not *resource-monotonicity*). \diamond

Remark 2. *Top-Trading Cycles Mechanisms*

Another mechanism that has also been playing an important role in market design is the so-called top-trading cycles mechanism (Shapley and Scarf, 1974). Any top-trading cycles (TTC) mechanism is based on a property rights structure (which is given by a priority structure) in order to determine endowments and satisfies *unavailable-type-invariance*, *individual-rationality*, *weak non-wastefulness*, *truncation-invariance*, and *strategy-proofness*. However, since priorities can be traded, it is easy to check that any TTC-mechanism violates *resource-monotonicity*, *population-monotonicity*, and *stability*. Any TTC-mechanism is *strategy-proof* and *efficient*. One can see easily that for the house allocation model for a given priority structure, the corresponding TTC-mechanism and DA-mechanism coincide if and only if the priority structure is acyclic. In other words, the TTC-mechanism satisfies the comparative statics properties if and only if it is a DA-mechanism with an acyclic priority structure. \diamond

By Theorem 1 and Corollary 4, in the house allocation model *population-monotonicity* is a “stronger” comparative statics requirement than *resource-monotonicity* because (i) *population-monotonicity* in conjunction with *strategy-proofness* and basic properties characterize responsive DA-mechanisms whereas (ii) *resource-monotonicity* in conjunction with *strategy-proofness* and basic properties does not characterize responsive DA-mechanisms. On the other hand, (i) *population-monotonicity* in conjunction with *truncation-invariance* and basic properties does not imply stability (Remark 1) whereas (ii) *resource-monotonicity* in conjunction with *truncation-invariance* and basic properties implies stability (Corollary 3). We summarize these findings in the table below

(where any mechanism φ satisfies our basic properties *individual-rationality*, *weak non-wastefulness*, and *unavailable-type-invariance*).

| φ is | stable | DA |
|---------------|------------|------------|
| PM and TI | <i>no</i> | <i>no</i> |
| PM and SP | <i>yes</i> | <i>yes</i> |
| RM | <i>no</i> | <i>no</i> |
| RM and TI | <i>yes</i> | <i>no</i> |
| RM, TI and SP | <i>yes</i> | <i>yes</i> |

Here RM stands for *resource-monotonicity*, PM stands for *population-monotonicity*, TI stands for *truncation-invariance*, and SP stands for *strategy-proofness*.

5.2 Non-Unit Capacities

In the general model *resource-monotonicity* is a “stronger” comparative statics requirement than *population-monotonicity* because (i) *resource-monotonicity* in conjunction with *strategy-proofness*, *truncation-invariance* and basic properties characterizes choice-based DA-mechanisms (Theorem 3) whereas (ii) *population-monotonicity* in conjunction with *strategy-proofness*, *truncation-invariance* and basic properties does not characterize choice-based DA-mechanisms (Example 1).

We now return to our general allocation model where more than one copy of each real type may be available and the set of agents may vary; a mechanism is a function $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \times \mathcal{Q} \rightarrow \mathcal{A}$.

The following are immediate corollaries from Theorems 2 and 3: if we add *population-monotonicity* to the properties in Theorems 2 and 3, then the choice structure will have to be the same across different populations.

Corollary 5. *If a mechanism φ satisfies unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, population-monotonicity, and truncation-invariance, then there exists a choice structure \mathcal{C} satisfying acceptance, monotonicity, and path-independence such that φ is \mathcal{C} -stable.*

Corollary 6. *Choice-deferred-acceptance-mechanisms (where the choice structure satisfies acceptance, monotonicity, and path-independence) are the only mechanisms satisfying unavailable-type-invariance, individual-rationality, weak non-wastefulness, resource-monotonicity, population-monotonicity, truncation-invariance, and strategy-proofness.*

It is important to notice that neither in Corollary 5 nor in Corollary 6 *truncation-invariance* can be dropped from the list of properties (even though both comparative statics properties are required). The following example establishes this fact.

Example 2 (Not truncation-invariant). Let $N = \{1, 2, 3, 4\}$, $O = \{x, y\}$, and there are at most two copies of x and at most one copy of y available, *i.e.*, $\bar{q}_x = 2$ and $\bar{q}_y = 1$.

Let $\succ_x: 1\ 2\ 3\ 4$, $\succ'_x: 1\ 2\ 4\ 3$, and $\succ_y: 1\ 2\ 3\ 4$. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each $S \in \mathcal{N}$ and each problem $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$,

$$\varphi(R, q) = \begin{cases} DA^\succ(R, q) & \text{if } q_y = 1, q_x = 2, S = N, \text{ and } R_1 = R_2 : y\ x\ \emptyset \text{ and} \\ DA^{\succ'}(R, q) & \text{otherwise.} \end{cases}$$

First, we show that φ violates *truncation-invariance*. Let $R \in \mathcal{R}^N$ be such that $R_1 = R_2 : y\ x\ \emptyset$, $R_3 = R_4 : x\ \emptyset\ y$, and $q_x = 2$ and $q_y = 1$. Then $\varphi_3(R, q) = x$ and $\varphi_4(R, q) = \emptyset$. However, for $R'_1 \in \mathcal{R}$ with $R'_1 : y\ \emptyset\ x$ and $R' = (R'_1, R_{-1})$ we have $\varphi_4(R', q) = x$ and $\varphi_3(R', q) = \emptyset$. Note that R'_1 is a truncation of R_1 and $\varphi(R, q) \neq \varphi(R', q)$, *i.e.*, φ violates *truncation-invariance*. Note that this argument also establishes that there exists no acceptant choice structure \mathcal{C} such that φ is \mathcal{C} -stable: (i) $\varphi_2(R, q) = x = \varphi_3(R, q)$ and $\varphi_4(R, q) = \emptyset$ imply $4 \notin \mathcal{C}_x(\{2, 3, 4\}, 2)$ and (ii) $\varphi_2(R', q) = x = \varphi_4(R', q)$ and $\varphi_3(R', q) = \emptyset$ imply $3 \notin \mathcal{C}_x(\{2, 3, 4\}, 2)$; now (i) and (ii) yield $|\mathcal{C}_x(\{2, 3, 4\}, 2)| < 2$ and \mathcal{C}_x is not acceptant.

Second, in order to see that φ satisfies *resource-monotonicity*, let $R \in \mathcal{R}^N$ and $q, q' \in \mathcal{Q}$ be such that $q \leq q'$ (*resource-monotonicity* is trivially satisfied when not all agents are present). If φ uses the same priority structure for (R, q) and (R, q') , then *resource-monotonicity* is satisfied. Thus, $q'_y = 1$, $q'_x = 2$, and $R_1 = R_2 : y\ x\ \emptyset$. Note that the difference in \succ_x and \succ'_x only matters for $R \in \mathcal{R}^N$ with $x \in A(R_3) \cap A(R_4)$. Then, for all the following specifications of q and q' we see that *resource-monotonicity* is satisfied (see table below).

| $q = (q_y, q_x)$ | $\varphi(R, q)$ | $\varphi(R, q')$ |
|------------------|--|------------------------|
| (0, 0) | $(\emptyset, \emptyset, \emptyset, \emptyset)$ | (y, x, x, \emptyset) |
| (1, 0) | $(y, \emptyset, \emptyset, \emptyset)$ | (y, x, x, \emptyset) |
| (0, 1) | $(x, \emptyset, \emptyset, \emptyset)$ | (y, x, x, \emptyset) |
| (1, 1) | $(y, x, \emptyset, \emptyset)$ | (y, x, x, \emptyset) |
| (0, 2) | $(x, x, \emptyset, \emptyset)$ | (y, x, x, \emptyset) |

Third, in order to see that φ satisfies *population-monotonicity*, let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and $S \subseteq N$ with $|S| = 2, 3$. If φ uses the same priority structure for (R, q) and (R_S, q) , then *population-monotonicity* is satisfied. Thus, $q_y = 1$, $q_x = 2$, and $R_1 = R_2 : y\ x\ \emptyset$. Furthermore (for different priorities) to be employed, we must have $3, 4 \in S$ and $S \in \{\{1, 3, 4\}, \{2, 3, 4\}, \{3, 4\}\}$.

For $S = \{3, 4\}$, $DA^\succ(R_S, q) = DA^{\succ'}(R_S, q)$ (because $\succ_y = \succ'_y$ and two copies of x are available). For $S = \{i, 3, 4\}$, $i \in \{1, 2\}$, $DA_i^\succ(R_S, q) = DA_i^{\succ'}(R_S, q) = y$, and $DA_3^\succ(R_S, q) = DA_3^{\succ'}(R_S, q)$, and $DA_4^\succ(R_S, q) = DA_4^{\succ'}(R_S, q)$ (because there are two copies of x for agents 3 and 4). Since the use of the two possible different priorities is outcome equivalent, *population-monotonicity* is satisfied. \diamond

Finally, we present an example of an acceptant and substitutable choice that is not responsive.

Example 3. Let $N = \{s_1, s_2, j_1, j_2\}$ be four economists looking for a new academic position. Furthermore, s_1, s_2 are seniors with specializations 1 or 2 and j_1, j_2 are juniors with specializations 1 or 2. The intuition behind the priorities that we define is the following. An economics department $x \in O$ has the following preferences for hiring. If only one economist can be hired the priority ranking for hiring is $s_1 \succ_x s_2 \succ_x j_1 \succ_x j_2$. However, if two positions can be filled, the department always would like to fill both positions. So, $\bar{q}_x = 2$. Furthermore, they would like to hire the two seniors; but if only one senior s_i can be hired, then they are interested in also hiring the junior j_i in the same field. To be more specific, we assume that the department's priority ranking for hiring two economists is $\{s_1, s_2\} \succ_x \{s_1, j_1\} \succ_x \{s_2, j_2\} \succ_x \{s_1, j_2\} \succ_x \{s_2, j_1\} \succ_x \{j_1, j_2\}$. Note that the department loosely speaking has lexicographic preferences. Priorities \mathcal{C}_x based on these preferences work as follows. For $q_x = 2$ (the case $\mathcal{C}_x(M, 1)$ is obvious) it follows that

$$\mathcal{C}_x(M, 2) = \begin{cases} M & \text{if } |M| \leq 2, \\ \{s_1, s_2\} & \text{if } \{s_1, s_2\} \subsetneq M, \\ \{s_i, j_i\} & \text{if } \{s_i, j_i\} \subsetneq M, i \in \{1, 2\}. \end{cases}$$

It is easily verified that \mathcal{C}_x is an acceptant, path independent and substitutable choice function. Since $\mathcal{C}_x(\{s_1, j_1, j_2\}, 2) = \{s_1, j_1\}$ and $\mathcal{C}_x(\{s_2, j_1, j_2\}, 2) = \{s_2, j_2\}$, \mathcal{C}_x is not responsive (because responsiveness would imply $j_1 \succ_x j_2 \Leftrightarrow \{s_2, j_1\} \succ_x \{s_2, j_2\}$). \diamond

Finally, we discuss the relation of our Theorem 3 with the main result in Ehlers and Klaus (2014, Theorem 1).

We denote a preference relation with only one acceptable type $x \in O$ by R^x , *i.e.*, $A(R^x) = \{x\}$. We denote the set of all preference relations that have $x \in O$ as the unique acceptable type by \mathcal{R}^x . A requirement which played a key role in Ehlers and Klaus (2014, Theorem 1) is the following.

Two-Agent Consistent Conflict Resolution: For all $R \in \mathcal{R}^N$, all $q, q' \in \mathcal{Q}$, and all $R^x \in \mathcal{R}^x$, if $R_i = R_j = R^x$ and $\{\varphi_i(R, q), \varphi_j(R, q)\} = \{\varphi_i(R, q'), \varphi_j(R, q')\} = \{x, \emptyset\}$, then for $k \in \{i, j\}$, $\varphi_k(R, q) = \varphi_k(R, q')$.

One can interpret this property as a weak tie-breaking property because it only imposes that the tie between two agents in very specific maximal conflict situations is broken consistently in favor of the same agent. It is obvious that in Example 3 the induced choice-based DA-mechanism violates this property (because the choice structure is not responsive).

Ehlers and Klaus (2014, Theorem 1) prove that responsive-deferred-acceptance-mechanisms are the only mechanisms satisfying *unavailable-type-invariance*, *individual rationality*, *weak non-wastefulness*, *two-agent consistent conflict resolution*, *truncation invariance*, and *strategy-proofness*. Hence, with Theorem 3 we demonstrate that by replacing *two-agent consistent conflict resolution* by *resource-monotonicity*, we obtain a larger class of mechanisms.

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A Appendix

A.1 Proof of Theorem 1

Recall that in the house allocation model at most one object per real type is available. It therefore suffices to denote a problem by $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$). Also recall that we consider variable-population mechanisms $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \rightarrow \mathcal{A}$ (with a fixed set of types O).

First, note that all responsive-deferred-acceptance mechanisms are stable and that *stability* implies *individual-rationality* and *weak non-wastefulness*. Dubins and Freedman (1981) and Roth (1982) proved *strategy-proofness* of all responsive-deferred-acceptance-mechanisms. Crawford (1991) studied comparative statics of deferred-acceptance-mechanisms (for a more general model than that of housing markets). From his results it follows that all responsive-deferred-acceptance-mechanisms are *population-monotonic*. Hence, responsive-deferred-acceptance-mechanisms satisfy all properties of Theorem 1.

Second, let φ be a mechanism satisfying *individual-rationality*, *weak non-wastefulness*, and *population-monotonicity* (we will add *strategy-proofness* later on). We construct a priority structure using maximal conflict preference profiles.

Let $x \in O$ and $R^x \in \mathcal{R}$ be such that $A(R^x) = \{x\}$. We denote the set of all preference relations that have $x \in O$ as the unique acceptable type by \mathcal{R}^x . For any $S \in \mathcal{N}$, let $R_S^x = (R_i^x)_{i \in S}$ such that for all $i \in S$, $R_i^x = R^x$.

Consider the problem R_N^x . By *weak non-wastefulness* and *individual-rationality*, for some $j \in N$, $\varphi_j(R_N^x) = x$, say $j = 1$. Then, for all $i \in N \setminus \{1\}$, we set $1 \succ_x i$.

Next consider the problem $R_{N \setminus \{1\}}^x$. By *weak non-wastefulness* and *individual-rationality*, for some $j \in N \setminus \{1\}$, $\varphi_j(R_{N \setminus \{1\}}^x) = x$, say $j = 2$. Then, for all $i \in N \setminus \{1, 2\}$, we set $2 \succ_x i$.

By induction, we obtain \succ_x for any real type x and thus a priority structure $\succ = (\succ_x)_{x \in O}$ for N . By *weak non-wastefulness*, *individual-rationality*, and *population-monotonicity*, for any $x \in O$ and any $i, j \in N$ such that $i \succ_x j$, $\varphi_i(R^x, R^x) = x$ and $\varphi_j(R^x, R^x) = \emptyset$.

Let mechanism φ satisfy *strategy-proofness*. With the following step, Theorem 1 follows. Note that mechanism DA^\succ satisfies all properties of Theorem 1.

Let $S \in \mathcal{N}$, $R \in \mathcal{R}^S$, and denote by $Z(R) := |\{i \in S : |A(R_i)| \leq 1\}|$ the number of agents who find at most one object acceptable. Assume that $R \in \mathcal{R}^S$ is such that $\varphi(R) \neq DA^\succ(R)$ and that $Z(R)$ is maximal, *i.e.*, for all $R' \in \mathcal{R}^S$ such that $\varphi(R') \neq DA^\succ(R')$, $Z(R) \geq Z(R')$.

We first show that for all $i \in S$ such that $\varphi_i(R) \neq DA_i^\succ(R)$, $|A(R_i)| = 1$. If $\varphi_i(R) P_i DA_i^\succ(R)$, by *individual-rationality*, $\varphi_i(R) := x \in O$. If $|A(R_i)| = 1$, then we are done. If $|A(R_i)| > 1$, then consider $(R^x, R_{-i}) \in \mathcal{R}^S$ where R^x as in the construction of \succ . By *strategy-proofness* (and *individual-rationality*), $\varphi_i(R^x, R_{-i}) = x$ and $DA_i^\succ(R^x, R_{-i}) = \emptyset$. Hence, (R^x, R_{-i}) is such that $\varphi(R^x, R_{-i}) \neq DA^\succ(R^x, R_{-i})$ and $Z(R^x, R_{-i}) > Z(R)$; contradicting our assumption that $Z(R)$ was maximal. If $DA_i^\succ(R) P_i \varphi_i(R)$, then $DA_i^\succ(R) := x \in O$ and the proof that $|A(R_i)| = 1$

proceeds as before (with DA^\succ in the role of φ and vice versa). Hence, for all $i \in S$ such that $\varphi_i(R) \neq DA_i^\succ(R)$, $|A(R_i)| = 1$. By *strategy-proofness*, we can assume that for all $i \in S$ such that $\varphi_i(R) \neq DA_i^\succ(R)$, $R_i = R^x$ where R^x is as in the construction of \succ .

If $x = DA_i^\succ(R) P_i \varphi_i(R) = \emptyset$, then by *weak non-wastefulness* there exists $j \in S \setminus \{i\}$ such that $\varphi_j(R) = x$. Hence, $\varphi_j(R) \neq DA_j^\succ(R)$ and $R_j = R^x$ where R^x is as in the construction of \succ . Thus, $(R_i, R_j) = (R^x, R^x)$ and $x = \varphi_j(R) P_j DA_j^\succ(R) = \emptyset$. Now $DA_i^\succ(R) = x$ and *population-monotonicity* (and *individual-rationality*) for DA^\succ imply $DA_i^\succ(R_i, R_j) = x$ and $DA_j^\succ(R_i, R_j) = \emptyset$. Hence, $i \succ_x j$. Next, $\varphi_j(R) = x$ and *population-monotonicity* (and *individual-rationality*) for φ imply $\varphi_j(R_i, R_j) = x$ and $\varphi_i(R_i, R_j) = \emptyset$. Hence, $j \succ_x i$; a contradiction.

If $x = \varphi_i(R) P_i DA_i^\succ(R) = \emptyset$, then by *weak non-wastefulness* there exists $j \in S \setminus \{i\}$ such that $DA_j^\succ(R) = x$. Hence, we obtain a contradiction as in the previous step.

A.2 Independence of Properties in Theorem 1

Recall that we denote a problem by $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and that we consider variable-population mechanisms $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \rightarrow \mathcal{A}$ (with a fixed set of types O). Without loss of generality we assume $N = \{1, \dots, n\}$ to establish independence.

For any strict order π of agents in N , we denote the corresponding *serial dictatorship mechanism* by f^π ; for example, if $\pi : 1 \ 2 \ \dots \ (n-1) \ n$, then f^π works as follows: for each problem $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$), first agent $\min S$ chooses his preferred object in $O \cup \{\emptyset\}$, then agent $\min(S \setminus \{\min S\})$ chooses his preferred object from the remaining objects $(O \setminus \{f_{\min S}^\pi(R)\}) \cup \{\emptyset\}$, etc. Note that for each strict order π of N , $f^\pi = DA^{\succ^\pi}$ where \succ^π equals the strict order π , i.e. for all $x \in O$, $\succ_x^\pi = \pi$. Thus, each serial dictatorship mechanism f^π satisfies *individual-rationality*, *weak non-wastefulness*, *population-monotonicity*, *truncation-invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily) in Theorem 1.

Not individually-rational: Let $\pi : 1 \ 2 \ \dots \ (n-1) \ n$. For each $R_n \in \mathcal{R}$, let \hat{R}_n be such that $A(\hat{R}_n) = O$ and $\hat{R}_n|O = R_n|O$. Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$),

$$\varphi(R) = \begin{cases} f^\pi(R_{-n}, \hat{R}_n) & \text{if } n \in S \text{ and} \\ f^\pi(R) & \text{otherwise.} \end{cases}$$

Not weakly non-wasteful: Fix a type $y \in O$ and $\pi : 1 \ 2 \ \dots \ (n-1) \ n$. Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$),

$$\varphi(R) = f^\pi(R, O \setminus \{y\}).$$

Not population-monotonic:¹⁰ Let π and π' be two distinct strict orders of agents in N . Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$),

$$\varphi(R) = \begin{cases} f^\pi(R) & \text{if } S = N \text{ and} \\ f^{\pi'}(R) & \text{otherwise.} \end{cases}$$

Not strategy-proof: Let \succ be a priority structure. Then, the responsive-deferred-acceptance-mechanism based on the type-optimal stable allocation that is obtained by using Gale and Shapley's (1962) type-proposing deferred-acceptance-algorithm satisfies all properties except *strategy-proofness*.

A.3 Proof of Corollary 1

Acyclic responsive-deferred-acceptance mechanisms satisfy all the properties listed in the corollary.

(a) Let φ be a mechanism satisfying *efficiency*, *population-monotonicity*, and *strategy-proofness*.

Efficiency implies *individual-rationality* and *weak non-wastefulness*. Then, by Theorem 1, there exists a priority structure \succ such that $\varphi = DA^\succ$. Finally, by *efficiency* and Ergin (2002, Theorem 1), \succ must be acyclic.

(b) Let φ be a mechanism satisfying *individual-rationality*, *weak non-wastefulness*, *population-monotonicity*, and *group-strategy-proofness*.

Group-strategy-proofness implies *strategy-proofness*. Hence, by Theorem 1, there exists a priority structure \succ such that $\varphi = DA^\succ$. Finally, by *group-strategy-proofness* and Ergin (2002, Theorem 1), \succ must be acyclic.

A.4 Proof of Theorems 2 and 3

Let $\mathcal{C} = (\mathcal{C}_x)_{x \in O}$ be an acceptant, monotonic, and path-independent (hence, substitutable and consistent) choice structure. First, note that the corresponding choice-deferred-acceptance-mechanism is \mathcal{C} -stable and that \mathcal{C} -stability implies *individual-rationality* and *weak non-wastefulness*. Furthermore, it is easy to check that $DA^{\mathcal{C}}$ satisfies *unavailable-type-invariance* and *truncation-invariance*. Hatfield and Milgrom (2005) proved *strategy-proofness* of the choice-deferred-acceptance-mechanisms based on an acceptant, substitutable, and consistent choice structure \mathcal{C} . The most general result implying *resource-monotonicity* (and *population-monotonicity*) for choice-deferred-acceptance-mechanism based on path-independent choice structures \mathcal{C} we are aware of is Chambers and Yenmez (2014, Corollary 1).¹¹ Hence, choice-deferred-acceptance-mechanisms that are based on acceptant, monotonic, and path-independent choice structures satisfy all properties of Theorem 3 (and hence of Theorem 2).

¹⁰Again note that here we could have chosen the TTC-mechanism as an example satisfying all properties in Theorem 1 except for *population-monotonicity*.

¹¹Chambers and Yenmez (2014, Theorem 2) extend the previous comparative statics results obtained by Crawford (1991, Theorems 1 and 2).

Second, let φ be a mechanism satisfying *unavailable-type-invariance*, *individual-rationality*, *weak non-wastefulness*, *resource-monotonicity*, and *truncation-invariance* (we will first complete the proof of Theorem 2 and then add *strategy-proofness* later on to complete the proof of Theorem 3). We construct a choice structure by using maximal conflict preference profiles.

For any $x \in O$ let $R^x \in \mathcal{R}$ be such that $A(R^x) = \{x\}$. For any $S \in \mathcal{N}$, let $R_S^x = (R_i^x)_{i \in S}$ such that for all $i \in S$, $R_i^x = R^x$. Given $l \in \{1, \dots, \bar{q}_x\}$, let l_x denote the capacity vector where l copies of x are available and no copy of any other real object is available. Let $R^\emptyset \in \mathcal{R}$ be such that $A(R^\emptyset) = \emptyset$. For any $S \in \mathcal{N}$, let $R_S^\emptyset = (R_i^\emptyset)_{i \in S}$ such that for all $i \in S$, $R_i^\emptyset = R^\emptyset$.

For any $x \in O$, let $\mathcal{C}_x : \mathcal{N} \times \{1, \dots, \bar{q}_x\} \rightarrow \mathcal{N}$ be such that for all $S \in \mathcal{N}$ and $l \in \{1, \dots, \bar{q}_x\}$, $\mathcal{C}_x(S, l) = \{i \in S : \varphi_i(R_S^x, R_{N \setminus S}^\emptyset, l_x) = x\}$ denotes the choice made at the maximal conflict problem (R_S^x, l_x) . By *weak non-wastefulness*, \mathcal{C}_x is *acceptant*, i.e., $|\mathcal{C}_x(S, l)| = \min\{|S|, l\}$. By Lemma 1 (a), \mathcal{C}_x satisfies the *law of aggregate demand* (LAD).

Lemma 2. *For all $x \in O$, \mathcal{C}_x is substitutable.*

Proof. Let $x \in O$. Let $s, s' \in S$ with $s \neq s'$ and $l \in \{1, \dots, \bar{q}_x\}$. To show substitutability of \mathcal{C}_x we need to show that $s \in \mathcal{C}_x(S, l)$ implies $s \in \mathcal{C}_x(S \setminus \{s'\}, l)$. By definition, $\mathcal{C}_x(S, l) = \{i \in S : \varphi_i(R_S^x, R_{N \setminus S}^\emptyset, l_x) = x\}$ and $\mathcal{C}_x(S \setminus \{s'\}, l) = \{i \in S \setminus \{s'\} : \varphi_i(R_{S \setminus \{s'\}}^x, R_{(N \setminus S) \cup \{s'\}}^\emptyset, l_x) = x\}$.

Let $y \in O \setminus \{x\}$ be such that $R_{s'} : y \succ x \succ \emptyset \dots$ and $R_{s'}|_{O \setminus \{y\}} = R_{s'}^x|_{O \setminus \{y\}}$, i.e., we obtain $R_{s'}$ from $R_{s'}^x$ by moving object y on top of agent s' 's preferences. By *unavailable-type-invariance*,

$$\varphi(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, R_{s'}, l_x) = \varphi(R_S^x, R_{N \setminus S}^\emptyset, l_x).$$

We now add object y . Let $l_x 1_y$ denote the capacity vector where l copies of object x are available and one copy of object y (and no copies of any other object). By *resource-monotonicity* and $\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, R_{s'}, l_x) = \varphi_s(R_S^x, R_{N \setminus S}^\emptyset, l_x) = x$, we have $\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, R_{s'}, l_x 1_y) = x$. By *weak non-wastefulness* and *individual-rationality*, $\varphi_{s'}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, R_{s'}, 1_y) = y$ which, together with *resource-monotonicity*, implies $\varphi_{s'}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, R_{s'}, l_x 1_y) = y$.

Next, let $\hat{R}_{s'} \in \mathcal{R}$ be such that $\hat{R}_{s'} : y \succ \emptyset \succ x \dots$, and $\hat{R}_{s'}|_O = R_{s'}|_O$, i.e., we obtain $\hat{R}_{s'}$ from $R_{s'}$ by moving the null object \emptyset between objects y and x . Note that $\hat{R}_{s'}$ is a truncation of $R_{s'}$. Hence, by *truncation-invariance*,

$$\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x 1_y) = x \text{ and } \varphi_{s'}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x 1_y) = y.$$

We now remove (the added) object y . By *individual-rationality*, $\varphi_{s'}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) = \emptyset$ and $\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) \in \{x, \emptyset\}$. If $\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) = \emptyset$, then by *weak non-wastefulness* and *individual-rationality* there exists $s'' \in S$ with preferences $R_{s''}^x$ such that $\varphi_{s''}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) = x$ and $\varphi_{s''}(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x 1_y) = \emptyset$; contradicting *resource-monotonicity*. Thus, $\varphi_s(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) = x$. By *unavailable-type-invariance*,

$$\varphi(R_{S \setminus \{s'\}}^x, R_{N \setminus S}^\emptyset, \hat{R}_{s'}, l_x) = \varphi(R_{S \setminus \{s'\}}^x, R_{(N \setminus S) \cup \{s'\}}^\emptyset, l_x).$$

Hence, $\varphi_s(R_{S \setminus \{s'\}}^x, R_{(N \setminus S) \cup \{s'\}}^\emptyset, l_x) = x$. Thus, $s \in \mathcal{C}_x(S \setminus \{s'\}, l)$. \square

So far we have proven that for all $x \in O$, \mathcal{C}_x is *substitutable* and satisfies the *law of aggregate demand* (LAD). By Lemma 1 (b), for all $x \in O$, \mathcal{C}_x is *consistent*. Hence, by Lemma 1 (c), for all $x \in O$, \mathcal{C}_x is *path-independent*. Furthermore, by *resource-monotonicity*, \mathcal{C}_x is *monotonic*, *i.e.*, $\mathcal{C}_x(S, l) \subseteq \mathcal{C}_x(S, l + 1)$.

Let mechanism φ satisfy *strategy-proofness*. With the following lemma, Theorem 2 follows.

Lemma 3. *For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q)$ is \mathcal{C} -stable.*

Proof. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Assume that $\varphi(R, q)$ is not \mathcal{C} -stable.

By *individual-rationality*, for all $i \in N$, $\varphi_i(R, q) \cap R_i = \emptyset$. Since for all $x \in O_+(q) \cup \{\emptyset\}$, \mathcal{C}_x is acceptant, we have $\mathcal{C}_x(\varphi(R, q)(x), q_x) = \varphi(R, q)(x)$ (recall that $\varphi(R, q)(x)$ denotes the set of agents that receive x at $\varphi(R, q)$). Hence, (s1) in the definition of \mathcal{C} -stability is satisfied.

Thus, (s2) in the definition of \mathcal{C} -stability is violated. Then, there exists an agent-type pair $(i, x) \in N \times O_+(q)$ such that $x \in \varphi_i(R, q)$ and $i \in \mathcal{C}_x(\varphi(R, q)(x) \cup \{i\}, q_x)$.

Let $\bar{R} \in \mathcal{R}^N$ be such that (i) for all $j \in N$ such that $\varphi_j(R, q) \neq \emptyset$, \bar{R}_j is a truncation of R_j such that there exists no $y \in O \setminus \{\varphi_j(R, q)\}$ with $\varphi_j(R, q) \bar{R}_j y \bar{R}_j \emptyset$ and (ii) for all $j \in N$ such that $\varphi_j(R, q) = \emptyset$, $\bar{R}_j = R_j$. (By *individual-rationality*, \bar{R}_j in (i) is well-defined as truncation of R_j .) By *truncation-invariance*, $\varphi(\bar{R}, q) = \varphi(R, q)$ and $(i, x) \in N \times O_+(q)$ is such that $x \in \bar{P}_i \varphi_i(\bar{R}, q)$ and $i \in \mathcal{C}_x(\varphi(\bar{R}, q)(x) \cup \{i\}, q_x)$.

Let $T = \{j \in N : x \in \bar{P}_j \varphi_j(\bar{R}, q)\}$. Note that $i \in T$. Let l_{q_x} denote the capacity vector where q_x copies of x are available and no copy of any other real object is available.

If for $j \in T$, $\varphi_j(\bar{R}, l_{q_x}) = x$, then by *resource-monotonicity*, $\varphi_j(\bar{R}, q) \bar{R}_j x$, contradicting $x \in \bar{P}_j \varphi_j(\bar{R}, q)$. Hence, for all $j \in T$, $\varphi_j(\bar{R}, l_{q_x}) = \emptyset$. If $|\{j \in N \setminus T : x \in \bar{P}_j \emptyset\}| < q_x$, then this contradicts *weak non-wastefulness*. Thus, $T \subsetneq S := \{j \in N : x \in \bar{P}_j \emptyset\}$ and $|S| \geq |T| + q_x$.

By construction of \bar{R} , for all $j \in N$ we have either (i) $x \in \bar{P}_j \varphi_j(\bar{R}, q)$ (and $j \in T$) or (ii) $\varphi_j(\bar{R}, q) = x$ or (iii) $\emptyset \in \bar{P}_j x$. Then, (i), (ii), and *individual-rationality* imply $|S| = |T| + q_x$ and there exist distinct $k_1, \dots, k_{q_x} \in N \setminus T$ such that $S = T \cup \{k_1, \dots, k_{q_x}\}$ and $\varphi_{k_1}(\bar{R}, q) = \dots = \varphi_{k_{q_x}}(\bar{R}, q) = x$. In other words, $\varphi(\bar{R}, q)(x) = \{k_1, \dots, k_{q_x}\}$.

Next, the fact that for all $j \in T$, $\varphi_j(\bar{R}, l_{q_x}) = \emptyset$, together with *weak non-wastefulness* and *individual-rationality*, implies that $\varphi_{k_1}(\bar{R}, l_{q_x}) = \dots = \varphi_{k_{q_x}}(\bar{R}, l_{q_x}) = x$. By *unavailable-type-invariance*, $\varphi((R_S^x, R_{N \setminus S}^\emptyset), l_{q_x}) = \varphi(\bar{R}, l_{q_x})$. Hence, $\mathcal{C}_x(S, q_x) = \{k_1, \dots, k_{q_x}\}$. By *substitutability*, $\mathcal{C}_x(\{k_1, \dots, k_{q_x}, i\}, q_x) = \{k_1, \dots, k_{q_x}\}$; contradicting $i \in \mathcal{C}_x(\varphi(\bar{R}, q)(x) \cup \{i\}, q_x) = \mathcal{C}_x(\{k_1, \dots, k_{q_x}, i\}, q_x)$. \square

So far we have established that for any mechanism φ that satisfies the properties of Theorem 2, there exists a choice structure \mathcal{C} (where the choice structure satisfies acceptance, monotonicity, and path-independence) such that for any $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q)$ is \mathcal{C} -stable. Hence, in the

terminology of two-sided matching, the mechanism φ picks a stable matching for the many-to-one market where types have preferences over agents who consume the types based on the choice structure \mathcal{C} and agents have strict preferences over types based on preferences R (see Hatfield and Milgrom, 2005). For these markets it is well-known that the choice-deferred-acceptance-mechanism is the only *strategy-proof* stable matching mechanism. Hence, Theorem 3 follows immediately from Theorem 2.

A.5 Independence of Properties in Theorem 3 and Corollary 6

We denote a problem by (R, q) where $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and $q \in \mathcal{Q}$. We consider variable-resource and variable-population mechanisms $\varphi : (\cup_{S \in \mathcal{N}} \mathcal{R}^S) \times \mathcal{Q} \rightarrow \mathcal{A}$. Without loss of generality we assume $N = \{1, \dots, n\}$ to establish independence.

For any strict order π of agents in N , we denote the corresponding *serial dictatorship mechanism* by f^π ; for example, if $\pi : 1 \ 2 \ \dots \ (n-1) \ n$, then f^π works as follows: for each problem $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$), first agent $\min S$ chooses his preferred object in $O_+(q) \cup \{\emptyset\}$, then agent $\min(S \setminus \{\min S\})$ chooses his preferred object among the remaining available objects (respecting the capacity constraints of q), *etc.* Note that for each strict order π of N , $f^\pi = DA^{\succ^\pi}$ where \succ^π equals the strict order π , *i.e.* for all $x \in O$, $\succ_x^\pi = \pi$. Thus, each serial dictatorship mechanism f^π satisfies *unavailable-type-invariance*, *individual-rationality*, *weak non-wastefulness*, *resource-monotonicity*, *population-monotonicity*, *truncation-invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily) in Corollary 6 and Theorem 3 (by restricting any mechanism to the domain $\mathcal{R}^N \times \mathcal{Q}$).

Not *unavailable-type-invariant*: Let $N = \{1, 2, 3\}$, $O = \{x, y\}$, and there are at most two copies of x and at most one copy of y available, *i.e.*, $\bar{q}_x = 2$ and $\bar{q}_y = 1$.

Let $\succ_x : 1 \ 2 \ 3$, $\succ'_x : 1 \ 3 \ 2$, and $\succ_y : 1 \ 2 \ 3$. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each $S \in \mathcal{N}$ and each problem $(R, q) \in \mathcal{R}^S \times \mathcal{Q}$,

$$\varphi(R, q) = \begin{cases} DA^\succ(R, q) & \text{if } q_y = 0, 1 \in S, \text{ and } R_1 : y \ x \ \emptyset \text{ and} \\ DA^{\succ'}(R, q) & \text{otherwise.} \end{cases}$$

Note that if for $R \in \mathcal{R}^N$ we have $R_1 : y \ x \ \emptyset$, $A(R_2) = A(R_3) = \{x\}$, and $q_x = 2$ and $q_y = 0$, then $\varphi_1(R, q) = x = \varphi_2(R, q)$ and $\varphi_3(R, q) = \emptyset$ whereas for $R'_1 \in \mathcal{R}$ with $A(R'_1) = \{x\}$ and $R' = (R'_1, R_{-1})$ we have $\varphi_1(R', q) = x = \varphi_3(R', q)$ and $\varphi_2(R', q) = \emptyset$. Thus, $R|_{O_+(q) \cup \{\emptyset\}} = R'|_{O_+(q) \cup \{\emptyset\}}$ and $\varphi(R, q) \neq \varphi(R', q)$, *i.e.*, φ violates *unavailable-type-invariance*. Note that this argument also establishes that there exists no acceptant choice structure \mathcal{C} such that φ is \mathcal{C} -stable: (i) $\varphi_1(R, q) = x = \varphi_2(R, q)$ and $\varphi_3(R, q) = \emptyset$ imply $3 \notin \mathcal{C}_x(\{1, 2, 3\}, 2)$ and (ii) $\varphi_1(R', q) = x = \varphi_3(R', q)$ and

$\varphi_2(R', q) = \emptyset$ imply $2 \notin \mathcal{C}_x(\{1, 2, 3\}, 2)$; now (i) and (ii) yield $|\mathcal{C}_x(\{1, 2, 3\}, 2)| < 2$ and \mathcal{C}_x is not acceptant.

In order to see that φ satisfies *resource-monotonicity*, let $R \in \mathcal{R}^N$ and $q, q' \in \mathcal{Q}$ be such that $q \leq q'$ (*resource-monotonicity* is trivially satisfied when not all agents are present). If φ uses the same priority structure for (R, q) and (R, q') , then *resource-monotonicity* is satisfied. Thus, $q_y = 0 < 1 = q'_y$ and $R_1 : y \ x \ \emptyset$. Note that the difference in \succ_x and \succ'_x only matters for $R \in \mathcal{R}^N$ with $x \in A(R_2) \cap A(R_3)$. Then, for all the following specifications of q and q' we see that *resource-monotonicity* is satisfied (see table below).

| $q = (q_y, q_x)$ | $q' = (q'_y, q'_x)$ | $\varphi(R, q)$ | $\varphi(R, q')$ |
|------------------|---------------------|-------------------------------------|-----------------------------|
| (0, 0) | (1, 0) | $(\emptyset, \emptyset, \emptyset)$ | $(y, \emptyset, \emptyset)$ |
| (0, 0) | (1, 1) | $(\emptyset, \emptyset, \emptyset)$ | (y, \emptyset, x) |
| (0, 0) | (1, 2) | $(\emptyset, \emptyset, \emptyset)$ | (y, x, x) |
| (0, 1) | (1, 1) | $(x, \emptyset, \emptyset)$ | (y, \emptyset, x) |
| (0, 1) | (1, 2) | $(x, \emptyset, \emptyset)$ | (y, x, x) |
| (0, 2) | (1, 2) | (x, x, \emptyset) | (y, x, x) |

In order to see that φ satisfies *population-monotonicity*, let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and $S \subseteq N$ with $|S| = 2$. If φ uses the same priority structure for (R, q) and (R_S, q) , then *population-monotonicity* is satisfied. Thus, $q_y = 0$, $R_1 : y \ x \ \emptyset$, and $1 \notin S$. Hence, $S = \{2, 3\}$. For $q_x = 2$, *population-monotonicity* is satisfied. For $q_x = 1$, $\varphi_1(R, q) = x$ and agent 1 leaving also implies *population-monotonicity*.

Not individually-rational: Let $\pi : 1 \ 2 \ \dots \ (n-1) \ n$. For each $S \in \mathcal{N}$ and each $R \in \mathcal{R}^S$, if $n \in S$, then let \hat{R}_n be such that $A(\hat{R}_n) = O$ and $\hat{R}_n|O = R_n|O$. Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and $q \in \mathcal{Q}$,

$$\varphi(R, q) = \begin{cases} f^\pi((R_{-n}, \hat{R}_n), q) & \text{if } n \in S \text{ and} \\ f^\pi(R, q) & \text{otherwise.} \end{cases}$$

Not weakly non-wasteful: Let $\pi : 1 \ 2 \ \dots \ (n-1) \ n$. Fix a type $y \in O$ and for each $q \in \mathcal{Q}$ define $\hat{q} \in \mathcal{Q}$ such that $\hat{q}_y = 0$ and for all $x \in O \setminus \{y\}$, $\hat{q}_x = q_x$. Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and $q \in \mathcal{Q}$,

$$\varphi(R, q) = f^\pi(R, \hat{q}).$$

Not resource-monotonic: Let π and π' be two distinct strict orders of agents in N . Fix a capacity vector $\tilde{q} \in \mathcal{Q}$. Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and $q \in \mathcal{Q}$,

$$\varphi(R, q) = \begin{cases} f^\pi(R, q) & \text{if } q = \tilde{q} \text{ and} \\ f^{\pi'}(R, q) & \text{otherwise.} \end{cases}$$

Not population-monotonic: Let π and π' be two distinct strict orders of agents in N . Then, for each $R \in \mathcal{R}^S$ ($S \in \mathcal{N}$) and $q \in \mathcal{Q}$,

$$\varphi(R) = \begin{cases} f^\pi(R, q) & \text{if } S = N \text{ and} \\ f^{\pi'}(R, q) & \text{otherwise.} \end{cases}$$

Not *strategy-proof*: Let \succ be a priority structure. Then, the responsive-deferred-acceptance-mechanism based on the type-optimal stable allocation that is obtained by using Gale and Shapley's (1962) type-proposing deferred-acceptance-algorithm satisfies all properties except *strategy-proofness*.

Not *truncation-invariant*: Example 2 establishes the independence of *truncation-invariance*.

A.6 Proof of Corollary 2

*-Acyclic choice-deferred-acceptance mechanisms satisfy all the properties listed in the corollary.

(a) Let φ be a mechanism satisfying *unavailable-type-invariance*, *efficiency*, *resource-monotonicity*, and *truncation-invariance*. Since *efficiency* implies *individual-rationality* and *weak non-wastefulness*, by Theorem 2, there exists a choice structure \mathcal{C} such that φ is \mathcal{C} -stable. Since the choice-deferred-acceptance-mechanism Pareto dominates any other stable mechanism (Kumano, 2009, Proposition 2), efficiency implies that $\varphi = DA^{\mathcal{C}}$. Finally, by *efficiency* and Kumano (2009, Theorem 1), \mathcal{C} must be *-acyclic.

(b) Let φ be a mechanism satisfying *individual-rationality*, *weak non-wastefulness*, *resource-monotonicity*, and *group-strategy-proofness*. By Ehlers and Klaus (2004, Lemma 1), *group-strategy-proofness* implies *unavailable-type-invariance*. Furthermore, *group-strategy-proofness* implies *truncation-invariance* and *strategy-proofness*. Hence, by Theorem 3, there exists a choice structure \mathcal{C} such that $\varphi = DA^{\mathcal{C}}$. Finally, by *group-strategy-proofness* and Kumano (2009, Theorem 1), \mathcal{C} must be *-acyclic.

A.7 Independence of Properties in Corollary 4

Recall that $S \in \mathcal{N}$ is fixed. Without loss of generality we assume $S = N = \{1, \dots, n\}$ to establish independence and we consider fixed-population mechanisms $\varphi : \mathcal{R}^N \times \mathcal{Q}_1 \rightarrow \mathcal{A}$. Again to simplify notation, we will at times denote a problem by (R, H) where $H \subseteq O$ denotes the real objects that are available with capacity 1.

For any strict order π of agents in N , we denote the corresponding *serial dictatorship mechanism* by f^π ; for example, if $\pi : 1 \ 2 \ \dots \ (n-1) \ n$, then f^π works as follows: for each problem (R, H) , first agent 1 chooses his preferred object in $H \cup \{\emptyset\}$, then agent 2 chooses his preferred object from the remaining objects $(H \setminus \{f_1^\pi(R, H)\}) \cup \{\emptyset\}$, etc. Note that for each strict order π of N , $f^\pi = DA^{\succ^\pi}$ where \succ^π equals the strict order π , i.e. for all $x \in O$, $\succ_x^\pi = \pi$. Thus, each serial dictatorship mechanism f^π satisfies *unavailable-type-invariance*, *individual-rationality*, *weak non-wastefulness*, *resource-monotonicity*, *truncation-invariance*, and *strategy-proofness*.

The following examples establish the independence of the properties (properties not mentioned in the examples follow easily) in Corollary 4.

Not *unavailable-type-invariant*: Let $n \geq 3$ and $\pi : 1\ 2\ 3 \dots (n-1)\ n$ and $\pi' : 1\ n\ (n-1) \dots 3\ 2$. Then, for each $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$),

$$\varphi(R, H) = \begin{cases} f^\pi(R, H) & \text{if } A(R_1) = \emptyset \text{ and} \\ f^{\pi'}(R, H) & \text{otherwise.} \end{cases}$$

Not *individually-rational*: Let $\pi : 1\ 2 \dots (n-1)\ n$. For each $R_n \in \mathcal{R}$, let \hat{R}_n be such that $A(\hat{R}_n) = O$ and $\hat{R}_n|O = R_n|O$. Then, for each $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$),

$$\varphi(R, H) = f^\pi((R_{-n}, \hat{R}_n), H).$$

Not *weakly non-wasteful*: Fix a type $y \in O$ and $\pi : 1\ 2 \dots (n-1)\ n$. Then, for each $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$),

$$\varphi(R, H) = f^\pi(R, H \setminus \{y\}).$$

Not *resource-monotonic*:¹² Let π and π' be two distinct strict orders of agents in N . Then, for each $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$),

$$\varphi(R, H) = \begin{cases} f^\pi(R, H) & \text{if } H = O \text{ and} \\ f^{\pi'}(R, H) & \text{otherwise.} \end{cases}$$

Not *strategy-proof*: Let \succ be a priority structure. Then, the responsive-deferred-acceptance-mechanism based on the type-optimal stable allocation that is obtained by using Gale and Shapley's (1962) type-proposing deferred-acceptance-algorithm satisfies all properties except *strategy-proofness*.¹³

Not *truncation-invariant*: Let $N = \{1, 2, 3\}$, $O = \{x, y\}$, $\succ_x : 1\ 2\ 3$, $\succ'_x : 1\ 3\ 2$, and $\succ_y : 1\ 2\ 3$. Let $\succ = (\succ_x, \succ_y)$ and $\succ' = (\succ'_x, \succ_y)$. Then, for each $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$),

$$\varphi(R, H) = \begin{cases} DA^\succ(R, H) & \text{if } \emptyset P_1 x \text{ and } x \in H \text{ and} \\ DA^{\succ'}(R, H) & \text{otherwise.} \end{cases}$$

Let $R_1 : y \emptyset x$, $R_2 : x \emptyset y$, $R_3 : x \emptyset y$, and $R'_1 : y x \emptyset$. Let $R = (R_1, R_2, R_3)$ and $R' = (R'_1, R_2, R_3)$. Note that R_1 is a truncation of R'_1 and $\varphi_1(R, \{x, y\}) = y = \varphi_1(R', \{x, y\})$. However, $\varphi_2(R, \{x, y\}) = x$ and $\varphi_3(R', \{x, y\}) = x$, *i.e.*, φ violates *truncation-invariance*. Next, we show *strategy-proofness* and *resource-monotonicity* for this mechanism.

¹²Note that here we could have chosen the TTC-mechanism as an example satisfying all properties in Theorem 4 except for *resource-monotonicity*.

¹³Given \succ , there exist many "artificial" stable mechanisms satisfying all properties except *strategy-proofness*. For instance, let $v \in \{1, \dots, |O| - 1\}$. For all $R \in \mathcal{R}^N$ and $H \subseteq O$ ($H \neq \emptyset$), let the mechanism ϕ choose the type-optimal stable allocation (for \succ) if $|H| \leq v$ and the agent-optimal stable allocation (for \succ) if $|H| > v$. The mechanism ϕ satisfies all properties except *strategy-proofness*.

For *strategy-proofness*, note that agents 2 and 3 cannot change the priority structure by reporting a false preference relation. Agent 1 always receives his most preferred object in H for any problem (R, H) . Thus, agent 1 cannot profitably manipulate by reporting a false preference relation.

For *resource-monotonicity*, let $|H| = 1$ and $R \in \mathcal{R}^N$. If there is a violation of *resource-monotonicity*, then φ must use different priority structures for (R, H) and $(R, \{x, y\})$. But then we must have $H = \{y\}$ and both $\varphi(R, \{y\}) = DA^{\succ'}(R, \{y\})$ and $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\})$.

If $y P_1 \emptyset$, then $\varphi_1(R, \{y\}) = y$ and $\varphi_2(R, \{y\}) = \varphi_3(R, \{y\}) = \emptyset$, and all agents weakly prefer $\varphi(R, \{x, y\})$ to $\varphi(R, \{y\})$.

If $\emptyset P_1 y$, then $\varphi_1(R, y) = \emptyset$. Since $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\})$, we have $\emptyset P_1 x$ and by *individual-rationality*, $\varphi_1(R, \{x, y\}) = \emptyset$. Note that $2 \succ_x 3$, $2 \succ_y 3$, and $2 \succ'_y 3$ (the latter because $\succ_y = \succ'_y$). Let $\pi : 2 \succ 3 \succ 1$. Then, $\varphi(R, \{y\}) = DA^{\succ'}(R, \{y\}) = f^\pi(R, \{y\})$ and $\varphi(R, \{x, y\}) = DA^{\succ}(R, \{x, y\}) = f^\pi(R, \{x, y\})$. Hence, *resource-monotonicity* is satisfied.