The Government Spending Multiplier in a (Mis-)Managed Liquidity Trap

Jordan Roulleau-Pasdeloup
University of Lausanne

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Abstract

I study the impact of a government spending shock in a New Keynesian model when monetary policy is set optimally. In this framework, the economy is at the Zero Lower Bound but expectations are well managed by the Central Bank. As such, the multiplier effect of government spending increases on expected inflation is small while the one on output can be larger than one. This is consistent with recent empirical evidence on the effects of the 2009 ARRA.

Keywords: Zero lower bound, New Keynesian, Government spending multiplier.
JEL Classification: E31, E32, E52, E62

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1 Introduction

After the effects of the American Recovery and Reinvestment Act of 2009 have passed and the dust has settled on the policy debate, there is still a lively debate among academics about the effectiveness of fiscal policy. Mostly after it had been enacted, the fiscal stimulus plan was hailed as effective by many academics because it was enacted at a time when the economy was in a liquidity trap with the Fed Funds rate close to zero. In the seminal contributions of Eggertsson [2011] and Christiano et al. [2011], such a policy was shown to be efficient in stimulating output in a depressed economy.

Because higher government spending would generate more employment and lower the level of slack in the economy, it would generate higher inflation. In turn, higher inflation would reduce the real interest rate and prompt and increase in aggregate demand, thereby boosting a depressed economy. This transmission mechanism is at the heart of the New Keynesian model with sticky prices. Along with the workhorse New Keynesian model itself, much doubt has been cast about the empirical relevance of the tight link between inflation and recession/recovery.

In particular, such a model will predict that the economy falls in a liquidity trap after a negative aggregate demand shock generates a large deflation. While the U.S. only experienced a mild deflation in 2009, in the following years inflation was hovering below the stated objective of 2% as the economy was still depressed. This has prompted a new wave of research trying to understand if the New Keynesian model could be amended to fit the facts.[1]

On the fiscal policy front, Dupor and Li [2015] show that the American Recovery and Reinvestment Act did not spur a rise in expected inflation. From the standard New Keynesian model, an absence of inflation after a government spending shock in a liquidity trap would mean that the multiplier effect of government spending is likely to be way smaller than what can be found in Eggertsson [2011] and Christiano et al. [2011].

The objective of this paper is to reconcile a baseline New Keynesian model with observed empirical evidence. To do so, I will show that in this model, there is no such thing as an invariant 'government spending multiplier at the zero lower bound'. It is well known that, outside a liquidity trap, the stimulative properties of government spending depend a lot on the stance of mon-

[1] See for example Ball and Mazumder [2011], Hall [2011], Gordon [2013], Del-Negro et al. [2014], Christiano et al. [2015] and Coibion and Gorodnichenko [2015].

etary policy. But this statement is also true in a liquidity trap. If the central bank is able to commit and the economy is hit by a large negative aggregate demand shock, it will be optimal to set the interest rate to zero for an extended period of time. While the economy will formally be at the zero lower bound, the central bank is able to commit and promises credibly to generate a boom in output and inflation when the natural interest rate becomes positive. As such, expected inflation is well anchored when the economy is in a liquidity trap.

As a consequence, it can be shown that an increase in government spending when the central bank is behaving optimally has much less effect on (expected) inflation than when the central bank is simply following a Taylor rule. Another result is that while the effect on (expected) inflation is smaller, this does not mean that the multiplier effect of government spending is negligible. In fact, while the effect on inflation will be very different in the two cases, the effect on output will be rather close. It follows that it is not enough to show that the American Recovery and Reinvestment Act did not generate expected inflation to conclude that such a policy is misguided and did not have large effects on output, at least in the short run.

The model is also able to replicate the fact that inflation did fall initially by a small amount and remained mostly positive in the subsequent quarters. As such, the absence of a long-lasting Great Depression style deflation is no *prima facie* evidence that the New Keynesian model is wrong. In sharp contrast with the early stages of the Great Depression, it could mean that the Fed did a good job in managing expectations. In any case, the model in which monetary policy is optimal produces a path for inflation (initial deflationary blip, then slightly positive inflation) that resembles very much actual U.S. experience after the Great Recession. As such, it makes for a better environment to study the impact of a government spending shock compared to a model with a Central Bank that follows a Taylor rule.[2]

2 The Model

In this section I develop a baseline New Keynesian model with wasteful government spending. As the presence of government spending in the utility function does not influence the dynamics of the model (as long as it is additively separable to private consumption, which is the common assumption in the literature), I leave it out for simplicity. This is also justified on the ground

[2]In recent work, Cochrane [2013] also argues that the Taylor rule might not be the best tool to study monetary policy.

that the focus of this paper is mainly on the positive aspect of government spending, and not the normative one.

This section will be divided in two subsections. In the first one, I present the decentralized equilibrium in which monetary policy is given by a standard Taylor rule. This is the case where the liquidity trap will be mis-managed because the Central Bank is not able to commit and inflation expectations will be deflationary. In the second subsection, I will assume that the nominal interest rate is set by a Ramsey planner. In this case, the Central Bank is able to commit and the liquidity trap will be well managed. In this setup, inflation expectations will be well anchored. In both setups, government spending is driven by a simple exogenous process.

2.1 Decentralized Equilibrium

Most of the assumptions and derivations of the baseline New Keynesian model are pretty standard, so I refer the interested reader to the appendix and only report the relevant first order conditions.

There is a representative consumer that gets utility from consumption C_t and disutility form working N_t, supplies his labor services to monopolistically competitive firms and earns a real wage in return as well as profits coming from monopolistically competitive firms. He pays lump-sum taxes, has access to one period riskless nominal bonds which yield a nominal interest rate of R_t. Importantly, the period utility function is disturbed by a 'preference shock' ξ_t that makes consumption more or less attractive today. A decrease in ξ_t means that consumers want to save more and consume less today. If this shock is large enough, it will send the economy at the zero lower bound. Optimal choice with respect to consumption and holding of bonds is given by the following Euler equation:

$$U_{C,t} = \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} (1+R_t), \tag{1}$$

where $U_{C,t}$ is the marginal utility one gets from consumping one more unit of the final good and π_t is the CPI inflation rate. Firms are monopolistically competitive and rent labor services from the representative household and operates a constant return technology using only labor. They face a quadratic cost when adjusting their prices as in Rotemberg [1982], which is proportional to final output. Since all firms are identical, they will choose the same price each period. In aggregate, otpimal choice for the new price at period t gives rise to the following New Keynesian Phillips Curve:

$$\psi\pi_t(1+\pi_t) = \beta\frac{\xi_{t+1}}{\xi_t}\frac{U_{C,t+1}}{U_{C,t}}\frac{N_{t+1}}{N_t}\psi\pi_{t+1}(1+\pi_{t+1}) - \theta\frac{U_{N,t}}{U_{C,t}} - \theta, \tag{2}$$

where θ is the elasticity of substitution across varieties and ψ governs the size of the price adjustment cost. I have assumed that an optimal subsidy corrects for the absence of a markup on leisure at the steady state. In equilibrium, combining the budget constraint of the government (Ricardian equivalence holds and the budget is balanced each period), the household and the definition of firm's profits, one gets the resource constraint of this economy :

$$N_t\left(1 - \frac{\psi}{2}\pi_t^2\right) = C_t + G_t. \tag{3}$$

Finally, the model is closed by specifying a rule for the setting of the nominal interest rate. Following much of the literature, I assume that it follows a standard Taylor rule of the form:

$$1 + R_t = \max\left\{1, R_t^n(1+\pi_t)^{\phi_\pi}N_t^{\phi_Y}\right\}, \tag{4}$$

where $R_t^n \equiv \beta^{-1}\xi_t/\xi_{t+1}$ is the natural rate of interest in this economy, *i.e* the real interest rate that would prevail under flexible prices.[3] Indeed, Cúrdia et al. [2015] show that a model in which the central bank targets the natural rate of interest is better able to fit the data than one with a standard Taylor rule. Given a path for government consumption expenses, equations (8)-(12) uniquely determine the value C_t, N_t, π_t and R_t. Government consumption expenses and the preference shock are given by the following $AR(1)$ processes:

$$\begin{aligned} G_t &= G_{t-1}^{\rho_g}G^{1-\rho_g}exp(\epsilon_t^g), \\ \xi_t &= \xi_{t-1}^{\rho_\xi}exp(\epsilon_t^\xi), \quad \epsilon_t^g, \epsilon_t^\xi \sim \mathcal{N}(0,1), \end{aligned}$$

where G is the steady state value of government expenses. As is by now well known,[4] if the preference shock is large enough, it will generate a negative response of inflation and output large enough to send the economy in a liquidity trap. Once the economy is there, government spending can have a large multiplier effect for two reasons. First, whether or not the economy is in a liquidity trap higher government spending pushes the real wage up due to the increase

[3]Note that there is no expectation term as ξ_{t+1} is known at date t, assuming that there are no more unanticipated shocks after that.

[4]See Eggertsson [2011] and Christiano et al. [2011] for two well known examples.

in labor demand. This raises the marginal cost of the firms and, as a consequence, generates inflation. Secondly, because the shock is persistent, *expected* inflation also increases after an increase in government spending. Since the nominal rate is stuck at zero, such an increase in expected inflation decreases the expected real interest rate. From the Euler equation, it follows that households wants to consume more, which in turn raises inflation, and so and so on.

As is shown in Eggertsson [2011], *expected* inflation plays a very important role there. First, because firms anticipate that inflation is going to be persistently low after the preference shock they decrease their price even more on impact; this aggravates the recession. At the end of the day, government spending does what monetary policy is not able to do anymore : generate expected inflation. What is important here is that, when stuck at the zero lower bound, monetary policy is completely passive. However, what we saw during the current Great Recession in the U.S. is that the successive presidents of the Fed were were not merely siting idle. They made announcements to try to influence inflation expectations in the private sector all along. As such, being in a liquidity trap is not equivalent with monetary policy being unable to control inflation expectations. As will be clear in the next subsection, it can be optimal for the Ramsey planner to stay at the zero lower bound for some period after the natural interest rate has become positive.

2.2 Ramsey Equilibrium

In this subsection, I drop the Taylor rule and assume that the Central Bank has the ability to commit to a certain path for the nominal interest rate.[5] Admittedly, this is a simple way of capturing the fact that the central bank may have some power to influence the private sector's expectations during a liquidity trap. The Ramsey planner solves the following maximization program, which

[5]There is a large literature investigating how government spending should be set optimally over the cycle, see for instance Woodford [2011], Werning [2011], Schmidt [2013], Nakata [2013], Bilbiie et al. [2014], and Nakata [2015]. In related work, Bouakez, Guillard, and Roulleau-Pasdeloup [2015] develop a more general setup in which both public consumption and investment can be set optimally over the cycle. Such a framework however prevents the analysis of discretionary increases in government spending, which is the focus of this paper.

I express directly as a Lagrangian:

$$
\begin{aligned}
\mathcal{L}_0 \equiv & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \xi_t U\left(C_t, N_t\right) \\
& + \phi_{1,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} (1+R_t) - U_{C,t} \right] \\
& + \phi_{2,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} \frac{Y_{t+1}}{Y_t} \psi \pi_{t+1}(1+\pi_{t+1}) - \psi \pi_t (1+\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta \right] \\
& + \phi_{3,t} \left[N_t \left(1 - \frac{\psi}{2} \pi_t^2 \right) - C_t - G_t \right] \\
& + \phi_{4,t} \left[R_t - 0 \right] \Bigg\}
\end{aligned}
$$

The equilibrium is then given by equations (8)-(12) together with the first order conditions with respect to C_t, N_t, π_t and R_t. The latter are quite complicated, so they are reported in the appendix.

2.3 Calibration and solution method

The utility function follows Christiano et al. [2011] and is given by:

$$
U\left(C_t, N_t\right) = \frac{\left(C_t^{\gamma}(1-N_t)^{1-\gamma}\right)^{1-\sigma}}{1-\sigma}, \quad \sigma > 1 \tag{5}
$$

The parameter σ is equal to 2 and γ is set so that $N = 1/3$ at steady state. This gives $\gamma = 0.2857$. The elasticity of substitution across goods is set to $\theta = 6$, which gives a steady state markup of 20%. Given these parameters, the price adjustment cost parameter is fixed so that it corresponds to a Calvo probability of 0.75 in log linear terms. This gives $\psi = 90$. The discount factor is set to $\beta = 0.99$. For the Taylor rule, I set $\phi_\pi = 1.5$ and $\phi_y = 0.25$ which are standard values. Finally, I set $\rho_\xi = 0.9$ for the preference shock so that the economy spends some time at the zero lower bound. Since increases in government spending are usually short-lived, I set $\rho_g = 0.7$ for the government spending shock. The share of government spending at steady state is calibrated to $g = G/Y = 0.2$ to match the post-WWII average in the U.S.

To solve the model with the occasionally binding constraint, I use the piecewise linear algorithm developed by Guerrieri and Iacoviello [2014]. It consists in estimating two different policy rules : one for the regime when the zero lower bound is binding and one for when it is not. The algorithm allows for endogenous exit of the zero lower bound. If a government spending shock is large enough, it can potentially get the economy out of the liquidity trap. This is more general than the two state markov process that is usually assumed in this type of framework. Furthermore, Guerrieri and Iacoviello [2014] show that their algorithm delivers approximation errors that are on par with state-of-the art global projection methods.

To visualize what optimal policy looks like after a large preference shock, I show the path of the main endogenous variables in this framework and compare it to the decentralized allocation in Figure 1

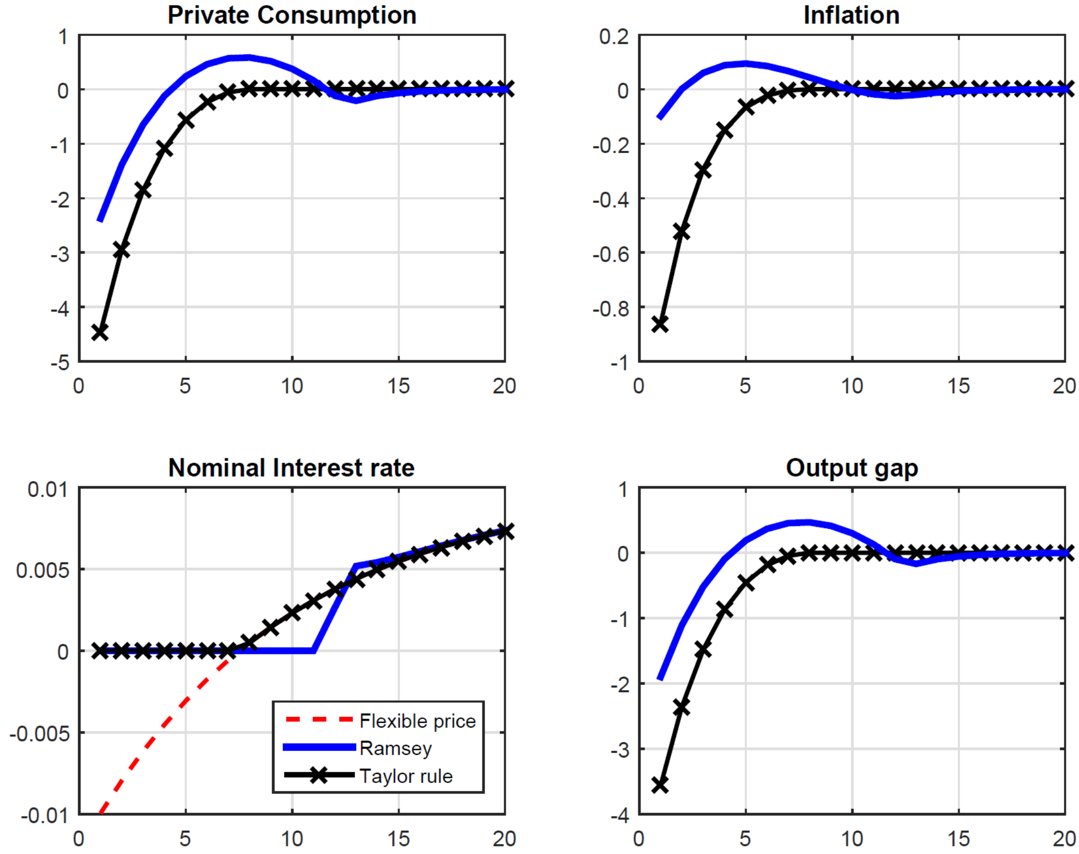

Figure 1: Ramsey optimal monetary policy and Taylor Rule after a negative preference shock

The main take-away from Figure 1 is that expected inflation is rather well managed under the optimal policy. As in Eggertsson and Woodford [2003], the central bank commits to keeping the nominal interest rate at its zero lower bound even after the natural interest rate becomes positive again.[6] In other words, monetary policy is too accomodative during this period, which generates a boom in consumption and inflation as soon as the natural rate of interest becomes positive. This percolates back to the first periods as agents anticipate higher inflation, and thus cut down less on consumption. Because the fall in aggregate demand is dampened, so is the fall in inflation.

What is important to note here is that whereas the economy is in a liquidity trap in both allocations in the first 7 periods, what happens is very different.[7] It follows that the effects of increasing government spending at the zero lower bound are poised to be different in the two allocations. As a result, there will be no such thing as '*the* government spending multiplier in a liquidity trap'. As will become clear, even for the same duration of the liquidity trap, government spending is likely to have different effects in both allocations. I take up this issue in the next section.

3 Government spending multipliers in (mis-)managed liquidity traps

The goal here is to compare the reaction of inflation and output to a government spending shock in a liquidity trap in the two setups. I proceed as follows : after a preference shock in period 1 (that may or may not be big enough to send the economy in a liquidity trap), government spending increases in period 2. For increasing values of the initial preference shock shock, I compute the following two objects:

$$\frac{dY_t}{dG_t} = \frac{(1-g)\Delta\hat{c}_t + g}{g} \quad \text{and} \quad \frac{d\mathbb{E}_t\pi_{t+1}}{dG_t} = 4 \cdot \frac{\Delta\hat{\mathbb{E}}_t\pi_{t+1}}{G},$$

where hatted variables denote log-deviations from their steady state values and $\Delta\hat{z}_t$ is the difference between the scenario with and without an increase in government spending—which is normalized to 1—for each variable z. Note that in the second expression the fraction is divided by the *level* of government spending at steady state. The impact multiplier on inflation is multiplied by

[6] See also Jung et al. [2005] and Adam and Billi [2006].

[7] Wu and Xia [2014] derive a shadow policy rate for the Federal Reserve and show that it conveys additional information about the real stance of monetary policy with respect to the Effective Federal Funds Rate. In particular, this shadow rate shows that monetary policy was quite accomodative despite being stuck at the ZLB. This is evidence that one has to go beyond the fact that the nominal rate is zero to gauge the real stance of monetary policy.

4 to give a number in annualized percent terms. To make sure that I do not compare apples and oranges, I plot the value of the impact multiplier as a function of the duration of the liquidity trap in each simulation in Figure 2.

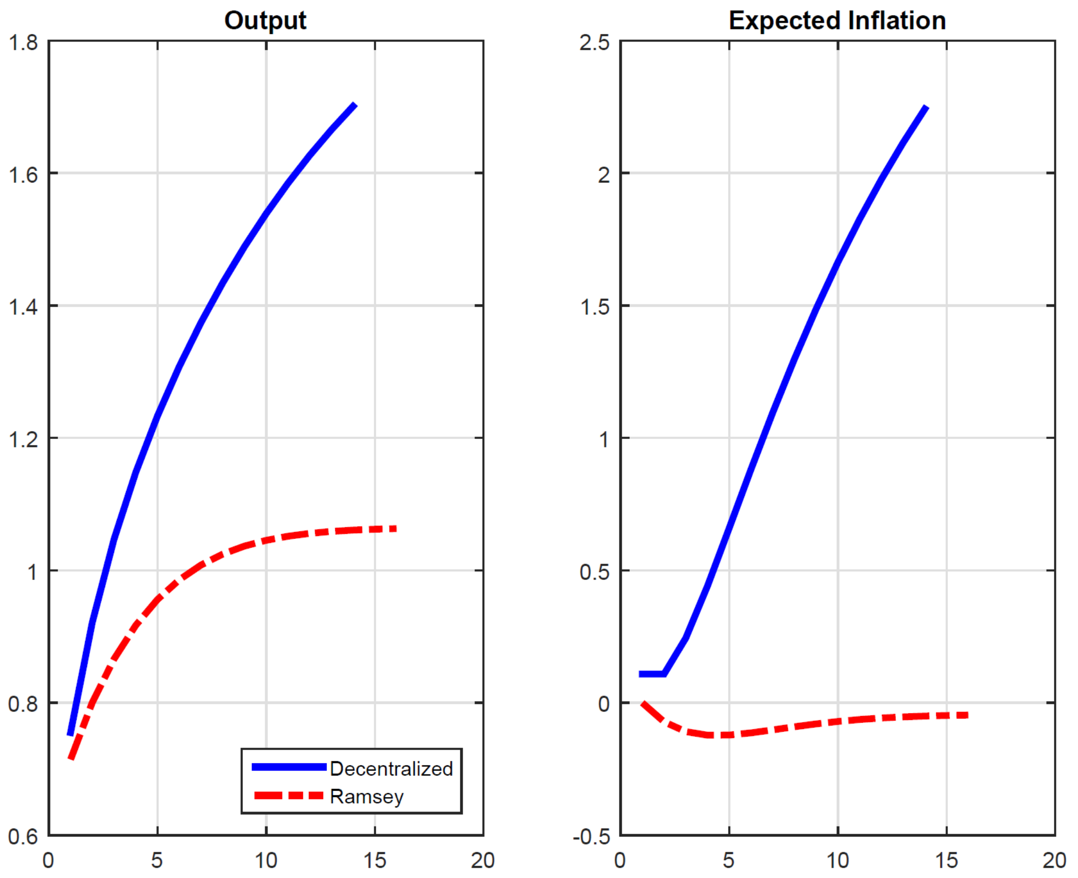

Figure 2: Output and Inflation multipliers on impact

Note that the multiplier effect on expected inflation is much higher in the decentralized case than for the Ramsey one. For this simulation, the impact multiplier effect on inflation is always positive in the decentralized equilibrium and negative in the Ramsey equilibrium. This comes from the fact that expected inflation depends a lot on the fact that the Central Bank is able to commit in the Ramsey equilibrium. In this allocation, over the simulation period inflation is actually *lower* when there is an increase in government spending.

In fact, it all depends on what happens in the liquidity trap; after that, inflation with and without government spending are roughly the same. Because government spending still persists after the zero lower bound has been lifted, the nominal rate should increase more in comparison to the case where government spending is kept constant. This increases the real interest rate and puts negative pressure on aggregate demand and inflation while the economy is in the liquidity trap. Since expected inflation depends a lot on monetary policy when the latter is optimally set, the short lived increase in inflation after government spending shoots up is quickly dominated by a relative decrease in inflation because of what happens when the economy gets out of the ZLB. This deflationary effect last for a major part of the time where the economy is at the ZLB so that higher government spending actually generates *less* inflation. This is in sharp contrast to what happens in the decentralized allocation. In this case, expectations are deflationary so that the increase in government spending makes a difference. This sets off a virtuous spiral whereby an increase in government spending decreases the expected real interest rate and crowds in consumption —if the ZLB spell is long enough.

These results carry over to the impact multiplier on output. Note that for a short duration of a liquidity trap, the multiplier on output is smaller than 1. This comes from the fact that since government spending is persistent, part of the increase is going to persist when the economy is out of the liquidity trap. During that time, higher than steady state government spending means a higher nominal rate set by the central bank now that the Taylor principle is active again. When that happens, the real interest rate increases. Since consumption today depends on the whole path of future expected interest rates,[8] the fact that government spending persists outside the liquidity trap has a negative effect on the stimulative properties of government spending inside the trap.

Eventually however, in the decentralized equilibrium the trap is going to last long enough so that the effects of government spending inside the trap will outweigh the ones outside : consumption is crowded in and the impact multiplier is higher than 1. For the case when monetary policy is given by the Ramsey planner, the two effects roughly cancel each other and the maximum impact multiplier effect of government spending is somewhat higher than 1. It should also be noted that for the same multiplier effect of 1, the decentralized model gives an inflation response of 0.63% in annualized terms, while the Ramsey model gives a negative one.

Finally, the difference between multipliers might be a direct consequence of the fact that the Ramsey policy is more efficient in mitigating the initial output

[8] Assuming separable preferences between consumption and leisure for simplicity, iterating forward the log linear Euler Equation yields $c_t = -\mathbb{E}_t \sum_{k=0}^{\infty} \left(i_{t+k} - \pi_{t+k+1} + \log(\beta) + (\rho_\xi - 1)\xi_{t+k+1}\right)$.

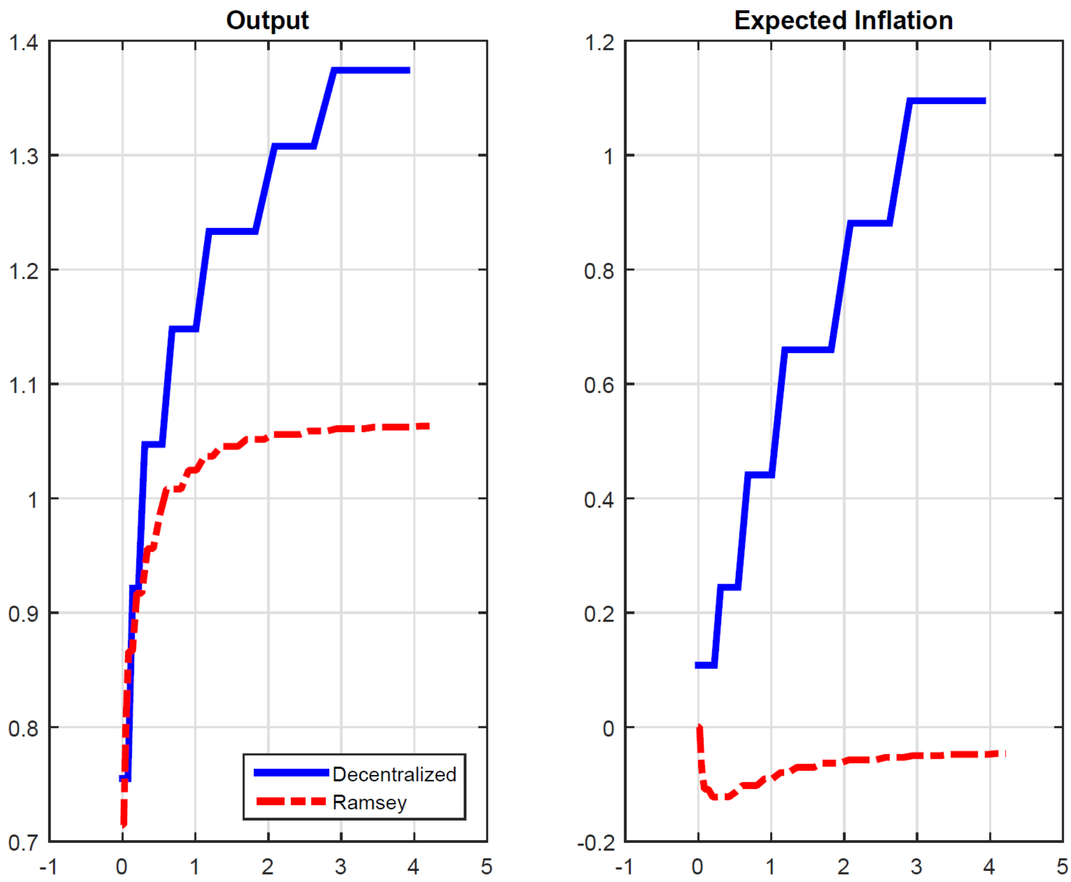

Figure 3: Output and Inflation multipliers on impact as a function of initial output gap
Note : A number of 4 in the x-axis means that the initial drop in output is equal to 4%.

gap. Therefore, both multiplier effects could be lower in this setup simply because the crisis is less severe, even for the same duration of the liquidity trap in both scenarios. To show that this is actually not the case, I plot in Figure 3 the impact multiplier effects on output and inflation as a function of the initial output gap after the preference shock. What clearly stands out is that both impact multipliers display the same pattern as before.

The impact multiplier might not convey all the information however. To consider all the dynamics in the response, I also consider the cumulative multiplier. Indeed, Uhlig [2010] puts the emphasis on the long run effects of government spending which can undermine the stimulative properties of the fiscal package taken as a whole. I compute the cumulative multiplier effects on output and inflation as follows:

$$M_{y,h} = \frac{1}{g} \frac{\sum_{t=1}^{h} (1-g)\Delta\hat{c}_t + g \cdot \Delta\hat{g}_t}{\sum_{t=1}^{h} \Delta\hat{g}_t}, \qquad M_{\pi,h} = 4 \cdot \frac{1}{G} \frac{\sum_{t=1}^{h} \Delta\hat{\mathbb{E}}_t \pi_t}{\sum_{t=1}^{h} \Delta\hat{g}_t}$$

where h is the horizon for the computation. I choose $h = 200$ for the simulation since all variables have returned to their steady state value after this time. I plot the cumulative multipliers as a function of the duration of the ZLB in Figure 4

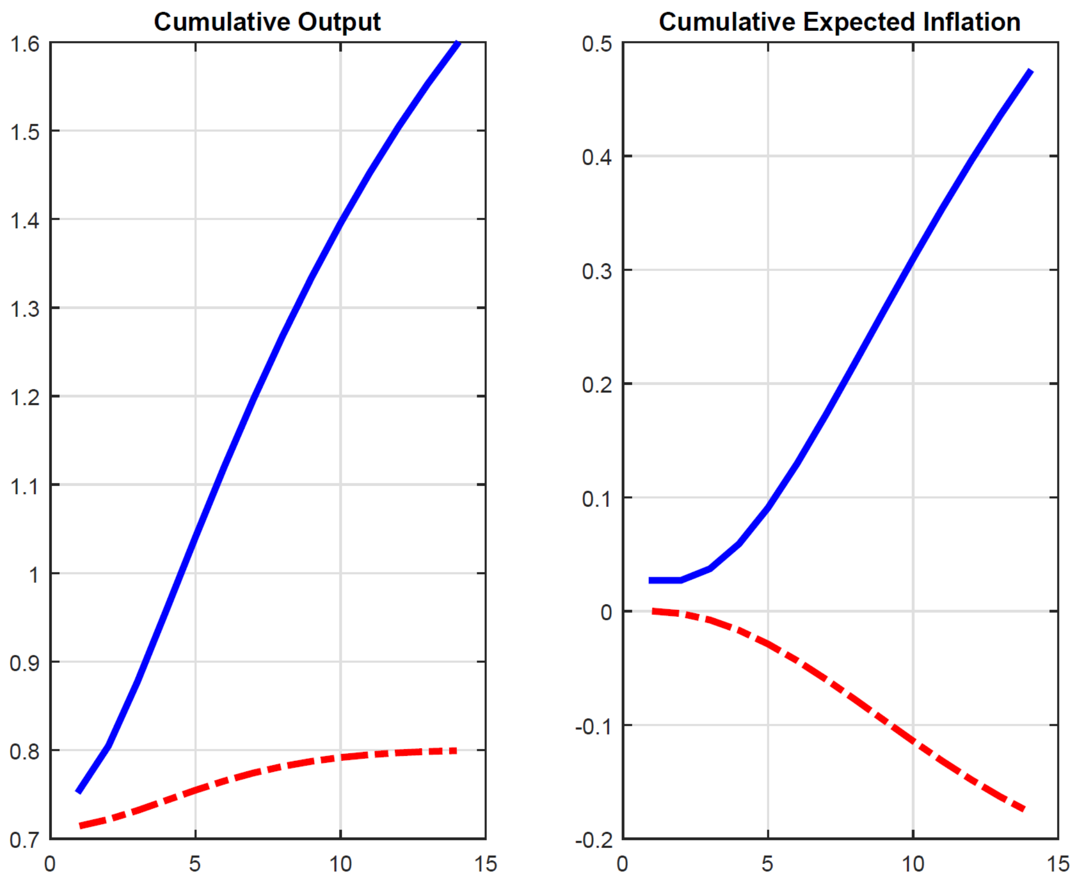

Figure 4: Cumulative Output and Inflation multipliers

Because inflation and output reach their respective maximum in both cases on impact, the impact multipliers are of a smaller magnitude. As before, eventually the cumulative multipliers effect of government spending in the decentralized case will be higher than 1 if the trap lasts long enough. That is not the case for the Ramsey allocation, which is consistent with the fact that the maximum impact multiplier is slightly higher than 1 at its maximum.

At the end of the day, the model in which monetary policy is run by a Ramsey planner seems to be more consistent with recent empirical evidence. On top of that, there are additional benefits associated with chosing this specification to study the impact of government spending policies. First, the model has a unique rational expectations equilibrium even at the Zero Lower Bound. In the model with a Taylor rule, the central bank loses the capacity to influence inflation altogether and this opens the door to sunspot driven equilibria. In the Ramsey equilibrium, the policymaker can still influence inflation expectations in a liquidity trap. At least under my baseline calibration, this is enough to rule out sunspot fluctuations.[9]

Then, the Ramsey allocation gives a much more reasonnable description of the Great Recession. To clarify this, I plot in Figure 5 the path of the Ramsey equilibrium with and without an increase in government spending. The preference shock is calibrated so that the economy experiences a trough at -5.65% for output as in the recent U.S. Great Recession. The government spending shock is calibrated so that it represents 1.58% of steady state GDP to mimic the actual size of the recent American Recovery and Reinvestment Act of 2009.

The main take-away from Figure 5 is that for such a simple model, it does a remarkably good job at describing the behavior of the main macroeconomic variables during the Great Recession. As in the data, the model economy experiences an initial drop in inflation. Subsequently, inflation turns positive and converges to its long run level of zero. Before it does so, there is a short deflationary blip. Output is initially depressed but recovers quickly, overshoots its steady state level and converges towards the latter from below. The nominal interest rate stays at its lower bound for 4 and a half years before jumping abruptlly.

With this in mind, the contribution of the fiscal stimulus is as follows : consistent with the results presented so far, the increase in government spending does not really matter for inflation dynamics. Inflation is somewhat larger with the fiscal stimulus initially, but the largest deviation actually coincides with inflation beeing *lower* with the fiscal stimulus. The reason for that is the same which have been described to explain the negative (cumulative) multiplier on expected inflation in the Ramsey equilibrium. Regarding the path of output however, the fiscal stimulus makes a clear contribution : output is noticeably lower without the discretionary increase in government spending during the early stages of the recession.

[9]While a systematic inquiry of the determinacy properties of the Ramsey equilibrium at the ZLB is beyond the scope of this paper, for all the calibrations employed in the baseline results and robustness tests there is a unique rational expectations equilibrium.

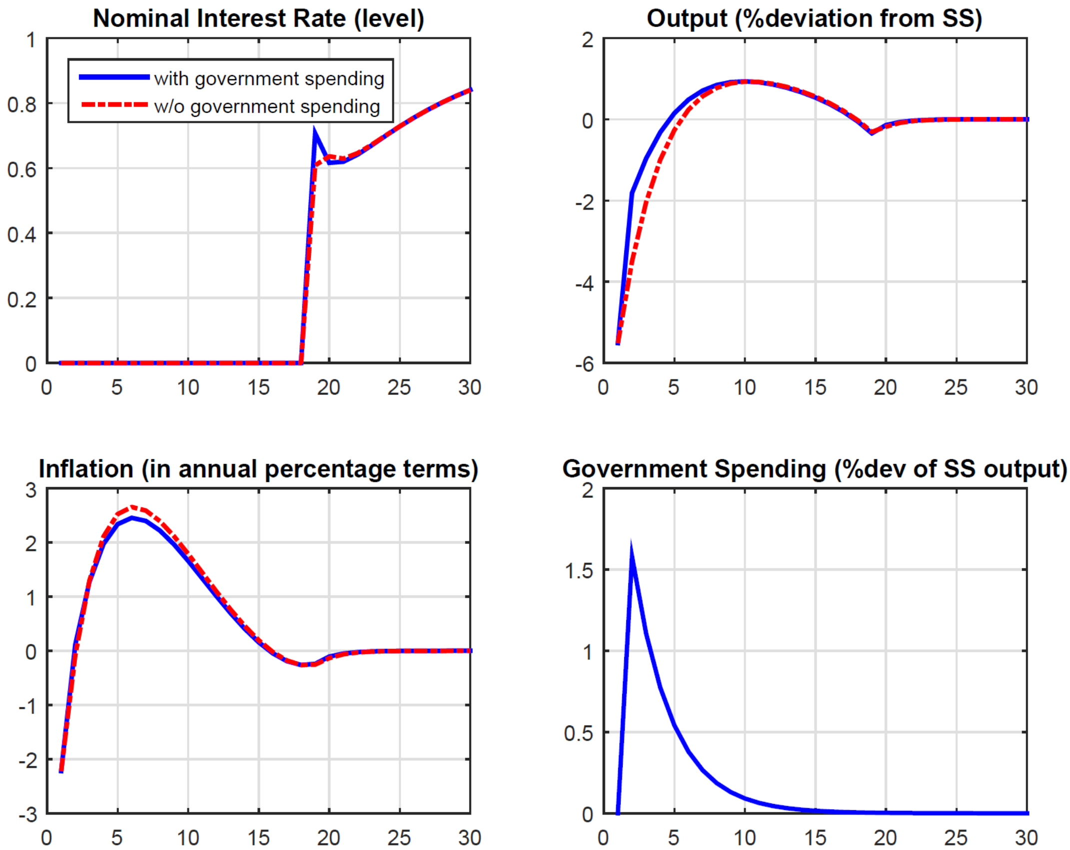

Figure 5: The Great Recession With and Without Fiscal Stimulus

In Figure 9 (in the appendix), I conduct the same exercise under the assumption that monetary policy is given by the Taylor rule described earlier. In this specification, two features are somewhat at odds with what actually happened during the Great Recession. First, the economy experiences a large deflation of roughly -6% in annual terms. Then, the liquidity trap lasts only for two years, that is a full two and a half years short of the length in the Ramsey equilibrium. With this in mind, one can see that fiscal policy still dampens the fall in output, but it has a larger (albeit rather small in absolute terms) impact on inflation compared to the Ramsey equilibrium.

4 Robustness and Sensitivity

4.1 Utility Function

In this section I study some variations of the model to show that the results do not depend on the exact specification that I chose in the last section. I begin with the utility function. The fact that hours worked and private consumption were assumed to be complements has important implications for the size of the multiplier (see Monacelli and Perotti [2008] and Bilbiie [2011]). In particular, in this setup such a feature works to increase the multiplier effect on output, which can be higher than 1 even in normal times (see Christiano et al. [2011]). To show that this is not crucial, I repeat the exercise of the last section with a separable utility function:

$$U(C_t, N_t) = log(C_t) - N_t^{\varphi}.$$

This utility function is very commonly used in the literature. With this functional form, $1/\varphi$ is the Frisch elasticity of labor supply. Consistent with much of the literature, I set $\varphi = 2$ so that the Frisch elasticity of labor supply is equal to 0.5. I plot both the cumulative and impact multipliers as a function of the duration of the liquidity trap in Figure 6:

What stands out of Figure 6 is that the results are qualitatively similar with the ones from last section. In particular, an increase in government spending when monetary policy is carried out optimally generate much less expected inflation. In this respect, the gap between the Ramsey and decentralized allocation is even more pronounced : the maximum impact on expected inflation in the decentralized case is as high as 3% in annual terms, whereas in the Ramsey allocation it is still negative. As was expected, the multiplier effects of government spending on output are now smaller without the complementarity between labor and consumption.

4.2 Taylor Rule

Another functional form that potentially has an impact on the result is the Taylor rule in the decentralized case. In the form that I have adopted up until now, the Central bank is assumed to be reactive in that it reacts to *current* variations in the natural rate of interest, inflation and output. However, most papers that try to fit a Taylor rule to the data assume some internal persistence : the nom-

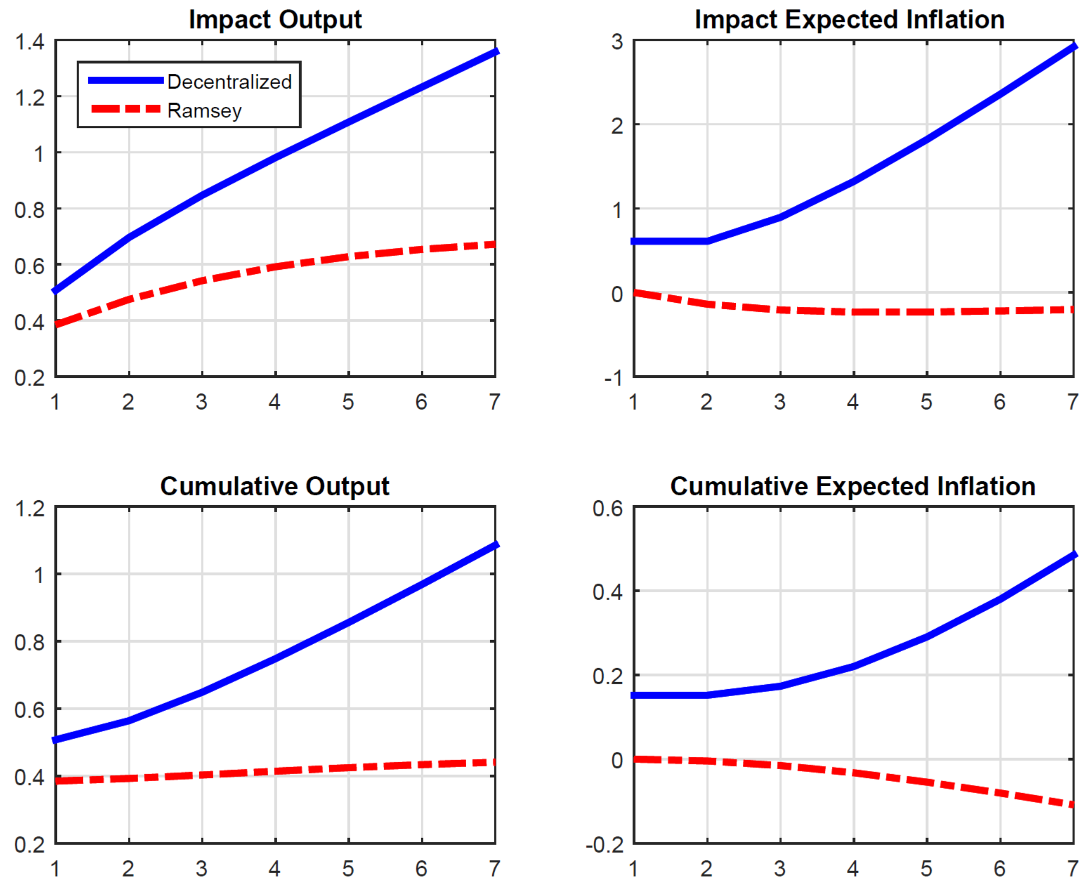

Figure 6: Cumulative/Impact Output and Inflation multipliers with separable utility

inal interest rate today depends on its last period value. Accordingly, I repeat the experiments of the last section with the following Taylor rule, that consists in two parts:

$$1 + R_t = \max\left\{1, (1 + R_{t-1})^{\rho_R} \left[R_t^n (1 + \pi_t)^{\phi_\pi} Y_t^{\phi_Y}\right]^{1-\rho_R}\right\}.$$

To the extent that $\rho_R > 0$, the current interest rate will depend on its past realization. I set a standard value of $\rho_R = 0.7$ and report the results of this experiment in Figure 7.

What stands out of Figure 7 is that the results are also the same qualitatively, except for the cumulative multiplier on expected inflation. Indeed, the latter is negative in the decentralized case for short liquidity traps. The rea-

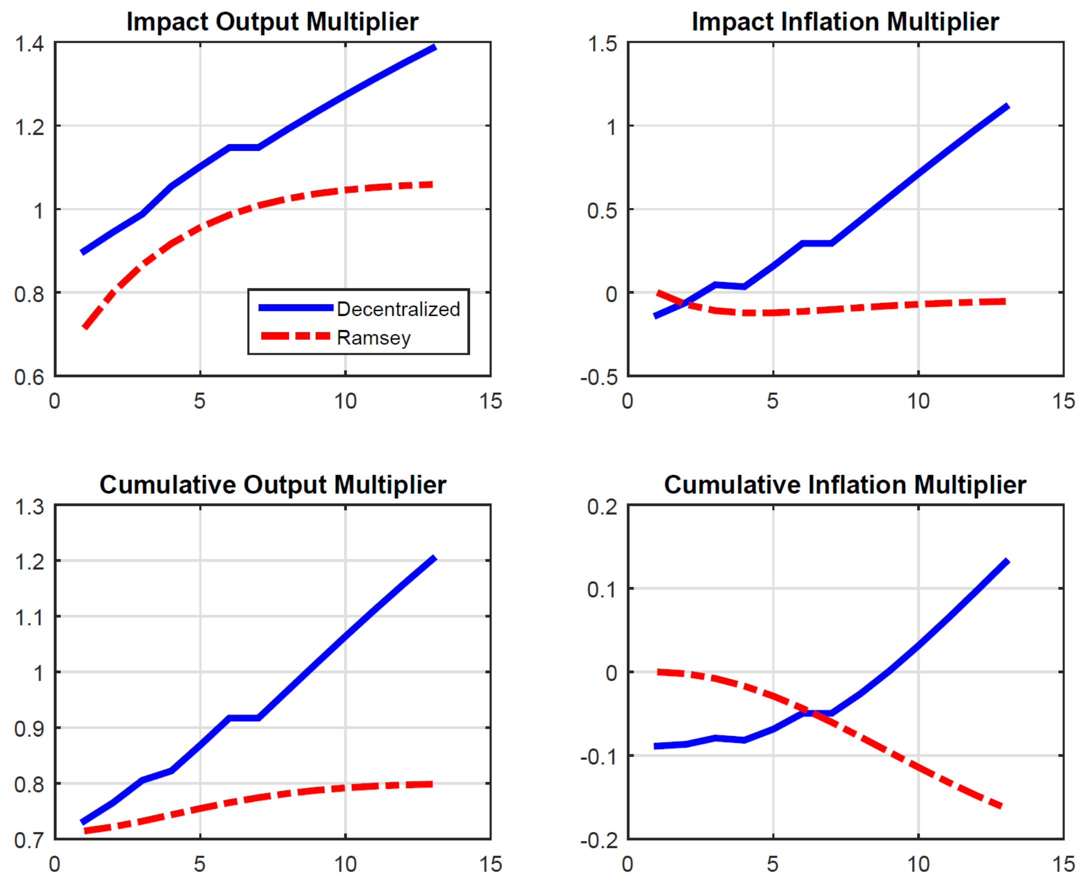

Figure 7: Cumulative/Impact Output and Inflation multipliers with persistent Taylor rule

son is the following : as before, part of the increase in government spending is going to occur when the economy is out of the liquidity trap, the more so the shorter the liquidity trap. As this will generate inflation and prompt the Central bank to raise the nominal interest rate, the latter will stay high for an extended period of time since it depends on its past value : as a result, during the liquidity trap agents anticipate a higher future real interest rate and this exerts a negative pressure on inflation. It follows that the cumulative output multiplier is lower in the decentralized case and closer to the one in the Ramsey allocation.

More generally, it is still the case that finding a low reaction of inflation in the data is not an indication that a government spending stimulus is not working.

4.3 A Model with Private Capital

In this subsection, I keep the augmented Taylor Rule and assume additionally that private capital is needed to produce the final good.[10] Further, the accumulation of private capital is subject to an investment adjustment cost in the same vein as Christiano et al. [2005]:

$$K_{t+1} = (1-\delta)K_t + \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t,$$

where K_t is the stock of capital at the beginning of period t and I_t is private investment. Since the model is linearized, what matters ultimately is the second derivative of the adjustment cost function with respect to its argument. Following Christiano et al. [2005], I set $S^{''}(1) = 2.5$ in the simulations.

I plot the results regarding the impact and cumulative multipliers of government spending as a function of the duration of the ZLB episode in Figure 8. As before, the impact multiplier on inflation is much smaller for the Ramsey allocation. The same can be said for the impact output multiplier, to a lesser extent. It should be noted that the magnitude of the impact on inflation for the decentralized case is much smaller than before. This is due to the fact that real marginal costs in this economy depend on the marginal product of capital. As a consequence, they react less on impact, which translates into the behavior of inflation. The same pattern holds for the cumulative multipliers, with a larger difference for the output multiplier in this case.

The reason is the following : as a discretionary increase in government spending generate less inflation, this puts upwards pressure on the real interest rate. The arbitrage condition between private capital and government bonds means that (modulo variations in Tobin's Q) both should give the same real return. Because the marginal productivity of capital is decreasing, a higher return means a lower level of capital, which is achieved through lower investment. In the decentralized equilibrium, the increase in (expected) inflation due to government spending mitigates this effect so that the cumulative multiplier is much higher.

[10] The de-centralized as well as the Ramsey allocations are described in full detail in the appendix.

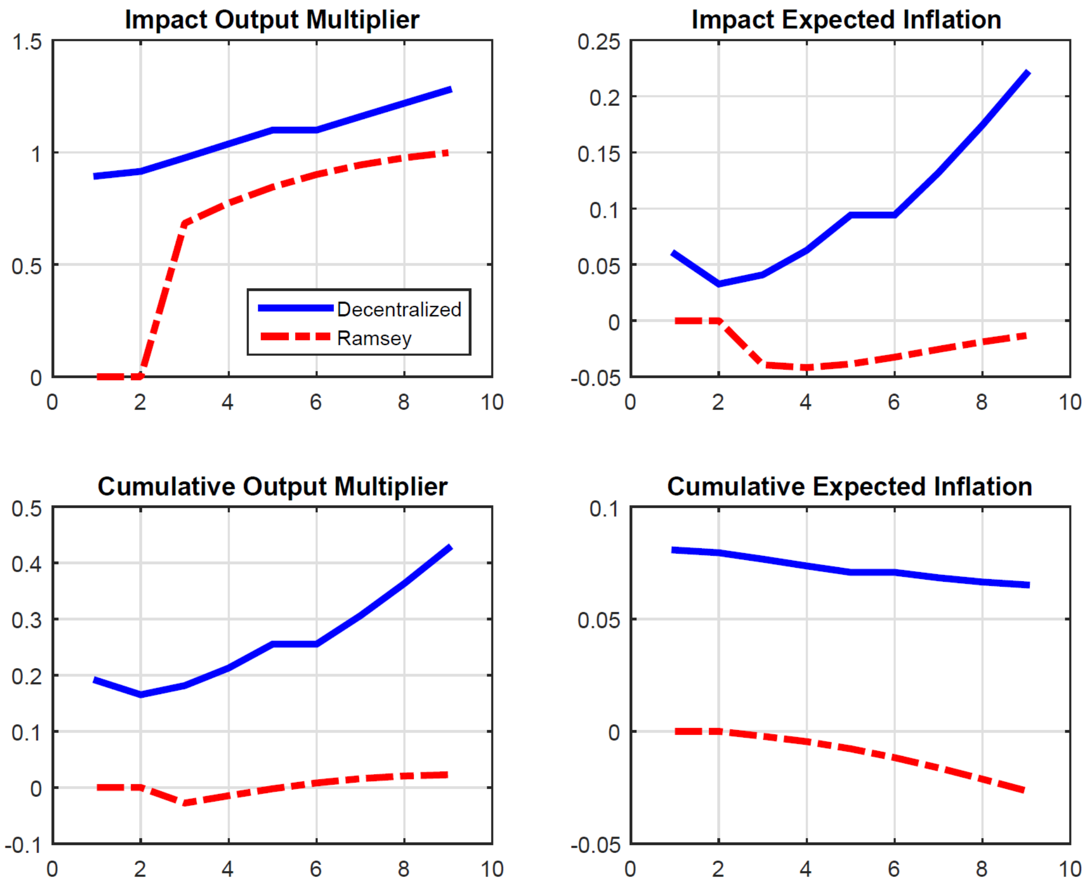

Figure 8: Impact Output and Inflation multipliers with persistent Taylor rule and private capital

At the end of the day, the inclusion of private capital calls for a qualification of the results obtained with simpler models. Indeed, a low cumulative response of expected inflation does not coincide with a sizable cumulative output response in the Ramsey allocation. As such, a fiscal stimulus is able to boost the economy in a short term fashion without generating an increase in expected inflation.

5 Conclusion

All ZLB episodes are not born equal. The dynamics of an economy at the ZLB are very dependent on whether the Central Bank is still actively managing expectations of the private sector. In this paper, I have formalized this by assuming that the monetary authority acted like a Ramsey planner. When this is the case, the economy will not experience sustained inflation and an increase in government spending will not generate a large increase in inflation.

As a consequence, the recent findings in Dupor and Li [2015] that the American Recovery and Reinvestment Act of 2009 did not generate a large increase in inflation cannot be taken as *prima facie* evidence that the New Keynesian transmission channel is wrong. Furthermore, such evidence is not necessarily inconsistent with the fact that government spending policy is ineffective at the ZLB. In the framework I have used in this paper, the multiplier effect on output when monetary policy is optimal can be higher than 1 if the recession is deep enough, while inflation still does not increase by a comparatively large amount.

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6 Appendix

6.1 Figures

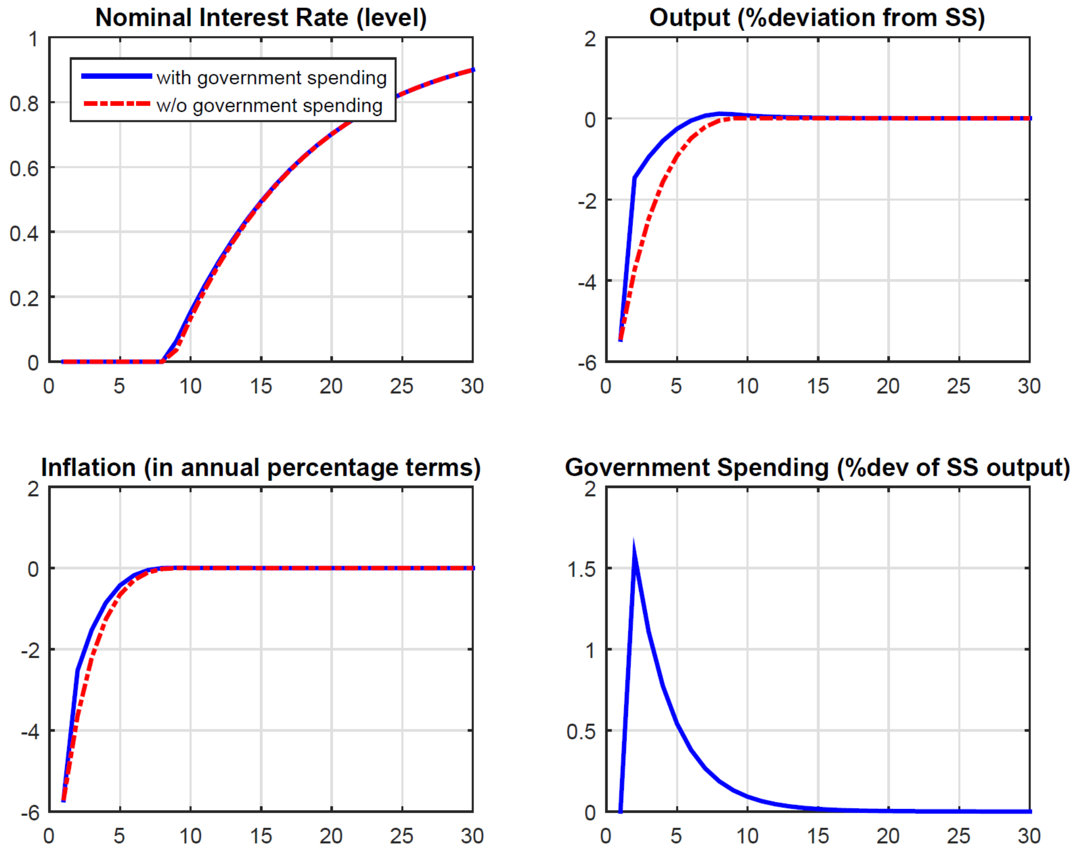

Figure 9: The Great Recession With and Without Fiscal Stimulus in the Decentralized Equilibrium

6.2 Model without Private Capital

6.2.1 Decentralized Equilibrium

The household's maximization program is the following:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \xi_{t+s} U\left(C_{t+s}, N_{t+s}\right), \tag{6}$$

subject to the budget constraint:

$$P_t C_t + \mathcal{T}_t + \frac{\mathcal{B}_t}{1+R_t} \leq \mathcal{W}_t N_t + \mathcal{D}_t + \mathcal{B}_{t-1}. \tag{7}$$

Optimal choice for private consumption, labor supply and the holdings of bonds yield:

$$U_{C,t} = \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} (1+R_t) \tag{8}$$

$$\frac{\mathcal{W}_t}{P_t} = -\frac{U_{N,t}}{U_{C,t}}, \tag{9}$$

where $\pi_t = \frac{P_t}{P_{t-1}} - 1$. The firms choose their individual price $P_t(z)$ to maximize the following objective function:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{\xi_{t+s}}{\xi_t} \frac{U_{C,t+s}}{U_{C,t}} \left\{ \frac{(1+\tau) P_t(z)}{P_t} Y_t(z) - \frac{\mathcal{W}_t}{P_t} N_t(z) - \frac{\psi}{2} \left(\frac{P_t(z)}{P_{t-1}(z)} - 1 \right)^2 N_t \right\},$$

where $\tau = 1/(\theta - 1)$. Using equation (9) and the fact that every firm is identical and will choose the same price, the first order condition with respect to $P_t(z)$ is given by:

$$\psi \pi_t (1+\pi_t) = \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} \frac{N_{t+1}}{N_t} \psi \pi_{t+1} (1+\pi_{t+1}) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta. \tag{10}$$

Using the definition of profits, plugging it in the household's budget constraint I obtain the resource constraint of this economy:

$$N_t \left(1 - \frac{\psi}{2} \pi_t^2\right) = C_t + G_t. \tag{11}$$

Finally, the model is closed by specifying a rule for the setting of the nominal interest rate. Following much of the literature, I assume that it follows a standard Taylor rule of the form:

$$1 + R_t = \max\left\{1, R^n_t(1+\pi_t)^{\phi_\pi} N_t^{\phi_Y}\right\}. \tag{12}$$

6.2.2 Steady State

The share of government spending at steady state is denoted by:

$$\tilde{g} = \frac{G}{Y}$$

Given the functional form for the utility function, equation (9) can be re-written as:

$$\begin{aligned} F_N &= \frac{1-\gamma}{\gamma}\frac{C}{1-N} \\ \Leftrightarrow 1 &= \frac{1-\gamma}{\gamma}\frac{N}{1-N}(1-\tilde{g}) \\ \Leftrightarrow \ N &= \frac{\gamma}{\gamma+(1-\gamma)(1-\tilde{g})} \end{aligned}$$

where we have used the resource constraint for the second equation. In practice, I choose γ so that $N = 1/3$ at steady state. In this case, I have

$$\gamma = \frac{(1-\tilde{g})}{2+(1-\tilde{g})}$$

The resource constraint can be re-written as:

$$C = (1-\tilde{g})N$$

6.2.3 Ramsey Equilibrium

I reproduce the Ramsey program for convenience:

$$
\begin{aligned}
\mathcal{L}_0 \equiv & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \xi_t U\left(C_t, N_t\right) \\
& + \phi_{1,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} (1+I_t) - U_{C,t} \right] \\
& + \phi_{2,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} \frac{N_{t+1}}{N_t} \psi \pi_{t+1}(1+\pi_{t+1}) - \psi \pi_t (1+\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} - \theta \right] \\
& + \phi_{3,t} \left[N_t \left(1 - \frac{\psi}{2}\pi_t^2\right) - C_t - G_t \right] \\
& + \phi_{4,t} \left[R_t - 0 \right] \Bigg\}
\end{aligned}
$$

The First Order Conditions are given by the following set of equations:

$$
\frac{\partial \mathcal{L}_0}{\partial C_t} = 0 \Leftrightarrow \xi_t U_{C,t} = \phi_{1,t} U_{CC,t} + \theta \phi_{2,t} \frac{1}{F_{N,t}} \left[\frac{U_{CN,t} U_{C,t} - U_{CC,t} U_{N,t}}{U_{C,t}^2} \right] + \phi_{3,t} \tag{13}
$$

$$
+ \beta \psi \mathbb{E}_t \phi_{2,t} \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1} U_{CC,t}}{U_{C,t}^2} \frac{N_{t+1}}{N_t} d(\pi_{t+1}) \tag{14}
$$

$$
- \frac{\xi_t}{\xi_{t-1}} U_{CC,t} \left\{ \phi_{1,t-1} \frac{1+R_{t-1}}{1+\pi_t} + \psi \phi_{2,t-1} \frac{N_t}{N_{t-1}} \frac{d(\pi_t)}{U_{C,t-1}} \right\}
$$

$$
\frac{\partial \mathcal{L}_0}{\partial N_t} = 0 \Leftrightarrow \xi_t U_{N,t} = \theta \phi_{2,t} \left\{ \frac{U_{NN,t} U_{C,t} - U_{CN,t} U_{N,t}}{U_{C,t}^2} \right\} + \phi_{1,t} U_{CN,t} \tag{15}
$$

$$
- \phi_{3,t} \left(1 - \frac{\pi_t^2}{2}\right) - \frac{\xi_t}{\xi_{t-1}} U_{CN,t} \left\{ \phi_{1,t-1} \frac{1+R_{t-1}}{1+\pi_t} + \psi \phi_{2,t-1} \frac{N_t}{N_{t-1}} \frac{d(\pi_t)}{U_{C,t-1}} \right\}
$$

$$
\frac{\partial \mathcal{L}_0}{\partial G_t} = 0 \Leftrightarrow \xi_t U_{G,t} = \phi_{3,t} \tag{16}
$$

$$
\frac{\partial \mathcal{L}_0}{\partial \pi_t} = 0 \Leftrightarrow \phi_{3,t} N_t \pi_t + \phi_{2,t} d^{'}(\pi_t) = \frac{\xi_t}{\xi_{t-1}} U_{C,t} \left(\phi_{2,t-1} \frac{N_t}{N_{t-1}} \frac{d^{'}(\pi_t)}{U_{C,t-1}} - \phi_{1,t-1} \frac{1+R_{t-1}}{\psi(1+\pi_t)^2} \right) \tag{17}
$$

$$
\frac{\partial \mathcal{L}_0}{\partial R_t} = 0 \Leftrightarrow \phi_{4,t} = -\beta \phi_{1,t} \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} \tag{18}
$$

To this set of equations I must add the complementary slackness condition:

$$R_t \phi_{4,t} = 0.$$

6.2.4 Steady State

The steady state value of C and N is still given by the values reported before. I derive here the steady state value for the Lagrange Mutlipliers of the Ramsey program. I focus on the zero inflation steady state in which the zero lower bound is not binding. As such, I have $R > 0$ and $\phi_4 = 0$. From equation (18), I get that

$$\phi_1 = 0.$$

From equation (13) and the fact that $\phi_2 = \phi_1 = 0$, I get:

$$\phi_3 = U_C.$$

6.2.5 Log linear approximation outside ZLB

I first focus on the equations that are also present in the decentralized equilibrium. The Euler equation for consumption is:

$$c_t = \mathbb{E}_t c_{t+1} + \frac{U_C}{CU_{CC}} \left(i_t - \mathbb{E}_t \pi_{t+1} + \log(\beta) + (\rho - 1)\hat{\xi}_t \right) + \frac{NU_{CN}}{CU_{CC}} (\mathbb{E}_t n_{t+1} - n_t) \tag{19}$$

The resource constraint is given by:

$$n_t = (1 - g)c_t + g \cdot g_t. \tag{20}$$

The log linear New Keynesian Phillips curve is:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa mc_t, \tag{21}$$

where

$$mc_t = \left(\frac{NU_{NN}}{U_N} - \frac{NU_{CN}}{U_C} \right) n_t + \left(\frac{CU_{CN}}{U_N} - \frac{CU_{CC}}{U_C} \right) c_t.$$

I now focus on the log linear equilibrium conditions for the Ramsey equilibrium when the economy is outside the Zero Lower Bound. As such, from equation (18) I have

$$\phi_{1,t} \equiv 0.$$

I use this equation to simplify the derivations from now on. I start with equation (17). When log linearized *around the zero inflation steady state* it gives:

$$\phi_3 N \cdot \pi_t + \hat{\phi}_{2,t} = \hat{\phi}_{2,t-1}, \tag{22}$$

where a hat denotes the log linear deviation of the Lagrange multiplier from its steady state value.[11] With this in mind, the log linear version of equation (13) is:

$$\frac{CU_{CC}}{U_C} c_t + \frac{NU_{CN}}{U_C} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \frac{\theta}{U_C}\left[\frac{U_{CN}U_C - U_{CC}U_N}{U_C^2}\right]\hat{\phi}_{2,t}$$

Likewise, the log linear version of equation (15) is:

$$\frac{CU_{CN}}{U_N} c_t + \frac{NU_{NN}}{U_N} n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \theta\left[\frac{U_{NN}U_C - U_{CN}U_N}{U_N U_C^2}\right]\hat{\phi}_{2,t} \tag{23}$$

6.2.6 Log linear approximation at ZLB

I now focus on the log linear equilibrium conditions when the economy is stuck at the Zero Lower Bound. When this is the case, the complementary slackness condition is binding and I have $\phi_{4,t} > 0$. As a consequence, from equation (18) I have

$$\hat{\phi}_{4,t} = -\beta\phi_3 \cdot \hat{\phi}_{1,t}.$$

When log linearized *around the zero inflation steady state* (17) gives:

$$\phi_3 N \cdot \pi_t + \hat{\phi}_{2,t} = \hat{\phi}_{2,t-1} - \frac{\phi_3}{\psi\beta}\hat{\phi}_{1,t-1}, \tag{24}$$

where a hat denotes the log linear deviation of the Lagrange multiplier from its steady state value. Now that ϕ_1 is no longer constant, the log linear version of equation (13) is:

$$\begin{aligned}\frac{CU_{CC}}{U_C} c_t + \frac{NU_{CN}}{U_C} n_t + \hat{\xi}_t = {} & \hat{\phi}_{3,t} + \frac{\theta}{U_C}\left[\frac{U_{CN}U_C - U_{CC}U_N}{U_C^2}\right]\hat{\phi}_{2,t} \\ & + \frac{U_{CC}}{U_C}\hat{\phi}_{1,t} - \frac{U_{CC}}{\beta U_C}\hat{\phi}_{1,t-1}\end{aligned} \tag{25}$$

[11] Note that since the steady state value of ϕ_2 is zero, a first order Taylor expansion of $\phi_{2,t}$ yields simply $\hat{\phi}_{2,t} = \phi_{2,t}$.

Likewise, the log linear version of equation (15) is:

$$\frac{CU_{CN}}{U_N}c_t + \frac{NU_{NN}}{U_N}n_t + \hat{\xi}_t = \hat{\phi}_{3,t} + \theta\left[\frac{U_{NN}U_C - U_{CN}U_N}{U_N U_C^2}\right]\hat{\phi}_{2,t} + \hat{\phi}_{1,t} - \frac{U_{CN}}{\beta U_N}\hat{\phi}_{1,t-1} \tag{26}$$

Finally, the Euler equation for consumption becomes:

$$c_t = \mathbb{E}_t c_{t+1} + \frac{U_C}{CU_{CC}}\left(\log(\beta) - \mathbb{E}_t \pi_{t+1} + (\rho - 1)\hat{\xi}_t\right) + \frac{NU_{CN}}{CU_{CC}}(\mathbb{E}_t n_{t+1} - n_t) \tag{27}$$

6.3 Model with Private Capital

6.3.1 De-centralized allocation

The production function now reads:

$$F(N_t, K_t) = N_t^a K_t^b.$$

With this modification, the equation for optimal pricing by the monopolistic firm takes the following form:

$$\psi d(\pi_t) = \psi\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} \frac{F(N_{t+1}, K_{t+1})}{F(N_t, K_t)} d(\pi_{t+1}) - \theta \frac{U_{N,t}}{U_{C,t}} \frac{1}{F_{N,t}} - \theta.$$

The Lagrangian associated with the Household maximization program is the following:

$$\begin{aligned}
\mathcal{L}_0 \equiv & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \xi_t U\left(C_t, N_t\right) \\
& + \Omega_{1,t} \left[W_t N_t + P_t r_t^k K_t + R_{t-1} B_t - P_t(C_t + I_t) - B_{t+1} - P_t Q_t X_t \right] \\
& + \Omega_{2,t} \left[X_t + (1-\delta) K_t + \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) I_t - K_{t+1} \right] \Bigg\},
\end{aligned}$$

where Q_t is the price of installed capital, r_t^k is the return from capital (which will be equal to the marginal product of capital in equilibrium) and X_t is the amount of private capital that households members by from one another. In equilibrium, $X_t \equiv 0$ and I just include it so that I can determine the price of installed capital. I end up with the following first order conditions:

$$\frac{\partial \mathcal{L}_0}{\partial N_t} = 0 \Leftrightarrow \frac{\mathcal{W}_t}{P_t} = -\frac{U_{N,t}}{U_{C,t}} \tag{28}$$

$$\frac{\partial \mathcal{L}_0}{\partial C_t} = 0 \Leftrightarrow \xi_t U_{C,t} = \Omega_{1,t} P_t \tag{29}$$

$$\frac{\partial \mathcal{L}_0}{\partial X_t} = 0 \Leftrightarrow \Omega_{1,t} P_t Q_t = \Omega_{2,t} \tag{30}$$

$$\frac{\partial \mathcal{L}_0}{\partial I_t} = 0 \Leftrightarrow U_{C,t} = Q_t U_{C,t} \left[1 - S\left(\frac{I_t}{I_{t-1}}\right) - S^{'}\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}} \right] \tag{31}$$

$$+ \beta \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} Q_{t+1} U_{C,t+1} S^{'}\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \tag{32}$$

$$\frac{\partial \mathcal{L}_0}{\partial K_{t+1}} = 0 \Leftrightarrow U_{C,t} = \beta \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} U_{C,t+1} \left\{ \frac{(1-\delta) Q_{t+1} + r^k_{t+1}}{Q_t} \right\} \tag{33}$$

The resource constraint of the economy is now:

$$F(N_t, K_t) \left(1 - \frac{\psi}{2} \pi_t^2\right) = C_t + G_t + I_t$$

6.3.2 Steady State

By construction, at steady state $S(1) = S^{'}(1) = 0$. From equation (32), I get $Q = 1$. Let me introduce the following notations for steady state ratios:

$$\tilde{\imath} = \frac{I}{Y} \quad , \quad \tilde{g} = \frac{G}{Y}$$

Using equation (33), I obtain:

$$\beta F_K = 1 - \beta(1-\delta)$$
$$\Leftrightarrow \tilde{\imath} = \frac{\delta K}{Y} = \frac{b \beta \delta}{1 - \beta(1-\delta)},$$

where I have also used the functional form of the production function to substitute for $F_K = b\frac{Y}{K}$. Given the functional form for the utility function, equation

(28) can be re-written as:

$$\begin{aligned}
F_N &= \frac{1-\gamma}{\gamma}\frac{C}{1-N} \\
\Leftrightarrow\ a\frac{Y}{N} &= \frac{1-\gamma}{\gamma}\frac{Y}{1-N}(1-\tilde{g}-\tilde{\imath}) \\
\Leftrightarrow\ N &= \frac{a\gamma}{a\gamma+(1-\gamma)(1-\tilde{g}-\tilde{\imath})}
\end{aligned}$$

where we have used the resource constraint for the second equation. In practice, we choose γ so that $N = 1/3$ at steady state. In this case, we have

$$\gamma = \frac{(1-\tilde{g}-\tilde{\imath})}{2a+(1-\tilde{g}-\tilde{\imath})}$$

The resource constraint can be re-written as:

$$\begin{aligned}
C &= (1-\tilde{g}-\tilde{\imath})N^a K^b \\
\Leftrightarrow\ C &= (1-\tilde{g}-\tilde{\imath})N^a \left(\frac{\tilde{\imath}}{\delta}\frac{C}{1-\tilde{g}-\tilde{\imath}}\right)^b \\
\Leftrightarrow\ C &= (1-\tilde{g}-\tilde{\imath})\left[N^a\left(\frac{\tilde{\imath}}{\delta}\right)^b\right]^{\frac{1}{1-b}}
\end{aligned}$$

Using the steady state value of C and N I can recover the steady state level output and the stock of capital:

$$K = \frac{\tilde{\imath}}{\delta}\frac{C}{1-\tilde{g}-\tilde{\imath}} \quad , \quad Y = N^a K^b.$$

6.3.3 Log-Linear Approximation

The accumulation equation for private capital is now given by:

$$k_t = (1-\delta)k_{t-1} + \delta i_t. \tag{34}$$

Approximated around the zero inflation steady state, the resource constraint is now:

$$\frac{NF_N}{F}n_t + \frac{KF_K}{F}k_t = (1-g)c_t + g\cdot g_t^c + \frac{K}{Y}(k_{t+1}-(1-\delta)k_t). \tag{35}$$

The log linear approximation of equation (32) reads:

$$\hat{q}_t = S''(1)(i_t - i_{t-1}) - \beta S''(1)(\mathbb{E}_t i_{t+1} - i_t) \tag{36}$$

Approximating equation (33), I get:

$$\hat{U}_{C,t} = \mathbb{E}_t \hat{U}_{C,t+1} + (\rho - 1)\xi_t - q_t + \beta(1-\delta)\mathbb{E}_t q_{t+1} + (1 - \beta(1-\delta))\ \mathbb{E}_t \left\{ mc_{t+1} + \hat{F}_{K,t+1} \right\} \tag{37}$$

where I have defined:

$$\hat{U}_{C,t} = \frac{CU_{CC}}{U_C} c_t + \frac{NU_{CN}}{U_C} n_t$$
$$\hat{F}_{K,t} = \frac{NF_{KN}}{F_K} n_t + \frac{KF_{KK}}{F_K} k_t.$$

Equations (19), (20) and (21) are still valid, where real marginal cost is now defined as

$$mc_t = \hat{U}_{N,t} - \hat{U}_{C,t} - \hat{F}_{N,t}$$

with

$$\hat{U}_{N,t} = \frac{NU_{NN}}{U_N} n_t + \frac{CU_{CN}}{U_C} c_t \tag{38}$$
$$\hat{F}_{N,t} = \frac{NF_{NN}}{F_N} n_t + \frac{KF_{NK}}{F_N} k_t. \tag{39}$$

6.3.4 The Ramsey allocation

$$
\begin{aligned}
\mathcal{L}_0 \equiv & \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \Bigg\{ \xi_t U\left(C_t, N_t\right) \\
& + \Gamma_{1,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{1+\pi_{t+1}} (1+R_t) - U_{C,t} \right] \\
& + \Gamma_{2,t} \left[\psi \beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{U_{C,t}} \frac{F(N_{t+1}, K_{t+1})}{F(N_t, K_t)} d(\pi_{t+1}) - \psi d(\pi_t) - \theta \frac{U_{N,t}}{U_{C,t}} \frac{1}{F_{N,t}} - \theta \right] \\
& + \Gamma_{3,t} \left[F(N_t, K_t) \left(1 - \frac{\psi}{2} \pi_t^2 \right) - C_t - G_t - I_t \right] \\
& + \Gamma_{4,t} \left[(1-\delta) K_t + (1 - S_{P,t}) I_t - K_{t+1} \right] \\
& + \Gamma_{5,t} \left[\beta \mathbb{E}_t \frac{\xi_{t+1}}{\xi_t} Q_{t+1} U_{C,t+1} S'_{P,t+1} \dot{I}_{t+1}^2 + Q_t U_{C,t} \left(1 - S_{P,t} - S'_{P,t} \dot{I}_t \right) - U_{C,t} \right] \\
& + \Gamma_{7,t} \left[\beta \frac{\xi_{t+1}}{\xi_t} \frac{U_{C,t+1}}{Q_t} \left((1-\delta) Q_{t+1} - \frac{U_{N,t+1}}{U_{C,t+1}} \frac{F_{K,t+1}}{F_{N,t+1}} \right) - U_{C,t} \right] \\
& + \Gamma_{7,t} \left[R_t - 0 \right] \Bigg\},
\end{aligned}
$$

where I have defined

$$
\dot{I}_t \equiv \frac{I_t}{I_{t-1}}, \quad S_{P,t} = S_P\left(\dot{I}_t\right), \quad S'_{P,t} = S'_P\left(\dot{I}_t\right), S''_{P,t} = S''_P\left(\dot{I}_t\right)
$$

After some algebra, the FOC with respect to I_t yields:

$$
\begin{aligned}
\Gamma_{3,t} I_t = & \Gamma_{5,t} I_t \left(1 - S_{P,t} - S'_{P,t} \dot{I}_t \right) + Q_t U_{C,t} \tilde{S}_{P,t} \left(\Gamma_{6,t-1} \frac{\xi_t}{\xi_{t-1}} \dot{I}_t - \Gamma_{6,t} \right) \\
& + \beta \mathbb{E}_t Q_{t+1} U_{C,t+1} \tilde{S}_{P,t+1} \left(\Gamma_{6,t+1} - \Gamma_{6,t} \frac{\xi_{t+1}}{\xi_t} \dot{I}_{t+1} \right) + \beta \mathbb{E}_t \Gamma_{5,t+1} S'_{P,t+1} \dot{I}_{t+1} I_{t+1},
\end{aligned} \tag{40}
$$

where I have defined

$$
\tilde{S}_{P,t} \equiv 2 \cdot S'_{P,t} \dot{I}_t + S''_{P,t} \dot{I}_t^2.
$$

The FOC with respect to private consumption now reads:

$$\xi_t U_{C,t} = \Gamma_{3,t} + U_{CC,t}\left[\Gamma_{1,t} + \Gamma_{6,t} + \Gamma_{5,t}\left(1 - Q_t\left(1 - S_{P,t} - S'_{P,t}\dot{I}_t\right)\right)\right] \tag{41}$$

$$\begin{aligned}
&+ \theta\Gamma_{2,t}\frac{1}{F_{N,t}}\left[\frac{U_{CN,t}U_{C,t} - U_{CC,t}U_{N,t}}{U_{C,t}^2}\right] + \psi\beta\Gamma_{2,t}\mathbb{E}_t\frac{\xi_{t+1}}{\xi_t}\frac{U_{C,t+1}}{U_{C,t}^2}\frac{Y_{t+1}}{Y_t}d(\pi_{t+1})U_{CC,t} \\
-&U_{CC,t}\frac{\xi_t}{\xi_{t-1}}\left[\Gamma_{1,t-1}\frac{1+R_{t-1}}{1+\pi_t} + \psi\Gamma_{2,t-1}\frac{Y_t}{Y_{t-1}}\frac{d(\pi_t)}{U_{C,t-1}} + \Gamma_{5,t-1}S'_{P,t}\dot{I}_t^2 Q_t\right] \\
&- U_{CC,t}\frac{\xi_t}{\xi_{t-1}}\frac{\Gamma_{6,t-1}}{Q_{t-1}}\left\{(1-\delta)Q_t - \frac{F_{K,t}U_{CN,t}}{F_{N,t}U_{CC,t}}\right\}
\end{aligned} \tag{42}$$

where I have defined

$$Y_t = F(N_t, K_t).$$

Similarly, the FOC with respect to hours worked now reads:

$$\begin{aligned}
\xi_t U_{N,t} = {}& \Gamma_{1,t}U_{CN,t} + \theta\Gamma_{2,t}\left\{\frac{1}{F_{N,t}}\left[\frac{U_{NN,t}U_{C,t} - U_{CN,t}U_{N,t}}{U_{C,t}^2}\right] - \frac{U_{N,t}}{U_{C,t}}\frac{F_{NN,t}}{F_{N,t}^2}\right\} - \Gamma_{3,t}F_{N,t}\left(1 - \psi\frac{\pi_t^2}{2}\right) \\
&+ \left[\Gamma_{6,t} + \Gamma_{5,t}\left(1 - Q_t\left(1 - S_{P,t} - S'_{P,t}\dot{I}_t\right)\right)\right]U_{CN,t} + \psi\beta\Gamma_{2,t}\mathbb{E}_t\frac{\xi_{t+1}}{\xi_t}\frac{U_{C,t+1}}{U_{C,t}^2}\frac{Y_{t+1}}{Y_t}d(\pi_{t+1})U_{CN,t} \\
-&\frac{\xi_t}{\xi_{t-1}}\left[\Gamma_{1,t-1}\frac{1+R_{t-1}}{1+\pi_t} + \psi\Gamma_{2,t-1}\frac{Y_t}{Y_{t-1}}\frac{d(\pi_t)}{U_{C,t-1}} + \Gamma_{5,t-1}S'_{P,t}\dot{I}_t^2 Q_t\right]U_{CN,t} \\
&- \frac{\xi_t}{\xi_{t-1}}\frac{\Gamma_{6,t-1}}{Q_{t-1}}\left[(1-\delta)Q_tU_{CN,t} - \frac{F_{K,t}}{F_{N,t}}U_{NN,t} - U_{N,t}\frac{F_{KN,t}F_{N,t} - F_{K,t}F_{NN,t}}{F_{N,t}^2}\right] \\
&+ \psi F_{N,t}\left[\beta\Gamma_{2,t}\mathbb{E}_t\frac{\xi_{t+1}}{\xi_t}\frac{U_{C,t+1}}{U_{C,t}}\frac{Y_{t+1}}{Y_t^2}d(\pi_{t+1}) - \Gamma_{2,t-1}\frac{\xi_t}{\xi_{t-1}}\frac{U_{C,t}}{U_{C,t-1}Y_{t-1}}d(\pi_t)\right]
\end{aligned} \tag{43}$$

The FOC with respect to private capital is:

$$\begin{aligned}
\Gamma_{4,t} = {}& \beta(1-\delta)\mathbb{E}_t\Gamma_{4,t+1} + \beta\mathbb{E}_t\left[\Gamma_{3,t+1}F_{K,t+1}\left(1 - \frac{\pi_{t+1}^2}{2}\right) + \theta\Gamma_{2,t+1}\frac{U_{N,t+1}}{U_{C,t+1}}\frac{F_{NK,t+1}}{F_{N,t+1}{}^2}\right] \\
&+\beta\mathbb{E}_t\frac{\xi_{t+1}}{\xi_t}U_{C,t+1}\left[\psi\Gamma_{2,t}\frac{F_{K,t+1}}{Y_t}\frac{d(\pi_{t+1})}{U_{C,t}} - \frac{\Gamma_{6,t}}{Q_t}\frac{U_{N,t+1}}{U_{C,t+1}}\left(\frac{F_{KK,t+1}F_{N,t+1} - F_{K,t+1}F_{NK,t+1}}{F_{N,t+1}^2}\right)\right] \\
&- \psi\beta^2\mathbb{E}_t\Gamma_{2,t+1}\frac{\xi_{t+2}}{\xi_{t+1}}\frac{U_{C,t+2}}{U_{C,t+1}}\frac{Y_{t+2}F_{K,t+1}}{Y_{t+1}^2}d(\pi_{t+2})
\end{aligned} \tag{44}$$

The FOC with respect to the price of installed private capital Q_t yields:

$$U_{C,t}\Gamma_{5,t}\left(1-S_{P,t}-S^{'}_{P,t}\dot{I}_t\right) = -\frac{\xi_t}{\xi_{t-1}}U_{C,t}\left[\Gamma_{5,t-1}S^{'}_{P,t}\dot{I}^2_t+\Gamma_{6,t-1}\frac{(1-\delta)}{Q_{t-1}}\right]$$
$$+\beta\Gamma_{6,t}\mathbb{E}_t\frac{\xi_{t+1}}{\xi_t}\frac{U_{C,t+1}}{Q^2_t}\left[(1-\delta)Q_{t+1}-\frac{U_{N,t+1}}{U_{C,t+1}}\frac{F_{K,t+1}}{F_{N,t+1}}\right] \tag{45}$$

Finally, the FOC with respect to inflation and the nominal interest rate are the same as before.

6.3.5 Steady State

From equations (40) and (42), I have that

$$\Gamma_4 = \Gamma_3 = U_C.$$

Evaluating equations (44) and (45) at steady state and using the identity $\beta F_K = 1-\beta(1-\delta)$, I obtain:

$$\Gamma_6 = \Gamma_5 = 0.$$

6.3.6 Log linear approximation

Approximating the bond Euler Equation around the zero inflation steady state, I get:

$$\hat{\xi}_t + \hat{U}_{C,t} = \hat{\Gamma}_{3,t} + \frac{U_{CC}}{U_C}\left(\hat{\Gamma}_{1,t}+\hat{\Gamma}_{6,t}\right) - \frac{\theta}{U_C}\frac{U_{CN}U_C - U_{CC}U_N}{U_N U_C}\hat{\Gamma}_{2,t}$$
$$-\frac{U_{CC}}{U_C}\left\{1-\delta-\frac{F_K U_{CN}}{F_N U_{CC}}\right\}\hat{\Gamma}_{6,t-1} - \frac{U_{CC}}{\beta U_C}\hat{\Gamma}_{1,t-1}, \tag{46}$$

where we have used the fact that $F_N \cdot U_C = -U_N$ at steady state. Similarly, for hours worked I get:

$$\hat{U}_{N,t} + \hat{\xi}_t = \hat{\Gamma}_{3,t} + \hat{F}_{N,t} + \frac{\theta}{U_N F_N}\left\{\frac{U_{NN}U_C - U_{CN}U_N}{U_C^2} + F_{NN}\right\}\hat{\Gamma}_{2,t} + \frac{U_{CN}}{U_N}\left[\hat{\Gamma}_{1,t} + \hat{\Gamma}_{6,t} - \beta^{-1}\hat{\Gamma}_{1,t-1}\right]$$
$$- \left\{(1-\delta)\frac{U_{CN}}{U_N} - \frac{F_K}{F_N}\frac{U_{NN}}{U_N} - \frac{F_{KN}F_N - F_K F_{NN}}{F_N^2}\right\}\hat{\Gamma}_{6,t-1} \tag{47}$$

The first order condition with respect to private capital (**??**) is now:

$$\hat{\Gamma}_{4,t} = \beta(1-\delta)\mathbb{E}_t\hat{\Gamma}_{4,t+1} + (1-\beta(1-\delta))\mathbb{E}_t\left[\hat{\Gamma}_{3,t+1} + \hat{F}_{K,t+1} + \theta\frac{F_{NK}}{U_N F_K}\hat{\Gamma}_{2,t+1}\right]$$
$$+ \beta\frac{F_{KK}F_N - F_K F_{NK}}{F_N}\hat{\Gamma}_{6,t} \tag{48}$$

Using the fact that $Q = 1$ and $\beta F_K = 1 - \beta(1-\delta)$, the log linear approximation of equation (40) can be written as follows :

$$\hat{\Gamma}_{3,t} = \hat{\Gamma}_{4,t} - S^{''}(1)\left((1+\beta)i_t - i_{t-1} - \beta\mathbb{E}_t i_{t+1}\right) + \frac{S^{''}(1)}{I}\left(\Gamma_{5,t-1} + \beta\mathbb{E}_t\Gamma_{5,t+1} - (1+\beta)\Gamma_{5,t}\right), \tag{49}$$

where I is the steady state value of the level of private investment. Similarly, the log linear approximation of equation (45) gives:

$$\hat{\Gamma}_{5,t} = \hat{\Gamma}_{6,t} - (1-\delta)\Gamma_{6,t-1}. \tag{50}$$