Virtual Demand and Stable Mechanisms

Jan Christoph Schlegel*
Faculty of Business and Economics, University of Lausanne, Switzerland
jschlege@unil.ch

Abstract

We study conditions for the existence of stable, strategy-proof mechanisms in a many-to-one matching model with salaries. Workers and firms want to match and agree on the terms of their match. Firms demand different sets of workers at different salaries. Workers have preferences over different firm-salary combinations. Workers' preferences are monotone in salaries. We show that for this model, a descending auction mechanism is the only candidate for a stable mechanism that is strategy-proof for workers. Moreover, we identify a maximal domain of demand functions for firms, such that the mechanism is stable and strategy-proof.

In the special case, where demand functions are generated by quasi-linear profit functions, our domain reduces to the domain of demand functions under which workers are gross substitutes for firms. We provide two versions of the results for the quasi-linear case. One for a discrete model, where salaries are restricted to discrete units and one for a continuous model, where salaries can take on arbitrary positive numbers. More generally, we show that several celebrated results (the existence of a worker-optimal stable allocation, the rural hospitals theorem, the strategy-proofness of the worker-optimal stable mechanism) generalize from the discrete to the continuous model.

JEL-classification: C78, D47
Keywords: Matching with contracts; Matching with salaries; Gross Substitutes; Virtual Demand

1 Introduction

Centralized clearing houses based on the deferred-acceptance mechanism are at the heart of many successful real-world matching markets (Roth, 1984a; Abdulkadiroglu and Sönmez, 2003; Sönmez and Switzer, 2013; Sönmez, 2013). Deferred-acceptance mechanisms are appealing, because they produce stable outcomes, meaning that no subgroup of

*I am grateful to my adviser Bettina Klaus for many useful comments that greatly improved the paper and to Sangram Kadam for many insightful discussions. I thank Battal Doğan, Federico Echenique, Fuhito Kojima, Maciej Kotowski, participants of the 2016 Meeting of the Social Choice and Welfare Society and the 17th ACM Conference on Economics and Computation for comments on a previous version of this paper. I gratefully acknowledge financial support by the Swiss National Science Foundation (SNSF) under project 100018-150086.

agents can find a mutually beneficial deviation and thus would have a reason to contract outside the market.[1] Moreover, it is safe for the applying side of the market to report their true preferences to the mechanism. Thus, the mechanism successfully aggregates the information in the market and levels the playing field for naive and sophisticated participants.

In some applications, the market does not only match agents, but determines also the contractual details of the match. In a labor market, firms and workers may have some discretion on how to set the salary. In the cadet-to-branch match (Sönmez and Switzer, 2013), cadets can choose between different lengths of service time in exchange for a higher priority in their branch of choice. These markets can be understood as hybrids between matching markets and auctions and have first been analyzed in the seminal paper of Kelso and Crawford (1982). Kelso and Crawford propose a generalization of the (firm-proposing) deferred-acceptance algorithm that they call the salary adjustment process. They identify a condition on firms' demand functions - workers have to be gross substitutes for the firms - that guarantees that the process converges to a stable allocation. In the salary adjustment mechanism, it is in general not optimal for workers to reveal their true preferences or for firms to reveal their true demand for workers. Subsequently, Hatfield and Milgrom (2005) have identified conditions on the demand, such that the worker-proposing (descending) version of the salary adjustment process is stable, *and* strategy-proof for the workers.[2] For this to be the case, workers have to be gross substitutes for firms and the law of aggregate demand[3] must hold for each firm.

In this paper, we extend the analysis of Kelso and Crawford (1982) and Hatfield and Milgrom (2005) and consider necessary, as well as sufficient conditions for the existence of stable and strategy-proof mechanisms. We show that gross substitutability and the law of aggregate demand are essentially necessary for the existence of a stable and strategy-proof mechanism. For this purpose, we introduce the notion of a *virtual demand* function. For a demand function, the corresponding virtual demand function is a closely related but in general more well-behaved demand function. Replacing demand functions by virtual demand functions will not change the outcome of the salary adjustment process. If workers are *virtual gross substitutes* for firms, i.e. workers are gross substitutes for firms according to the virtual demand, then the salary adjustment process converges to an allocation that is stable, both in the original market, and in the virtual market where we replace demand by virtual demand. The domain of demand functions, under which workers are virtual gross substitutes for firms and the virtual law of aggregate demand holds, turns out to be a maximal domain for the stability and strategy-proofness of the

[1] See Roth (1991) for evidence that clearing houses using unstable mechanisms tend to fail in practice.

[2] Hatfield and Milgrom actually go beyond the Kelso-Crawford model, by allowing for multi-dimensional contract-terms, instead of just salaries in their model. Different contracts can be ranked arbitrarily, whereas in the Kelso-Crawford model workers' preferences are monotone in the contract-dimension, i.e. workers prefer higher salaries to lower salaries. We will discuss this issue in more detail later. See also Echenique (2012) and Schlegel (2015) for a discussion on what this adds in generality to the model.

[3] This means that if we enlarge the choice set of a firm by lowering salaries, an equal or smaller number of workers will be chosen.

salary adjustment process. More generally, it is a maximal domain for the existence of any stable and strategy-proof mechanism. The class contains demand functions under which workers are gross substitutes for firms and the law of aggregate demand holds. For these demand functions the virtual demand and the original demand are the same. However, it also contains many other demand functions for which the virtual demand and the original demand differ.

In the first part of the analysis, we treat demand functions of firms as given and do not make any assumption on how they are generated. In particular, we do not assume that firms are profit maximizers and salaries enter their profit linearly, as it is often assumed in the literature (Kelso and Crawford, 1982; Gul and Stacchetti, 1999; Hatfield et al., 2014).[4] Our modeling choice is motivated by real-world matching markets with endogenous contracting. In these markets, the demand side often does not consist of profit-maximizing firms, but rather of e.g. nonprofit hospitals (Roth, 1984a) or branches of the US-military (Sönmez and Switzer, 2013). Moreover, the salary is often non negotiable, whereas other contract terms are negotiated during the match. These contract terms could, e.g., be a particular job description, the working hours, or the length of the contract (as in the US-military match). As long as preferences are monotone in the contract dimension, the more general analysis in this paper applies to these cases.[5]

In the second part of the analysis, we consider the important special case, where demand functions are obtained by the maximization of quasi-linear profit functions. For this special case, we show that demand and virtual demand agree. Moreover, the law of aggregate demand is now implied by the gross substitutes condition. Thus, for the quasi-linear case, our domain of demand functions reduces to the domain of demand functions under which workers are gross substitutes. It turns out that an even stronger result holds for the quasi-linear case. For firms maximizing quasi-linear profit functions, the domain of gross substitutes profit functions, is not only maximal for the existence of a stable and strategy-proof mechanism, but more generally for the existence of stable allocations. We provide two versions of the results for the quasi-linear case. One for a discrete model, where salaries are restricted to discrete units and one for a continuous model, where salaries can take on arbitrary positive numbers. Importantly, our results only assume quasi-linearity on the firm-side of the market but not necessarily on the worker-side of the market. Thus, for the continuous model, our results generalize previous results due to Gul and Stacchetti (1999) and Hatfield et al. (2014).

Additionally, we provide generalizations of two celebrated results for the discrete model to the continuous model. By using a limit argument, we show that the existence of a worker-optimal stable allocation in the discrete model implies the existence of a worker-optimal stable allocation in the continuous model. Moreover, the “rural hospitals theorem” generalizes to the continuous model by a limit argument.

[4]Note however that Kelso and Crawford (1982), point out that their analysis can be generalized beyond the quasi-linear model since “all arguments are completely ordinal” (Kelso and Crawford, 1982, p.1492).

[5]The original analysis of the military match by Sönmez and Switzer (2013) uses a model that does not directly fit into the Kelso-Crawford model. However, it can be shown (Jagadeesan, 2016) that the problem can be rephrased in an equivalent way using a Kelso-Crawford model.

1.1 Related Literature

Stable many-to-one matching mechanisms and their incentive properties have been extensively studied (Hatfield and Kojima, 2010; Chen et al., 2016; Hirata and Kasuya, 2015; Kominers and Sönmez, 2016; Hatfield et al., 2015). Most papers focus on the pure matching model or on the matching with contracts model (Hatfield and Milgrom, 2005; Roth, 1984b; Fleiner, 2003). The latter model allows for multidimensional contract-terms that can be ranked arbitrarily. A labor contract could, for example, contain a particular job description, as well as a salary and it might not be a priori clear how a worker ranks different job characteristics and salary combinations. In contrast to this, we consider the more restricted case where there is a natural ordering on the contract set and the workers have monotone preferences with respect to this ordering. If a worker agrees to work for some firm under some salary, then he will also agree to work for the firm under a higher salary. Cadets prefer short service times over long service times etc.

Since we are considering a more restricted model, all sufficient conditions for stability and the existence of a stable and strategy-proof mechanism from the literature on matching with contracts also apply to our model. However, conditions that are necessary for the model with contracts are not necessary conditions for the model with salaries. For strategy-proofness, this is because certain preference manipulations are ruled out by the model. A worker must report monotone preferences. Thus, he cannot rank working for a firm under a low salary above working for the same firm under a high salary to manipulate the outcome of the mechanism in his favor. Similarly, weaker conditions are sufficient to guarantee the existence of stable allocations than those for markets with contracts.[6]

[6] To illustrate this point, consider a market with two firms f_1, f_2 and three workers w_1, w_2, w_3. Suppose there are two kinds of contracts: A firm and a worker can match under a low salary 1 or under a high salary 2. The firms have preferences

$$\{(w_1,1),(w_2,1),(w_3,1)\} \succ_{f_1} \{(w_1,1)\} \succ_{f_1} \{(w_1,2)\} \succ_{f_1} \emptyset \succ_{f_1} \dots$$
$$\{(w_2,1)\} \succ_{f_2} \{(w_3,1)\} \succ_{f_2} \emptyset \succ_{f_2} \dots$$

that induce choice functions in the usual way. Suppose workers always prefer to work for a firm under the high salary to working for the same firm under the low salary. Going through all different cases, one can show that for any preferences satisfying this monotonicity assumption a stable allocation (in the matching with contracts sense) exists. This changes if workers can report non-monotonic preferences. Consider the following preferences:

$$(f_1,1) \succ_{w_1} (f_1,2) \succ_{w_1} \emptyset \succ_{w_1} \dots$$
$$(f_1,2) \succ_{w_2} (f_2,2) \succ_{w_2} (f_1,1) \succ_{w_2} (f_2,1) \succ_{w_2} \emptyset$$
$$(f_2,2) \succ_{w_3} (f_1,2) \succ_{w_3} (f_2,1) \succ_{w_3} (f_1,1) \succ_{w_3} \emptyset$$

Worker w_1 has non-monotone preferences in salaries. He prefers to work for firm f_1 under a low salary to working for the same firm under a high salary. Thus, in a stable allocation it will never be the case that w_1 works for f_1 under the high salary. This in turn implies that no stable allocation exists: The allocation that matches all three workers to f_1 under the low salary is blocked by worker w_3 and firm f_2. Any allocation that matches w_2 to f_2 under the low salary is blocked by workers w_1, w_2 and w_3 and firm f_1. Any allocation that matches w_3 to f_2 under the low salary is blocked by worker w_2 and firm f_2. Finally, all other allocations are either not individually rational or blocked by workers w_1, w_2 and w_3 and firm f_1.

Recently, a maximal domain of choice functions for the existence of a stable and strategy-proof mechanism for matching markets with contracts has been identified (Hatfield et al., 2015). We note that the result of Hatfield et al. (2015) and our result on stable and strategy-proof mechanisms are logically independent. Our domain is larger than the domain of demand functions whose induced choice functions satisfy the conditions of Hatfield et al. (2015).[7] On the other hand, their result applies to a broader class of problems. We think that one advantage of studying the less general model is that a characterization becomes easier to state and the condition on the demand functions is easier to interpret. Furthermore, many practically relevant problems fit into the framework with salaries. On the other hand, some problems like the airline seat-upgrade problem of Kominers and Sönmez (2016), where the true preferences are likely to violate monotonicity in the contract-term do not fit well into the model with salaries. In this sense, we think that the results are complementary and both contribute to our understanding of stable and strategy-proof mechanisms in matching markets with endogenous contracting.

For the case of quasi-linear profit functions, we establish counterparts to both existence results of Kelso and Crawford (1982), by showing that the sufficient conditions from their paper are in fact maximal domain conditions. For the case of a continuous salary space, a similar result was obtained by Gul and Stacchetti (1999) in the slightly different context of an exchange economy with combinatorial demand. As far as we know, the present paper is the first that studies stable and strategy-proof mechanisms in the fully general job matching model with continuous salaries. Classical results for the one-to-one case can be found in Demange and Gale (1985). We generalize these results to the many-to-one case. Results for the many-to-one case where both sides of the market have quasi-linear preferences can be found in Hatfield et al. (2014). We generalize their result by relaxing the quasi-linearity for the applying side of the market.

2 Model and Known Results

2.1 Model

The following model is based on the job matching model of Kelso and Crawford (1982). There are two finite disjoint sets of agents, a set of **firms** F and a set of **workers** W. There is a finite set of possible **salaries** $S = \{1, 2, \ldots, \bar{\sigma}\}$.[8] Firms can hire workers and pay them salaries. Each firm f has a demand function $D_f : S^W \to 2^W$ that for a vector of salaries $s = (s_w)_{w \in W}$ specifies a set of workers $D_f(s) \subseteq W$ that the firm wants to hire under these salaries. Each worker w has preferences $\succeq_w$ over different firm-salary combinations and an outside option which we denote by "$\emptyset$". An ordinal **market** is a pair $(D, \succeq)$ consisting of a **demand profile** $D = (D_f)_{f \in F}$ and a **preference profile** $\succeq = (\succeq_w)_{w \in W}$.

[7]The preferences of firm f_1 in Footnote 6 induce, for example, a demand function that satisfies our condition but not the condition of Hatfield et al. (2015).

[8]The use of integer salaries is for notational convenience. Alternatively, we could use any finite and totally ordered set of salaries.

We make the following assumptions on demand functions:

1. The maximal salary $\bar{\sigma}$ is prohibitively high so that no firm will ever hire a worker under this salary:
$$s_w = \bar{\sigma} \Rightarrow w \notin D_f(s).$$
It will be convenient to extend demand functions to incomplete salary vectors as follows: Suppose we have a salary vector that specifies salaries only for a subset $W' \subseteq W$ of the workers, $s = (s_w)_{w \in W'} \in S^{W'}$. Define $\tilde{s} \in S^W$ by $\tilde{s}_w = s_w$ for $w \in W'$ and $\tilde{s}_w = \bar{\sigma}$ for $w \notin W'$. Then we let $D_f(s) := D_f(\tilde{s})$.

2. Demand functions satisfy **irrelevance of rejected contracts (IRC)**:[9] Suppose some worker-salary combination was not chosen at some salary vector and we increase the worker's salary further. Then the firm will make the same choice after the salary has increased. Formally, let $w \in W$ and $s, s' \in S^W$ with $s_{-w} = s'_{-w}$[10] and $s_w < s'_w$. Then
$$w \notin D_f(s) \Rightarrow D_f(s) = D_f(s').$$

We make the following assumption on workers' preferences:

1. Preferences are **strict**,
$$(f, \sigma) \neq (f', \sigma') \Rightarrow (f, \sigma) \succ_w (f', \sigma') \text{ or } (f', \sigma') \succ_w (f, \sigma),$$
and
$$(f, \sigma) \succ_w \emptyset \text{ or } \emptyset \succ_w (f, \sigma).$$

2. Preferences are **increasing in salaries**,
$$\sigma < \sigma' \Rightarrow (f, \sigma) \prec_w (f, \sigma').$$

We denote the set of all strict and increasing preferences by $\mathcal{R}$.

A **matching** is a function $\mu : F \cup W \to F \cup 2^W$ such that

1. for each $f \in F$, we have $\mu(f) \in 2^W$,
2. for each $w \in W$, we have $\mu(w) \in F \cup \{\emptyset\}$,
3. for each $f \in F$ and $w \in W$, we have $f = \mu(w)$ if and only if $w \in \mu(f)$.

A **salary schedule** for a matching μ is a salary vector $s \in S^{\mu(F)}$ that for each matched worker w specifies a salary s_w paid by $\mu(w)$ to w. For each $f \in F$, we let $s_f := (s_w)_{w \in \mu(f)}$. An **allocation** is a pair (μ, s) consisting of a matching μ and a salary schedule s for μ. We denote the set of allocations by $\mathcal{A}$. For notational convenience, we extend workers'

[9] The requirement is a form of the weak axiom of revealed preferences and is an adaption of the IRC condition from the matching with contracts literature (Aygün and Sönmez, 2013) to our set-up (see Subsection 2.1.1.).

[10] Here and in the following we let $s_{-w} := (s_{w'})_{w' \in W \setminus \{w\}}$.

preferences over firm-salary pairs to preferences over allocations in the usual way; for each $w \in W$ we let

$$(\mu, s) \succ_w (\mu', s') :\Leftrightarrow (\mu(w), s_w) \succ_w (\mu'(w), s'_w).$$

Let $(D, \succeq)$ be a market. Allocation (μ, s) is

individually rational in $(D, \succeq)$ if for each $f \in F$ we have $D_f(s_f) = \mu(f)$ and for each $w \in W$ we have $(\mu, s) \succeq_w \emptyset$,

blocked in $(D, \succeq)$ by firm $f \in F$ and group of workers $W' \neq \mu(f)$ if there is a salary vector $s' \in S^W$ with $s'|_{\mu(f)} = s_f$, such that $D_f(s') = W'$ and $(f, s'_w) \succeq_w (\mu, s)$ for each $w \in W'$,[11]

stable in $(D, \succeq)$ if it is individually rational and not blocked by any firm and group of workers.

We denote the set of all stable allocations in $(D, \succeq)$ by $\mathcal{S}(D, \succeq)$. The following lemma provides a reformulation of the stability condition that will be useful in some of the proofs. The proof as well as all subsequent proofs are in the appendix.

Lemma 1. *For $(\mu, s) \in \mathcal{A}$, $f \in F$ and $\succeq \in \mathcal{R}^W$ define the* **minimal potential blocking vector** $\tilde{s}_f = (\tilde{s}_{fw})_{w \in W} \in S^W$ *by*

$$\tilde{s}_{fw} := \min(\{\sigma \in S : (f, \sigma) \succeq_w (\mu, s)\} \cup \{\bar{\sigma}\}).$$

Let D be a demand profile. Then $(\mu, s) \in \mathcal{S}(D, \succeq)$ if and only if (μ, s) is individually rational in $(D, \succeq)$ and for each $f \in F$ we have $D_f(\tilde{s}_f) = \mu(f)$.

A **mechanism** (for the workers) is a mapping from preference profiles to allocations $\mathcal{M} : \mathcal{R}^W \to \mathcal{A}$. Mechanism $\mathcal{M}$ is **strategy-proof** if it is a weakly dominant strategy for workers to report their true preferences to the mechanism, i.e. for each $w \in W$, $\succeq_{-w} \in \mathcal{R}^{W \setminus \{w\}}$ and $\succeq_w, \succeq'_w \in \mathcal{R}$ we have

$$\mathcal{M}(\succeq_w, \succeq_{-w}) \succeq_w \mathcal{M}(\succeq'_w, \succeq_{-w}).$$

Let D be a demand profile. Mechanism $\mathcal{M}$ is **D-stable** if for each $\succeq \in \mathcal{R}^W$ we have $\mathcal{M}(\succeq) \in \mathcal{S}(D, \succeq)$.

[11]Note that, when blocking, the firm pays the workers that it will keep after the blocking the same salary as before (since $s'|_{\mu(f)} = s_f$). It appears natural to also allow for blocking such that the firm pays some of its current workers a higher salary. However, we make the implicit assumption that a firm will always choose a low salary contract over a high salary contract with the same worker. Hence, if the firm blocks by offering higher salaries to some of the workers that it currently employs and hiring some additional workers, then it could also block by offering the same salary to the workers that it keeps and otherwise hiring the new workers. In Section 2.1.1, we will show that our blocking notion corresponds to the usual blocking notion for the matching with contracts model, if we assume that a firm will always choose a low salary contract over a high salary contract with the same worker.

2.1.1 Matching with Contracts

Our model can be mapped into the matching with contracts model as follows (here we follow Hatfield and Milgrom, 2005): The set of possible contracts is $X = F \times W \times S$. Thus, a contract (f, w, σ) is a bilateral agreement between a firm f and a worker w to match under a salary σ. For each $f \in F$ we define a choice functions $C_f : 2^X \to 2^X$ as follows. For each set of contracts $X' \subseteq X$ define a salary vector $s_f(X') = (s_{fw}(X'))_{w \in W}$, such that salary $s_{fw}(X')$ is the lowest salary for w with firm f occurring in a contract in X', i.e.

$$s_{fw}(X') := \min\{\sigma \in S : (f, w, \sigma) \in X' \text{ or } \sigma = \bar{\sigma}\}.$$

Firm f chooses from X' the contracts with the workers that it demands under the minimal salaries $s_f(X')$:

$$C_f(X') := \{(f, w, s_{fw}(X')) : w \in D_f(s_f(X'))\}.$$

The IRC condition on demand functions translates to the IRC condition on choice functions (Aygün and Sönmez, 2013):

$$(f, w, \sigma) \notin C_f(X') \Rightarrow C_f(X' \setminus \{(f, w, \sigma)\}) = C_f(X').$$

Worker w's preferences over $X \cup \{\emptyset\}$ are given by $(f, w, \sigma) \succ_w (f', w', \sigma') \Leftrightarrow (f, \sigma) \succ_w (f', \sigma')$ and $(f, w, \sigma) \succ_w \emptyset \Leftrightarrow (f, \sigma) \succ_w \emptyset$ with the convention that $\emptyset \succ_w (f, w', \sigma)$ for $w' \neq w$. It is easy to check that our definition of stability is equivalent to the usual stability condition in the matching with contracts literature. In this sense, our model is just a matching with contracts model with the additional restriction that preferences are monotone in the contract terms.

2.2 Stable Allocations

In general, a stable allocation does not need to exist for our model. A sufficient condition for stability is that workers are gross substitutes for firms, i.e. increasing the salary of some worker will not decrease the demand for other workers whose salaries have not changed.

Gross Substitutability: For workers $w, w' \in W, w \neq w'$ and salary vectors $s, s' \in S^W$ with $s'_{-w'} = s_{-w'}$ and $s_{w'} < s'_{w'}$,

$$w \in D_f(s) \Rightarrow w \in D_f(s').$$

Not only is gross substitutability sufficient for the existence of a stable allocation but it also guarantees that the set of stable allocations has a lattice structure. If workers are gross substitutes for firms, then the set of stable allocation forms a lattice with respect to the preferences of workers (Blair, 1988). In particular, there is a unique stable allocation that is most preferred by all workers among all stable allocations. We call this allocation the **worker-optimal stable** allocation. It can be found by the **salary adjustment process**[12] that is defined for a market $(D, \succeq)$ as follows:

1. The process proceeds in rounds $t = 0, 1, \ldots$ In each round t, each worker w faces an **offer vector** $s_w(t) \in (S \cup \{0\})^F$ of salaries under which the workers can apply to firms. The salary 0 is added for notational convenience with the convention that workers never apply for a salary of 0 to any firm. We start with salaries $s_w(0) := (\bar{\sigma})_{f \in F}$ for each $w \in W$. Furthermore, in each round t, each firm f faces an **offer vector** $s_f(t) \in S^W$ of salaries with the best applications that it has received up to and including round t.

2. In round t, each worker w applies to his favorite firm under salaries $s_w(t)$ or stays alone, if he finds no firm acceptable under these salaries. After workers have applied, we update the offer vectors for the firms. For each firm f, we let $s_f(t) \in S^W$ be the vector of lowest salaries under which the workers have applied to f up to and including round t (with $(s_f(t))_w = \bar{\sigma}$ if w never has applied to f).

3. Each firm tentatively accepts the workers $D_f(s_f(t))$ and rejects all other workers. If all applications are accepted, then we go to Step 4. Otherwise, for each worker w we let $s_w(t+1)$ be the vector of the highest salaries under which w has not been rejected up to and including round t (with $(s_w(t+1))_f = 0$, if w has been rejected by f under all salaries). Then we repeat Step 3, with salary vectors $(s_w(t+1))_{w \in W}$.

4. We match each firm to the workers whose application it has accepted in the last round under the offered salary.

In general, the salary adjustment process does not need to converge to a feasible allocation. It could be the case that a worker w applies in some round to some firm f which tentatively accepts the worker, but another firm f' wants to accept an application made by w to f' in an earlier round that f' had previously rejected. Thus, in the final allocation multiple firms could be matched to the same worker. Gross substitutability rules out this possibility, since it guarantees that firms will never want to recall applications made in previous rounds. Moreover, by the definition of the process, if the outcome of the process is feasible then it is stable as well. Later we will see that weaker conditions than gross substitutability guarantee the convergence to a feasible (and stable) allocation. Let D be a demand profile, such that the salary adjustment process converges to a feasible outcome for any preference profile. Then the **salary adjustment mechanism** for D assigns to each $\succeq \in \mathcal{R}^W$ the outcome of the salary adjustment process in $(D, \succeq)$.

Worker-optimality is related to strategy-proofness. Under gross substitutability and the following additional condition on the firms' demand functions the salary adjustment

[12] For the more general model with arbitrary contracts, this is called the **cumulative offer process**. As Hatfield and Milgrom (2005), we consider a version of the process were multiple workers per round make new applications. We could also consider a version of the process were applications are made subsequently. If workers are gross substitutes for firms, then this will yield the same outcome (Hirata and Kasuya, 2014).

mechanism is strategy-proof (Hatfield and Milgrom, 2005).

Law of Aggregate Demand. For salary vectors $s, s' \in S^W$

$$s \leq s' \Rightarrow |D_f(s)| \geq |D_f(s')|.$$

The following proposition summarizes known results about side-optimal stable allocations, the invariance of the set of matched workers in stable allocations (the rural hospitals theorem), and strategy-proofness.

Proposition 1 (Kelso and Crawford, 1982; Blair, 1988; Hatfield and Milgrom, 2005)**.**

1. *If workers are gross substitutes for firms, then the salary adjustment process converges to a stable allocation that is most preferred by all workers among all stable allocations.*
2. *If demand functions satisfy, moreover, the law of aggregate demand, then*
   (a) *the set of employed workers is the same in all stable allocations,*
   (b) *the salary adjustment mechanism is strategy-proof.*

3 Results

3.1 Virtual Demand Functions

It is a natural question, whether the conditions of Section 2.2 for the stability and strategy-proofness of the salary adjustment process are also necessary. Next we provide a counter example showing that gross substitutability and the law of aggregate demand are not necessary for the salary adjustment mechanism to be stable and strategy-proof. The example will have the following structure: For each but one firm, workers are gross substitutes and the law of aggregate demand holds. There is one firm f for which workers are not gross substitutes. However, f's demand function can be replaced by another demand function, such that the outcome of the salary adjustment process is the same under the original demand profile and the profile where f's demand function is replaced. Under the second demand function - we will subsequently call it a virtual demand function - workers are gross substitutes for f and the law of aggregate demand holds. Thus, the salary adjustment mechanism is stable and strategy-proof both for the original market and the market where we have replaced f's demand function by the virtual demand function.

Example 1. Let $W = \{w_1, w_2, w_3\}$ and $f \in F$ be a firm. Suppose firms $F \setminus \{f\}$ have demand functions $D_{-f} = (D_{f'})_{f' \neq f}$ under which workers are gross substitutes and the law of aggregate demand holds. For firm f we consider two different demand functions

D_f and $D^{\vee}_f$ that are defined as follows:

$$D_f(s) = \begin{cases} \{w_1, w_2, w_3\}, & \text{if } s = (1,1,1), \\ \{w_2\}, & \text{if } s \neq (1,1,1) \text{ and } s_2 \leq 2, \\ \emptyset, & \text{else.} \end{cases}$$

$$D^{\vee}_f(s) = \begin{cases} \{w_2\}, & \text{if } s_2 \leq 2, \\ \emptyset, & \text{else.} \end{cases}$$

Note that under D_f workers are not gross substitutes as $w_3 \in D(1,1,1) = \{w_1, w_2, w_3\}$ but $w_3 \notin D_f(2,1,1) = \{w_2\}$ and that under $D^{\vee}_f$ workers are gross substitutes.

Let $\succeq \in \mathcal{R}^W$. We show that the salary adjustment in the market $(D_f, D_{-f}, \succeq)$ and the salary adjustment process in the market $(D^{\vee}_f, D_{-f}, \succeq)$ converge to the same allocation. Observe that the demand functions D_f and $D^{\vee}_f$ differ only at the salary vector $(1,1,1)$. Thus, for the salary adjustment processes to differ in the two markets, workers w_1, w_2 and w_3 must all apply to f under salary 1 during the salary adjustment process in $(D, \succeq)$. Note however that before w_2 applies to f under salary 1, he applies to f under salary 2. But once the process tentatively matches w_2 to f under salary 2, the firm will not subsequently drop the worker w_2 or accept additional workers. Thus, w_2 will never apply to f under salary 1. Hence, the salary adjustment processes in the two markets converge to the same allocation, which is the worker-optimal stable allocation in $(D^{\vee}_f, D_{-f}, \succeq)$. By Proposition 1, the salary adjustment mechanism for $(D^{\vee}_f, D_{-f})$ is strategy-proof and $(D^{\vee}_f, D_{-f})$-stable. The argument above implies that the mechanism is D-stable as well. Thus, there is a D-stable and strategy-proof mechanism.

The argument in the example can be made more generally. Subsequently, we will define for each demand function a corresponding virtual demand function. To define the virtual demand function $D^{\vee}_f$ for a demand function D_f, we consider a market with only one firm f and workers W. Thus, we have an auction rather than a matching market. Since there is only one firm, the workers' preferences are determined by their **reservation salaries**, i.e. the smallest salaries under which the workers are willing to work for f. Now suppose we run the salary adjustment process in the market consisting of firm f with demand function D_f and workers with reservation salaries $s \in S^W$. Let $s^{\vee}$ be the terminal offer vector for firm f in the salary adjustment process under reservation salaries s. In the terminal allocation f is matched to $D_f(s^{\vee})$. We define the **virtual demand function** for D_f to be the demand function $D^{\vee}_f : S^W \to 2^W$ where $D^{\vee}_f(s) := D_f(s^{\vee})$.

Example 1 (cont.). To see that

$$D^{\vee}_f(s) = \begin{cases} \{w_2\}, & \text{if } s_2 \leq 2, \\ \emptyset, & \text{else} \end{cases}$$

satisfies the definition of a virtual demand function, note that for each $s = (s_1, s_2, s_3)$ with $s_2 \leq 2$ a descending auction will terminate in an allocation that matches w_2 to f

under salary 2 i.e. for $s_2 \leq 2$ we have $s^\vee = (s_1, 2, s_3)$ and $D_f(s^\vee) = \{w_2\}$. Otherwise, the terminal assignment is empty, i.e. for $s_2 > 2$ we have $s^\vee = s$ and $D_f(s^\vee) = \emptyset$.

In the following, we indicate for each property of a demand function that the property holds for the virtual demand function by adding the adjective "virtual". Thus, we say that workers are **virtual gross substitutes** for firm f if they are gross substitutes according to the virtual demand function. Similarly, we talk about the **virtual law of aggregate demand**, etc. For a demand profile $D = (D_f)_{f\in F}$ we call $D^\vee = (D_f^\vee)_{f\in F}$ the **virtual demand profile** for D and for a market $(D, \succeq)$ we call the market $(D^\vee, \succeq)$ the **virtual market**.

If workers are virtual gross substitutes for a firm, then the virtual demand function of that firm is well-behaved in the sense that it satisfies the properties required for demand functions in Section 2: By definition, a worker is never virtually demanded under the maximal salary $\bar{\sigma}$. Moreover, the virtual demand function satisfies IRC. More generally it will satisfy a stronger regularity condition that implies IRC. We say that a demand function D_f satisfies the **law of demand invariance** if for $s, s' \in S^W$ with $s_{-w} = s'_{-w}$ and $s_w < s'_w$ we have

$$w \in D_f(s') \Rightarrow D_f(s) = D_f(s').$$

The law of demand invariance implies IRC: Let $w \notin D_f(s)$ and suppose for the sake of contradiction that $w \in D_f(s')$. Since $w \in D_f(s')$ the law of demand invariance implies that $D_f(s) = D_f(s')$. But then $w \in D_f(s) = D_f(s')$, contradicting $w \notin D_f(s)$.

If workers are virtual gross substitutes for a firm, then the virtual demand function of that firm satisfies the law of demand invariance and thus IRC.

Lemma 2. *If workers are virtual gross substitutes for a firm, then the virtual demand function of the firm satisfies the law of demand invariance. In particular, the virtual demand function of the firm satisfies IRC.*

As illustrated by Example 1, demand and virtual demand can differ. However, in two important cases they coincide. Later, in Section 3.2.1, we will show that demand and virtual demand agree for demand functions induced by quasi-linear profit function. Moreover, demand and virtual demand agree for demand functions under which workers are gross substitutes and the law of aggregate demand holds. Both results follow from the following more general result.

Proposition 2. *If the demand function D_f satisfies the law of demand invariance, then the demand function and the virtual demand function are the same $D_f = D_f^\vee$. If the demand function satisfies gross substitutability and the law of aggregate demand, then it satisfies the law of demand invariance.*

3.1.1 Stability

Next we relate stability in the virtual market to stability in the original market.

Proposition 3. *Let D be a demand profile, $D^\vee$ its virtual version and $\succeq$ a preference profile. If workers are virtual gross substitutes for firms, then the outcome of the salary* *adjustment process in the original market $(D, \succeq)$ and in the virtual market $(D^{\vee}, \succeq)$ is the same and stable in both markets.*

3.2 A Maximal Domain Result

In this section, we show that the domain of demand functions, such that workers are virtual gross substitutes for firms and the virtual law of aggregate demand holds, is a maximal Cartesian domain for the existence of a stable and strategy-proof mechanism. This means that if we choose a demand profile $D = (D_f)_{f \in F}$ such that each D_f has the two properties, then the salary adjustment mechanism is well-defined, stable, and strategy-proof. On the other hand, if either of the conditions fails for the demand function of a firm f, then we can define unit demand functions $D_{-f} = (D_{f'})_{f' \neq f}$[13] for the other firms, such that for the profile $D = (D_f, D_{-f})$ there is no D-stable, strategy-proof mechanism.

The following lemma will be useful in the proof of the maximal domain result. It states that stable and strategy-proof mechanisms are unique whenever they exist. Similar results are known for the classical matching model (Alcalde and Barberà, 1994) and the model with contracts (Hirata and Kasuya, 2015).

Lemma 3. *Let D be a demand profile. If there is a D-stable and strategy-proof mechanism, then it is unique. If workers are, moreover, virtual gross substitutes for firms, then the stable and strategy-proof mechanism implements the worker-optimal stable allocation in the virtual market.*

With the lemma we can proof our main result for ordinal markets.

Theorem 1. *The domain of demand functions under which workers are virtual gross substitutes and the virtual law of aggregate demand holds is maximal for the existence of a stable and strategy-proof mechanism.*

1. *Let D be a demand profile such for each firm workers are virtual gross substitutes and the virtual law of aggregate holds. Then, there is a D-stable and strategy-proof mechanism. The mechanism implements for each preference profile $\succeq$ the worker-optimal stable allocation in the virtual market $(D^{\vee}, \succeq)$.*

2. *Let D_f be a demand function such that either workers are not virtual gross substitutes or the virtual law of aggregate demand fails. Then, there are unit demand functions D_{-f} for the other firms, such that no D-stable and strategy-proof mechanism exists.*

[13] A **unit demand function** is a demand function, such that the firm demands at most one worker at each salary vector, i.e. for each $s \in S^W$ we have $|D_{f'}(s)| \leq 1$. This implies in particular that workers are gross substitutes for the firm.

4 Quasi-linear Profit Functions

Next, we spell out the implications of the previous result for the quasi-linear case. In the quasi-linear set-up, demand functions are obtained by the maximization of profit functions where salaries enter linearly (and negatively) in the profit. We do not assume quasi-linearity for workers' preferences. If a worker prefers to work for a firm f under a salary σ to working for another firm f' under a salary of σ', he may nevertheless prefer to work for f' instead of f, if both salaries σ and σ' are raised by the same amount.

For the special case of quasi-linear profit functions, we will see that demand and virtual demand agree. It is then a straightforward consequence of the proof of Theorem 1 that gross substitutes profit functions are a maximal domain of quasi-linear profit functions for the existence of a stable allocation. We state two versions of the maximal domain result: one discrete version where salaries are restricted to discrete units and one continuous version where salaries can be arbitrary positive real numbers. Thus, we prove a converse to both existence results of Kelso and Crawford (1982). For the continuous model, a similar result was obtained by Gul and Stacchetti (1999).

In contrast to the result of the previous section, the domain is maximal for the existence of a stable allocation. However, in the quasi-linear case, gross substitutability is also sufficient for the existence of a stable and strategy-proof mechanism. Thus, the domain is maximal for the existence of a stable and strategy-proof mechanism as well. Again we obtain two versions of this result, one for the discrete model and one for the continuous model. The result for the continuous model generalizes previous results by Hatfield et al. (2014). On the way to proving the result for the continuous model, we will establish continuous counter-part to the existence result for a worker-optimal stable allocation and the rural hospitals theorem. The results are derived by a limit argument from their discrete counterparts.

4.1 Model

The following model is the original job matching model of Kelso and Crawford (1982). As before, we consider a finite set of **firms** F and a finite set of **workers** W. Each firm f has a **production function** $y_f : 2^W \to \mathbb{R}$ that assigns to each set of workers the **gross product** of the workers working for the firm f measured in the same unit as salaries. We assume that $y_f(\emptyset) = 0$. If f hires workers W' under salaries $s \in \mathbb{R}^{W'}_{++}$, then f's **(net) profit** is

$$\pi_f(W', s) := y_f(W') - \sum_{w \in W'} s_w.$$

Each worker w has a **utility function** $u_w : F \times \mathbb{R}_{++} \cup \{\emptyset\} \to \mathbb{R}$ that is continuous and strictly increasing in salaries.

A **continuous (quasi-linear) market** is a pair (y, u) consisting of a production profile $y = (y_f)_{f \in F}$ and a utility profile $u = (u_w)_{w \in W}$. A **discrete (quasi-linear) market** is a triple (y, u, σ_0) consisting of a production profile y, a utility profile u, and a smallest salary increment $\sigma_0 \in \mathbb{R}_{++}$.

Continuous and discrete markets only differ in so far as salaries in discrete markets are restricted to integer-multiples of the smallest salary increment.[14] More precisely, an **allocation** in a continuous market, is any pair (μ, s) consisting of a matching μ and a salary schedule $s \in \mathbb{R}_{++}^{\mu(F)}$. We denote the set of allocations in continuous markets by $\mathcal{A}$. An allocation in a discrete market with smallest increment σ_0 is a $(\mu, s) \in \mathcal{A}$ such that for each $w \in \mu(F)$ we have $s_w = k \cdot \sigma_0$ for some $k \in \mathbb{N}$. We denote the set of allocations for discrete markets with increment σ_0 by $\mathcal{A}(\sigma_0)$. For notational convenience, we extend the domains of firms' profit functions and of workers' utility functions to the domain of allocations in the usual way; for each $f \in F$ we let

$$\pi_f(\mu, s) = \pi_f(\mu(f), s_f),$$

and for each $w \in W$ we let

$$u_w(\mu, s) = u_w(\mu(w), s_w).$$

A discrete market (y, u, σ_0) has **no ties** if agents are never indifferent between different assignments, i.e. for allocations $(\mu, s), (\mu', s') \in \mathcal{A}(\sigma_0)$ and each $f \in F$ we have

$$\mu(f) \neq \mu'(f) \Rightarrow \pi_f(\mu, s) \neq \pi_f(\mu', s')$$

and for each $w \in W$ we have

$$\mu(w) \neq \mu'(w) \Rightarrow u_w(\mu, s) \neq u_w(\mu', s').$$

Discrete market without ties are the generic case in the sense that almost every discrete market has no ties. A discrete market (y, u, σ_0) without ties can be considered to be a special case of the ordinal markets studied in the previous sections. Let $\bar{\sigma} > \max_{f \in F, W' \subseteq W} y_f(W')$ be an integer-multiple of σ_0 and define a set of possible salaries by $S := \{\sigma_0, 2\sigma_0, 3\sigma_0, \ldots, \bar{\sigma}\}$. Define for each $f \in F$ a demand function $D_f : S^W \to 2^W$ by

$$D_f(s) := \text{argmax}_{W' \subseteq W} \pi_f(W', s) \tag{1}$$

and for each $w \in W$ a strict preference relation $\succ_w$ over $F \times S \cup \{\emptyset\}$ by

$$(\mu, s) \succ_w (\mu', s') :\Leftrightarrow u_w(\mu, s) > u_w(\mu', s').$$

We call $(D, \succeq)$ the **ordinal market corresponding to** (y, u, σ_0).

4.1.1 Stability

We introduce two stability notions for quasi-linear markets. Strict core stability corresponds to the stability notion that we have considered earlier, in the sense that for discrete markets without ties, strict core allocations are the stable allocations in the

[14] More generally, all subsequent results would hold for arbitrary discrete salary spaces. We restrict ourselves to the case with uniform salary increments for notational convenience.

corresponding ordinal market. We also introduce the weaker stability notion of core stability, because for markets with ties, strict core allocations sometimes fail to exist while core allocations do exist. For continuous markets the two stability notions agree.

We define the stability notions for a continuous market. The definitions carry over to a discrete market by requiring that all salaries occurring in the definitions are integer-multiples of the salary increment. Let (y, u) be a continuous market. Allocation (μ, s) is

> **individually rational** if for each $f \in F$ and $W' \subseteq \mu(f)$ we have $\pi_f(\mu, s) \geq \pi_f(W', s)$ and for each $w \in W$ we have $u_w(\mu, s) \geq u_w(\emptyset)$.
>
> **strictly blocked** by firm $f \in F$ and workers $W' \subseteq W$ with salaries $s' \in \mathbb{R}^{W'}_{++}$ if $\pi_f(W', s') > \pi_f(\mu, s)$ and $u_w(f, s'_w) > u_w(\mu, s)$ for each $w \in W'$.
>
> **blocked** by firm $f \in F$ and workers $W' \subseteq W$ with salaries $s' \in \mathbb{R}^{W'}_{++}$ if $\pi_f(W', s') \geq \pi_f(\mu, s)$ and $u_w(f, s'_w) \geq u_w(\mu, s)$ for each $w \in W'$ and at least one of the inequalities is strict.
>
> in the **core** of (y, u) if it is individually rational and not strictly blocked by any firm and group of workers
>
> in the **strict core** of (y, u) if it is individually rational and not blocked by any firm and group of workers.

We denote the set of core allocations in (y, u) by $\mathcal{C}(y, u)$ and the set of core allocations in (y, u, σ_0) by $\mathcal{C}(y, u, \sigma_0)$. As utility and profit functions are continuous in salaries, the set $\mathcal{C}(y, u)$ is also the set of strict core allocation in (y, u).

One readily checks that strict core allocations in a discrete market without ties are just the stable allocations in the corresponding ordinal market.

Lemma 4. *For discrete markets without ties, an allocation is in the strict core if and only if it is stable in the corresponding ordinal market.*

Core allocations in discrete and continuous markets are related as follows. If an allocation is in the core of a continuous market and the salaries in the allocation are integer-multiple of some increment σ_0, then it is also in the core of the discrete market with minimal salary increment σ_0. On the other hand, core allocations in continuous markets can be approximated by core allocations in discrete markets in the following sense. We say that a sequence of allocation $(\mu^t, s^t)_{t=0,1,\ldots}$ **converges** to an allocation (μ, s) if the sequence of matrices $(M^t)_{t=0,1,\ldots} \subseteq \mathbb{R}^{F \times W}$ defined by

$$m^t_{fw} := \begin{cases} s^t_w, & \text{if } \mu^t(w) = f, \\ 0, & \text{if } \mu^t(w) = f, \end{cases}$$

converges to the matrix M defined by

$$m_{fw} := \begin{cases} s_w, & \text{if } \mu(w) = f, \\ 0, & \text{if } \mu(w) = f, \end{cases}$$

in $\mathbb{R}^{F\times W}$. With this definition, we have the following lemma that will be useful when deriving continuous analogs to result for the discrete model.

Lemma 5. *Let $(\sigma_0(t))_{t=0,1,\ldots}$ be a sequence of salary increments with $\lim_{t\to\infty}\sigma_0(t) = 0$ and $\{(\mu^t, s^t)\}_{t=0,1,\ldots}$ be a sequence of allocations with $(\mu^t, s^t) \in \mathcal{C}(y, u, \sigma_0(t))$. If the sequence $\{(\mu^t, s^t)\}_{t=0,1,\ldots}$ converges, then $\lim_{t\to\infty}(\mu^t, s^t) \in \mathcal{C}(y, u)$.*

4.2 Maximal Domain Results

4.2.1 Discrete Markets without Ties

A maximal domain result for markets without ties follows from the proof of Theorem 1 by observing that for quasi-linear profit functions, demand and virtual demand agree.

Lemma 6. *Let π_f be a quasi-linear profit function that has no ties for salaries that are integer-multiples of σ_0. Then the demand and virtual demand induced by π_f agree.*

The following theorem is a counterpart to Theorem 4 of Kelso and Crawford (1982). It follows from the previous lemma and Claim 3 in the proof of Theorem 1.

Theorem 2. *The domain of profit functions such that workers are gross substitutes is a maximal domain of quasi-linear profit functions that guarantee the existence of a strict core allocation in discrete markets without ties.*

1. *In each discrete market without ties such that workers are gross substitutes for firms, there is a strict core allocation.*

2. *Let y_f be a production function such that π_f has no ties when salaries are integer-multiples of σ_0 and such that workers are not gross substitutes for firm f. Then there exists unit demand production functions[15] y_{-f} and utility functions u such that (y, u, σ_0) has no ties and no strict core allocation in (y, u, σ_0) exists.*

4.2.2 Continuous Markets and Discrete Markets with Ties

The construction in the proof of Theorem 3 generalizes straightforwardly to the continuous model. To state and prove the result for the continuous model, we have to generalize the gross substitutes condition to the continuous model. The only difference is that we now have to deal with demand correspondences rather than demand functions. A production function y_f induces a **demand correspondence** $D_f : \mathbb{R}^W_{++} \rightrightarrows 2^W$ that is defined for each $s \in \mathbb{R}^W_{++}$ by Equation (1). Note that there now can be multiple profit maximizing bundles of workers, i.e. $D_f(s)$ can have multiple values $D_f(s) \subseteq 2^W$. Gross substitutability for the demand correspondence is defined as follows.

[15] Unit demand production functions are those that induce unit demand functions. Alternatively, we can now define a **unit demand production function** as a production function, such that $y_f(W') = \max_{w\in W'} y_f(\{w\})$ for each $W' \subseteq W$ with $W' \neq \emptyset$. This definition will carry over to the case with ties that we discuss later.

Gross Substitutability with Ties: For each worker $w' \in W$ and salary vectors $s, s' \in \mathbb{R}^W_{++}$ with $s'_{-w'} = s_{-w'}$ and $s_{w'} < s'_{w'}$ the following holds: For each $W' \in D_f(s)$ there exists a $W'' \in D_f(s')$ such that $W' \setminus \{w'\} \subseteq W''$.

The following theorem is a counterpart to Theorem 2 of Kelso and Crawford (1982).

Theorem 3. *The domain of profit functions such that workers are gross substitutes is a maximal domain of quasi-linear profit functions that guarantee the existence of a (strict) core allocation in continuous markets.*

1. *In each continuous market such that workers are gross substitutes for firms, there is a (strict) core allocation.*
2. *Let y_f be a production function such that workers are not gross substitutes for firm f. Then there exist unit demand production functions y_{-f} and utility functions u such that no (strict) core allocation in (y, u) exists.*

In discrete markets with ties, the strict core can be empty, even if workers are gross substitutes for firms.[16] However, core allocations still exist in this case (see Theorem 1 of Kelso and Crawford, 1982). It is a natural question, whether gross substitutes profit functions form a maximal domain of quasi-linear profit functions for the existence of core allocations in discrete markets. In general, the answer to this question is negative as the following example demonstrates.

Example 2. Let $f \in F$ and $W = \{w_1, w_2, w_3\}$. Define a production function $y_f : 2^W \to \mathbb{R}$ by $y_f(\{w_1, w_2, w_3\}) = 3.4$, $y_f(\{w_1\}) = 1.3$ and $y_f(W') = 0$ for all other $W' \subseteq W$. Workers are not gross substitutes for f. Under salaries $(1, 1, 1)$, the firm chooses all workers, but under salaries $(1, 1, 2)$, it chooses only worker w_1 and in particular not worker w_2.

Suppose salaries are restricted to integer values, $\sigma_0 = 1$. Consider production functions y_{-f} such that workers are gross substitutes for the other firms, and a utility profile u. We claim that there is a core allocation in $(y, u, 1)$. To see this, consider the modified production function $\tilde{y}_f : 2^W \to \mathbb{R}$ such that $\tilde{y}_f(\{w_1\}) = 1.3$ and $\tilde{y}_f(W') = 0$ for all other $W' \subseteq W$. Under $\tilde{y}_f$ workers are gross substitutes for firm f. Thus, there is a core allocation (μ, s) in $(\tilde{y}_f, y_{-f}, u, 1)$. We show that (μ, s) is in the core of $(y, u, 1)$ as well.

Suppose for the sake of contradiction that the allocation (μ, s) is strictly blocked in $(y, u, 1)$. The only possible blocking coalition consists of f and workers w_1, w_2, w_3. By

[16] The following market is a simple example. Let $F = \{f\}$ and $W = \{w_1, w_2\}$. Salaries are restricted to integers, $\sigma_0 = 1$. Define y_f by $y_f(\{w_1\}) = y_f(\{w_2\}) = 1.1$ and $y_f(W') = 0$ for all other $W' \subseteq W$. Choose utility functions such that for $w = w_1, w_2$ we have $u_w(f, 1) > u_w(\emptyset)$. Since salaries are integer-valued, there are three individually rational allocations. The empty allocation, the allocation where worker w_1 is matched to the firm under salary 1, and the allocation where worker w_2 is matched to the firm under salary 1. The first allocation is blocked by the firm and either of the two workers with salary 1, the second allocation is blocked by the firm (which is indifferent) and worker w_2, the third allocation is blocked by the firm (which is indifferent) and the worker w_1. Hence there is no strict core allocation. There are, however, two core allocation: One where f is matched to w_1 under salary 1 and one where f is matched to w_2 under salary 1.

individual rationality of (μ, s) in $(\tilde{y}_f, y_{-f}, u, 1)$ we either have $\mu(f) = \{w_1\}$ and $s_{w_1} = 1$ or $\mu(f) = \emptyset$. In the first case, firm f has to pay worker w_1 a salary of 2 to make w_1 strictly better off than in (μ, s). In the second case, we have $u_{w_1}(\mu, s) \geq u_{w_1}(f, 1)$ since otherwise f and w_1 would strictly block (μ, s) in $(\tilde{y}_f, y_{-f}, u, 1)$. Thus, in the second case, firm f has to pay worker w_1 a salary of at least 2 to make w_1 strictly better off than in (μ, s). Moreover, in both cases, firm f has to pay the workers w_2 and w_3 salaries of at least 1 to block. But $\pi_f(\{w_1, w_2, w_3\}, 2, 1, 1) = -0.6 < 0 = \min\{\pi_f(\emptyset), \pi_f(\{w_1\}, 1)\} \leq \pi_f(\mu, s)$. Thus, f and w_1, w_2, w_3 block in neither case. Hence (μ, s) is in the core of $(y, u, 1)$.

The above example shows that there can exist core allocations in markets with complementarities, provided that the salary space is very coarse. However, we obtain a maximal domain result, if we require additionally that salary increments are "small enough". The following result is a counterpart to Theorem 2 of Kelso and Crawford (1982) and follows directly from Theorem 3.

Corollary 1. *The domain of profit functions such that workers are gross substitutes is a maximal domain of quasi-linear profit functions for the existence of a core allocation in discrete markets with small salary increments.*

1. *In each discrete market such that workers are gross substitutes for firms, there is a core allocation.*
2. *Let y_f be a production function such that workers are not gross substitutes for firm f. Then there exist unit demand production functions y_{-f}, utility functions u and a $\sigma_0' \in \mathbb{R}_{++}$ such that for each $\sigma_0 < \sigma_0'$ no core allocation in (y, u, σ_0) exists.*

4.3 Strategy-proofness

In the previous section, we dealt with stability alone and did not consider strategy-proofness. In the following, we show that for quasi-linear markets, stable and strategy-proof mechanisms exist, if workers are gross substitutes for firms.

First we adapt the notion of a stable and strategy-proof mechanism to the modified set-up. Let U be the set of all utility functions that are continuous and strictly increasing in salaries. A **mechanism** (for the workers) is a mapping from utility profiles to allocations $\mathcal{M} : U^W \to \mathcal{A}$. Mechanism $\mathcal{M}$ is **strategy-proof**, if it is a weakly dominant strategy for workers to report their true utility function to the mechanism, i.e. for each $w \in W$, $u_{-w} \in U^{W \setminus \{w\}}$ and $u_w, u_w' \in U$ we have

$$u_w(\mathcal{M}(u_w, u_{-w})) \geq u_w(\mathcal{M}(u_w', u_{-w})).$$

Let y be a production profile and $\sigma_0 \in \mathbb{R}_{++}$ a salary increment. Mechanism $\mathcal{M}$ is **(y, σ_0)-core-stable**, if for each $u \in U^W$ we have $\mathcal{M}(u) \in \mathcal{C}(y, u, \sigma_0)$. Mechanism $\mathcal{M}$ is **y-core-stable** if for each $u \in U^W$ we have $\mathcal{M}(u) \in \mathcal{C}(y, u)$.

The intuition, why we get strategy-proofness "for free" in the quasi-linear case, lies in the fact that gross substitutability implies the law of aggregate demand for quasi-linear profit functions. More precisely, we can define the law of aggregate demand for correspondences and then observe that in the quasi-linear case, gross substitutability implies the law of aggregate demand.

Law of Aggregate Demand with Ties. For salary vectors $s, s' \in \mathbb{R}^W_{++}$ with $s \leq s'$ and each $W' \in D_f(s)$ there exist a $W'' \in D_f(s')$ such that $|W'| \geq |W''|$.

With this definition, we have the following result due to Hatfield and Milgrom (2005).

Proposition 4 (Hatfield and Milgrom, 2005)**.** *For each quasi-linear profit function such that workers are gross substitutes, the induced demand correspondence satisfies the law of aggregate demand.*

4.3.1 Discrete Mechanisms

First we consider the discrete model. To define a mechanism we have to deal with tie-breaking. We show that if workers are gross substitutes for firms, then for each $\sigma_0 \in \mathbb{R}_{++}$ there exists a profile D^{σ_0} of well behaved demand functions defined for salaries that are integer multiples of σ_0 that select from the demand correspondences induced by y_f. The demand functions are well-behaved in the sense that for each firm, workers are gross substitutes and the law of aggregate demand holds. The demand profile D^{σ_0} can be used to define a mechanism: First, if necessary, ties in workers' reported utilities u are broken according to some prescribed tie-breaking procedure to obtain a profile of strict worker preferences $\succeq$. Second, the salary adjustment process is run in the market $(D^{\sigma_0}, \succeq)$. Now we show that Proposition 4 implies that there exists a well-behaved and single-valued selection from the demand correspondence.[17]

Lemma 7. *Let y_f be a production profile such that workers are gross substitutes for firm f and let $S \subseteq \mathbb{R}_{++}$ be finite. Then there exists a demand function $\tilde{D}_f : S^W \to 2^W$ such that*

1. *$\tilde{D}_f$ is a selection from the demand correspondence D_f induced by y_f, i.e. for each $s \in S^W$ we have $\tilde{D}_f(s) \in D_f(s)$.*
2. *under $\tilde{D}_f$, workers are gross substitutes and the law of aggregate demand holds.*

Lemma 7 allows to define a core-stable and strategy-proof mechanism.

Proposition 5. *Let y be a production function such that workers are gross substitutes for firms, then for each $\sigma_0 \in \mathbb{R}_{++}$ there exists a (y, σ_0)-core-stable and strategy-proof mechanism.*

[17] We remark, however, that not every single-valued selection has to be well-behaved. Consider, for example, the following production function for a market with two workers w_1, w_2: $y_f(\{w_1, w_2\}) = 5, y_f(\{w_1\}) = 3, y_f(\{w_2\}) = 3, y_f(\emptyset) = 0$. Workers are gross substitutes for f. Suppose salaries are restricted to integer values. The demand correspondence induced by y_f can be multi-valued for integer salaries. We have e.g. $D_f(1, 2) = \{\{w_1\}, \{w_1, w_2\}\}$ and $D_f(2, 2) = \{\{w_1\}, \{w_2\}, \{w_1, w_2\}\}$. Now suppose we choose a single valued selection $\tilde{D}_f$ from D_f, such that $\tilde{D}_f(2, 1) = \{w_2\}$ but $\tilde{D}_f(2, 2) = \{w_1, w_2\}$. The selection violates the law of aggregate demand.

4.3.2 Continuous Mechanisms

Finally, we consider the continuous model. In this case, we can define a core stable and strategy-proof mechanism by using limit arguments. First we show that the existence of a worker-optimal strict core allocation in discrete markets without ties (Part 1 of Proposition 1) implies the existence of a worker-optimal core allocation in continuous markets. Similarly, we can derive a version of the rural hospitals theorem for continuous markets from its discrete counterpart (Part 2.(a) of Proposition 1).

Theorem 4. *In each continuous market such that workers are gross substitutes for firms,*

1. *there exists a core allocation that is most preferred by all workers among all core allocations,*

2. *if a worker is unemployed in one core allocation, then in all other core allocations he is either unemployed or indifferent between his assignment and being unemployed.*

In contrast to the discrete model, worker-optimal core allocations do not need to be unique. There can be continuous markets, where there are multiple worker-optimal core allocations. However, all workers are indifferent between all of these worker-optimal allocations. A simple example is the following.

Example 3. Let $F = \{f_1, f_2\}$ and $W = \{w_1, w_2\}$. Both firms have the same production function $y_f : 2^W \to \mathbb{R}$ for $f = f_1, f_2$ defined by $y_f(\{w_1\}) = y_f(\{w_2\}) = 1$ and $y_f(\emptyset) = y_f(\{w_1, w_2\}) = 0$. Both workers have the same utility function $u_w(f, \sigma) = \sigma$ for $f = f_1, f_2$, and $w = w_1, w_2$. There are two core allocations that are both worker-optimal. The first allocation matches w_1 to f_1 under salary 1 and w_2 to f_2 under salary 1. The second allocation matches w_1 to f_2 under salary 1 and w_2 to f_1 under salary 1. Both workers are indifferent between the two allocations.

Finally, we show that we can define a mechanism by selecting for each utility profile, one of the worker-optimal core allocations in such a way that the so-defined mechanism is strategy-proof. The strategy-proofness of the mechanism will follow by a limit argument from Proposition 5. First we show that if the sequence of allocations selected by the discrete mechanisms for a utility profile converges as salary increments go to 0, then the limit allocation is a worker-optimal allocation in the continuous market.

Lemma 8. *Let y be a production profile such that workers are gross substitutes for firms. Let $(\sigma_0(t))_{t=0,1,2,\ldots}$ be a sequence of salary increments with $\lim_{t\to\infty} \sigma_0(t) = 0$ and let $(\mathcal{M}_t)_{t=0,1,2,\ldots}$ be a sequence of $(y, \sigma_0(t))$-core-stable and strategy-proof mechanisms. If for $u \in U^W$ the sequence of allocations $(\mathcal{M}_t(u))_{t=0,1,2,\ldots}$ converges, then it converges to a worker-optimal core allocation in (y, u).*

In general, the sequence of allocations selected by the discrete mechanisms does not need to converge as salary increments go to 0. The sequence could for example oscillate between allocations that are close to different worker-optimal core allocations if the market has several worker-optimal core allocations as in Example 3. However, we can always choose (by the Bolzano-Weierstrass Theorem) a converging subsequence and use the limit allocation of the subsequence to define the mechanism. The strategy-proofness follows from the strategy-proofness of the discrete mechanisms.

Theorem 5. *Let y be a production profile such that workers are gross substitutes for firms, then there exists a y-core-stable and strategy-proof mechanism.*

Bibliography

Abdulkadiroglu, A. and Sönmez, T. (2003): "School Choice: A Mechanism Design Approach." *American Economic Review*, 93(3): 729–747.

Alcalde, J. and Barberà, S. (1994): "Top Dominance and the Possibility of Strategy-proof Stable Solutions to Matching Problems." *Economic Theory*, 4(3): 417–435.

Aygün, O. and Sönmez, T. (2013): "Matching with Contracts: Comment." *American Economic Review*, 103(5): 2050–2051.

Blair, C. (1988): "The Lattice Structure of the Set of Stable Matchings with Multiple Partners." *Mathematics of Operations Research*, 18(4): 619–628.

Chambers, C. P. and Echenique, F. (2016): "A Characterization of Combinatorial Demand." Technical report.

Chen, P., Egesdal, M., Pycia, M., and Yenmez, M. B. (2016): "Manipulability of Stable Mechanisms." *American Economic Journal: Microeconomics*, 8(2): 202–14.

Demange, G. and Gale, D. (1985): "The Strategy Structure of Two-Sided Matching Markets." *Econometrica*, 53: 873–888.

Echenique, F. (2012): "Contracts versus Salaries in Matching." *American Economic Review*, 102(1): 594–601.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and some Applications." *Mathematics of Operations Research*, 28(1): 103–126.

Gul, F. and Stacchetti, E. (1999): "Walrasian Equilibrium with Gross Substitutes." *Journal of Economic Theory*, 87(1): 95–124.

Hatfield, J. W. and Kojima, F. (2010): "Substitutes and Stability for Matching with Contracts." *Journal of Economic Theory*, 145(5): 1704–1723.

Hatfield, J. W., Kojima, F., and Kominers, S. D. (2014): "Investment Incentives in Labor Market Matching." *American Economic Review*, 104(5): 436–41.

Hatfield, J. W., Kominers, S. D., and Westkamp, A. (2015): "Stability, Strategy-Proofness, and Cumulative Offer Mechanisms." Technical report, mimeo.

Hatfield, J. W. and Milgrom, P. R. (2005): "Matching with Contracts." *American Economic Review*, 95(4): 913–935.

Hirata, D. and Kasuya, Y. (2014): "Cumulative Offer Process is Order-Independent." *Economics Letters*, 124(1): 37–40.

Hirata, D. and Kasuya, Y. (2015): "On Stable and Strategy-proof Rules in Matching Markets with Contracts." Technical report, mimeo.

Jagadeesan, R. (2016): "Cadet-Branch Matching in a Quasi-Linear Labor Market." In V. Conitzer, D. Bergemann, and Y. Chen, editors, *Proceedings of the 2016 ACM Conference on Economics and Computation, EC '16*, pages 61–62. ACM.

Kelso, A. and Crawford, V. P. (1982): "Job Matching, Coalition Formation, and Gross Substitutes." *Econometrica*, 50(6): 1483–1504.

Kominers, S. D. and Sönmez, T. (2016): "Matching with Slot-Specific Priorities: Theory." *Theoretical Economics*, 11: 683–710.

Roth, A. E. (1984a): "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory." *Journal of Political Economy*, 92(6): 991–1016.

Roth, A. E. (1984b): "Stability and Polarization of Interests in Job Matching." *Econometrica*, 52(1): 47–57.

Roth, A. E. (1991): "A Natural Experiment in the Organization of Entry-Level Labor Markets: Regional Markets for New Physicians and Surgeons in the United Kingdom." *American Economic Review*, 81: 415–440.

Schlegel, J. C. (2015): "Contracts versus Salaries in Matching: A General Result." *Journal of Economic Theory*, 159(A): 552–573.

Sönmez, T. (2013): "Bidding for Army Career Specialities: Improving the ROTC Branching Mechanism." *Journal of Political Economy*, 121(1): 186–219.

Sönmez, T. and Switzer, T. B. (2013): "Matching With (Branch-of-Choice) Contracts at the United States Military Academy." *Econometrica*, 81(2): 451–488.

A Proof of Lemma 1

Proof. Let $(\mu, s) \in \mathcal{S}(D, \succeq)$. By definition (μ, s) is individually rational. Moreover, if there is a $f \in F$ with $D_f(\tilde{s}_f) \neq \mu(f)$, then f and $D_f(\tilde{s}_f)$ block (μ, s) via $\tilde{s}_f$.

For the opposite direction, let (μ, s) be individually rational and let $D_f(\tilde{s}_f) = \mu(f)$ for each $f \in F$. Suppose there is a firm f, group of workers $W' \neq \mu(f)$ that block (μ, s) via salary vector $s' \in S^W$. We have $s' \geq \tilde{s}_f$ with $\tilde{s}|_{\mu(f)} = s'|_{\mu(f)}$. If $s'_f = \tilde{s}_f$, then $W' = D_f(s'_f) = \mu(f)$ contradicting the assumption that $\mu(f) \neq W'$. Otherwise, we can apply IRC repeatedly and obtain a contradiction: Let $s'_w > \tilde{s}_{fw}$. Then $w \notin D_f(\tilde{s}_f) = \mu(f)$ and, by IRC, $D_f(\tilde{s}_{f,-w}, s'_w) = D_f(\tilde{s}_f) = \mu(f)$. Repeatedly adjust salaries of undemanded workers to obtain, $D_f(s') = D_f(\tilde{s}_f) = \mu(f)$. Since $\mu(f) \neq W'$, this is a contradiction. □

B Proof of Lemma 2

The following lemma collects for future reference some observations about virtual demand functions that follow immediately from their definition.

Lemma 9. *Let D_f be a demand function and $s \in S^W$ a vector of reservation salaries. Let $(s(t))_{t=0,\dots T}$ be the sequence of offer vectors for f in a market with $F = \{f\}$ and workers with reservation salaries s. Then*

1. *for each $0 \leq t \leq T$, we have $s(t)^\vee = s(t)$ and therefore $D_f^\vee(s(t)) = D_f(s(t))$,*
2. *for $s^\vee = s(T)$ we have $D_f^\vee(s^\vee) = D_f^\vee(s) = D_f(s^\vee)$. Moreover, for each $w \notin D_f^\vee(s)$ we have $s_w^\vee = s_w$.*

Now we prove Lemma 2:

Proof. It suffices to show that the law of demand invariance holds for a salary change of one unit. The more general case, where the salary changes by more than one unit, follows by repeated application of the case where the salary is changed by one unit. Let $w \in W$ and $s, s' \in S^W$ such that $s_{-w} = s'_{-w}$ and $s'_w = s_w + 1$. We want to show that $w \in D_f^\vee(s) \Rightarrow D_f^\vee(s) = D_f^\vee(s')$. We show the equivalent statement that $D_f^\vee(s) \neq D_f^\vee(s') \Rightarrow w \notin D_f^\vee(s')$.

Let $(s(t))_{t=0,\dots,T}$ and $(s'(t))_{t=0,\dots,T'}$ be the sequences of offer vectors for f in the salary adjustment process for reservation salaries s and s'. Suppose $D_f^\vee(s) \neq D_f^\vee(s')$. Since $D_f^\vee(s) \neq D_f^\vee(s')$, there is a τ such that for $t \leq \tau$ we have $s(t) = s'(t)$ and such that $s(\tau+1) \neq s'(\tau+1)$. Under reservation salaries s, firm f rejects an application of w under salary s'_w in round τ or a previous round and worker w makes a new application to f in round $\tau+1$ under salary s_w, i.e. $s_w(\tau) = s'_w > s_w(\tau+1) = s_w$. Under reservation salaries s' reapplying is not individually rational for w. Thus, under reservation salaries s', salary s'_w is the lowest salary under which f receives an application from w. Thus, if we let $\tau' \leq \tau$ be the round in which firm f rejects an application of w under salary s'_w, the salary for w remains unchanged for all subsequent rounds, $s'_w(t) = s'_w$ for $t \geq \tau'$. By

Lemma 9, we have $D_f(s'(t)) = D_f^\vee(s'(t))$ for each $0 \le t \le T'$. Thus, as w's application in round τ' is rejected, we have $w \notin D_f^\vee(s'(\tau')) = D_f(s'(\tau'))$. If $\tau' \neq T'$, then, by virtual gross substitutability and the previous observation that $s'_w(t) = s'_w$ for $t \geq \tau'$, it follows that $w \notin D_f^\vee(s'(\tau')) = D_f(s'(\tau'))$ implies $w \notin D_f(s'(\tau'+1)) = D_f^\vee(s'(\tau'+1))$. In the same way, if $\tau'+1 \neq T'$, then $w \notin D_f(s'(\tau'+1)) = D_f^\vee(s'(\tau'+1))$ implies $w \notin D_f(s'(\tau'+2)) = D_f^\vee(s'(\tau'+2))$ and so on. Iterating in this way, we obtain $w \notin D_f^\vee(s'(T)) = D_f(s'(T)) = D_f^\vee(s')$. □

C Proof of Proposition 2

Proof. Let $s \in S^W$ and $s^\vee \geq s$ be the outcome of the salary adjustment process for reservation salaries s. By definition of the salary adjustment process, for $w \notin D_f(s^\vee)$ we have $s_w = s_w^\vee$. Now transform $s^\vee$ into s by sequentially lowering salaries of the workers w with $s_w^\vee > s_w$. In each step, the demand remains unchanged by the law of demand invariance. Thus, $D_f(s^\vee) = D_f(s)$.

Next we show that the law of demand invariance is satisfied if workers are gross substitutes and the law of aggregate demand holds. Let $s, s' \in S^W$ with $s_{-w} = s'_{-w}$ and $s_w < s'_w$. If $w \in D_f(s')$, then, by gross substitutability, we have $D_f(s) \subseteq D_f(s')$. By the law of aggregate demand this implies $D_f(s) = D_f(s')$. □

D Proof of Proposition 3

Proof. We prove the result by induction on the length of workers' preference lists. For $\succeq \in \mathcal{R}^W$ let $\ell(\succeq) := \sum_{w \in W} |\{(f,\sigma) \in F \times S : (f,\sigma) \succ_w \emptyset\}|$.
Induction Basis: For each $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) = 0$, the only individually rational allocation is the empty matching. Both the salary adjustment process in $(D, \succeq)$ and in $(D^\vee, \succeq)$ converge to the empty matching which is stable in both markets.
Induction Assumption: For each $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) \le n$ the outcome of the salary adjustment process in the market $(D, \succeq)$ and in the virtual market $(D^\vee, \succeq)$ is the same and stable in both markets.
Induction Step: Let $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) = n+1$. Since workers are virtual gross substitutes for each firm, Lemma 2 and Proposition 1 imply that the salary adjustment process in the virtual market converges to the worker-optimal stable allocation (μ, s) in $(D^\vee, \succeq)$. Note that Lemma 2 is necessary for this argument, since Proposition 1 requires that the demand functions satisfy IRC.

First, we consider the case that during the salary adjustment process in the virtual market $(D^\vee, \succeq)$, some worker w does not apply to some firm f under the acceptable salary σ, i.e.

$$\ldots \succ_w (\mu(w), s_w) \succ_w (f,\sigma) \succ_w \ldots \succ_w \emptyset \succ_w \ldots.$$

Let $\succeq'_w$ be obtained from $\succeq_w$ by truncating $\succeq_w$ after $(\mu(w), s_w)$, i.e.

$$\ldots \succ'_w (\mu(w), s_w) \succ'_w \emptyset \succ'_w (f,\sigma) \succ'_w \ldots.$$

Note that (μ, s) is also the worker-optimal stable allocation in the truncated virtual market $(D^\vee, \succeq'_w, \succeq_{-w})$. By the induction assumption, the salary adjustment process in the market $(D, \succeq'_w, \succeq_{-w})$ converges to (μ, s). We show that the salary adjustment process in $(D, \succeq)$ converges to (μ, s) as well. Suppose not. Then during the salary adjustment process in $(D, \succeq)$, firm $\mu(w)$ will reject an application of w under salary s_w. But up to that point, the salary adjustment process in $(D, \succeq)$ and that in $(D, \succeq'_w, \succeq_{-w})$ and therefore, by the induction assumption, that in $(D^\vee, \succeq'_w, \succeq_{-w})$ agree. However, if $\mu(w)$ rejects w under s_w during the salary adjustment process in $(D^\vee, \succeq'_w, \succeq_{-w})$, then, by virtual gross substitutability, $\mu(w)$ will never accept an application by w under s_w afterwards. This contradicts the fact that $\mu(w)$ is matched to w under s_w in the outcome of the salary adjustment process (μ, s) in $(D^\vee, \succeq'_w, \succeq_{-w})$. Thus, (μ, s) is also the outcome of the salary adjustment process in $(D, \succeq)$. By the definition of the salary adjustment process, the outcome of the salary adjustment process is stable whenever it is feasible. Thus, (μ, s) is stable in $(D, \succeq)$.

Next we consider the case that during the salary adjustment process in the virtual market $(D^\vee, \succeq)$, each worker applies to all firms under all acceptable salaries. First we show that then, also during the salary adjustment in the original market $(D, \succeq)$, each worker applies to all firms under all acceptable salaries. Suppose not. Then there is a worker w for which the last firm f and salary σ under which he applies during the salary adjustment process is ranked above the least-preferred acceptable firm-salary combination (f', σ'), i.e.

$$(f, \sigma) \succ_w (f', \sigma') \succ_w \emptyset.$$

Let $\succeq'_w$ be obtained from $\succeq_w$ by truncating $\succeq_w$ after (f, σ). Since w applies only to acceptable firm-salary combinations according to the truncated preferences $\succeq'_w$ during the salary adjustment process in $(D, \succeq)$, the salary adjustment process in the original market $(D, \succeq)$ and that in the truncated market $(D, \succeq'_w, \succeq_{-w})$ agree. Thus, the salary adjustment process in $(D, \succeq)$ converges, by the induction assumption, to the worker optimal stable allocation (μ', s') in $(D^\vee, \succeq'_w, \succeq_{-w})$ and $(\mu', s') \succeq_w (f, \sigma) \succ_w (f', \sigma')$. But, then (μ', s') is also stable in the un-truncated virtual market $(D^\vee, \succeq)$. Now recall that each worker applies to all firms under all possible salaries during the salary adjustment process in $(D^\vee, \succeq)$. Thus, either $\mu(w) = f'$ with $s_w = \sigma'$ or $\mu(w) = \emptyset$. Therefore $(\mu', s') \succ_w (\mu, s)$, contradicting the worker-optimality of (μ, s) in $(D^\vee, \succeq)$. Thus, in the following we can assume that each worker applies to all firms under all acceptable salaries during the salary adjustment process in both the original and the virtual market.

For each $f \in F$, define a salary vector $s_f = (s_{fw})_{w \in W} \in S^W$ by

$$s_{fw} = \min\{\sigma \in S : (f, \sigma) \succ_w \emptyset\}.$$

Since each worker applies to all firms under all acceptable salaries in the virtual and in the original market, the vector s_f is the terminal offer vector for f in the salary adjustment processes in both markets. Since the salary adjustment process in the virtual market converges to (μ, s), we have for each $f \in F$ that $D^\vee_f(s_f) = \mu(f)$. It remains to show that for each $f \in F$ we have $D_f(s_f) = \mu(f)$ and thus (μ, s) is also the outcome of the salary adjustment process in the original market.

Suppose for the sake of contradiction that there is a $f' \in F$ with $D_{f'}(s_{f'}) \neq D^{\vee}_{f'}(s_{f'}) = \mu(f')$. Consider the modified salary schedule $\tilde{s}$ where for each $w \in \mu(f')$ we let $\tilde{s}_w = (s_{f'})^{\vee}_w$ and for $w \notin \mu(f')$ we let $\tilde{s}_w = s_w$. Since $\mu(f') = D^{\vee}_{f'}(s_{f'}) \neq D_{f'}(s_{f'})$ we have $\tilde{s} \neq s$. Next we show that $(\mu, \tilde{s}) \in \mathcal{S}(D^{\vee}, \succeq)$. Since $\tilde{s} \geq s$ and $\tilde{s} \neq s$, this will be a contradiction with (μ, s) being the worker-optimal stable allocation in $(D^{\vee}, \succeq)$. For each $f \in F$ let $\tilde{s}_f$ be the minimal potential blocking vector for $(\mu, \tilde{s})$ under preferences $\succeq$ as defined in Lemma 1. For f' we have $\tilde{s}_{f'} = s^{\vee}_{f'}$ and by definition of $D^{\vee}_{f'}$ we have $D^{\vee}_{f'}(s^{\vee}_{f'}) = D^{\vee}_{f'}(s_{f'}) = \mu(f')$. For each $f \neq f'$ we have $\tilde{s}_f \geq s_f$ with $\tilde{s}_{fw} = s_{fw}$ for $w \in \mu(f)$. Thus, by the virtual IRC we have $D^{\vee}_f(\tilde{s}_f) = D^{\vee}_f(s_f)$. Therefore Lemma 1 (again we implicitly use the virtual IRC) implies $(\mu, \tilde{s}) \in \mathcal{S}(D^{\vee}, \succeq)$. We have reached a contradiction. Thus, for each $f \in F$ we have $D_f(s_f) = D^{\vee}_f(s_f) = \mu(f)$ and the allocation (μ, s) is also the outcome of the salary adjustment process and stable in $(D, \succeq)$.

□

E Proof of Lemma 3

Proof. Let $\mathcal{M}, \mathcal{M}'$ be D-stable and strategy-proof mechanisms. We show that for each $\succeq \in \mathcal{R}^W$ we have $\mathcal{M}(\succeq) = \mathcal{M}'(\succeq)$ and if workers are virtual gross substitutes under D for each firm, then $\mathcal{M}(\succeq)$ is the worker-optimal stable allocation in the virtual market $(D^{\vee}, \succeq)$. We prove the result by induction on the length of workers' preference lists. For $\succeq \in \mathcal{R}^W$ let $\ell(\succeq) := \sum_{w \in W} |\{(f, \sigma) \in F \times S : (f, \sigma) \succ_w \emptyset\}|$.
Induction Basis: For each $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) = 0$, the empty matching is the only stable allocation in $(D, \succeq)$ and $(D^{\vee}, \succeq)$ and the lemma trivially holds.
Induction Assumption: For each $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) \leq n$ we have $\mathcal{M}(\succeq) = \mathcal{M}'(\succeq)$. If workers are, moreover, virtual gross substitutes for each firm, then $\mathcal{M}(\succeq)$ is the worker-optimal stable allocation in $(D^{\vee}, \succeq)$.
Induction Step: Let $\succeq \in \mathcal{R}^W$ with $\ell(\succeq) = n + 1$. Let $(\mu, s) := \mathcal{M}(\succeq)$ and $(\mu', s') := \mathcal{M}'(\succeq)$. Suppose $(\mu, s) \neq (\mu', s')$. Since $(\mu, s), (\mu', s') \in \mathcal{S}(D, \succeq)$, there is a $w \in W$ such that either $(\mu', s') \succ_w (\mu, s) \succ_w \emptyset$ or $(\mu, s) \succ_w (\mu', s') \succ_w \emptyset$.

If $(\mu', s') \succ_w (\mu, s) \succ_w \emptyset$, then truncate w's preferences after $(\mu'(w), s'_w)$. If w submits the truncated preferences $\succeq'_w$ and everyone else submits preferences $\succeq_{-w}$, then - by strategy-proofness of $\mathcal{M}'$ - w will receive the same assignment $(\mu'(w), s'_w)$ in $\mathcal{M}'(\succeq)$ and $\mathcal{M}'(\succeq'_w, \succeq_{-w})$. Furthermore, by the induction assumption, $\mathcal{M}'(\succeq'_w, \succeq_{-w}) = \mathcal{M}(\succeq'_w, \succeq_{-w})$. But then $\mathcal{M}(\succeq'_w, \succeq_{-w}) \succ_w \mathcal{M}(\succeq)$, contradicting strategy-proofness. Thus, there is no $w \in W$ with $(\mu', s') \succ_w (\mu, s) \succ_w \emptyset$. A completely analog argument shows that there is no $w \in W$ with $(\mu, s) \succ_w (\mu', s') \succ_w \emptyset$.

Next assume that workers are virtual gross substitutes for each firm under D. Let $(\mu, s) := \mathcal{M}(\succeq)$ and (μ', s') be the worker-optimal stable allocation in $(D^{\vee}, \succeq)$. Suppose $(\mu, s) \neq (\mu', s')$. By Proposition 3, (μ', s') is stable in $(D, \succeq)$ as well. Since $(\mu, s), (\mu', s') \in \mathcal{S}(D, \succeq)$, there is a $w \in W$ such that either $(\mu', s') \succ_w (\mu, s) \succ_w \emptyset$ or $(\mu, s) \succ_w (\mu', s') \succ_w \emptyset$.

If $(\mu', s') \succ_w (\mu, s) \succ_w \emptyset$, then truncate w's preferences after $(\mu'(w), s'_w)$. Note that (μ', s') is also stable in the markets $(D, \succeq'_w, \succeq_{-w})$ and $(D^{\vee}, \succeq'_w, \succeq_{-w})$. By the induction assumption, $\mathcal{M}(\succeq'_w, \succeq_{-w})$ is the worker-optimal stable allocation in $(D^\vee, \succeq'_w, \succeq_{-w})$. By strategy-proofness of $\mathcal{M}$, worker w is unmatched in $\mathcal{M}(\succeq'_w, \succeq_{-w})$ since otherwise $\mathcal{M}(\succeq'_w, \succeq) \succ_w \mathcal{M}(\succeq)$. However, then $(\mu', s') \succ_w \mathcal{M}(\succeq'_w, \succeq)$ contradicting the worker-optimality of $\mathcal{M}(\succeq'_w, \succeq)$ in $(D^\vee, \succeq'_w, \succeq_{-w})$.

If $(\mu, s) \succ_w (\mu', s') \succ_w \emptyset$, then truncate w's preferences after $(\mu(w), s_w)$. By the induction assumption, $\mathcal{M}(\succeq'_w, \succeq_{-w})$ is the worker-optimal stable allocation in $(D^\vee, \succeq'_w, \succeq_{-w})$. By strategy-proofness of $\mathcal{M}$, $\mathcal{M}_w(\succeq'_w, \succeq_{-w}) = \mathcal{M}_w(\succeq)$. Thus, $\mathcal{M}(\succeq'_w, \succeq_{-w})$ is stable in $(D^\vee, \succeq)$ as well. But this contradicts the worker-optimality of (μ', s') in $(D^\vee \succeq)$. □

F Proof of Theorem 1

Proof of 1. By Proposition 3, the salary adjustment mechanism for D and for $D^\vee$ is the same and stable with respect to both profiles. By Proposition 1, it is strategy-proof.

Proof of 2. Let D_f be a demand function such that either workers are not virtual gross substitutes or the virtual law of aggregate demand fails. We first consider the case where virtual gross substitutability is violated and afterward the case where virtual gross substitutability holds but the virtual law of aggregate demand is violated.

Case 1. Violation of virtual gross substitutability.

In the following, a **violation** is a pair of salary vectors (s, s') such that there are workers $w, w' \in W$ with $w \neq w'$ such that $s_{-w'} = s'_{-w'}$, $s'_{w'} = s_{w'} + 1$ and $w \in D^\vee_f(s)$ but $w \notin D^\vee_f(s')$. By assumption there exists at least one violation. In the following, we let $(\bar{s}, \bar{\bar{s}}) \in S^W \times S^W$ be a violation such that there is no violation (s, s') with $s \geq \bar{s}$ and $s \neq \bar{s}$. By finiteness of S^W at least one such maximal violation exits. First we show the following facts about the maximal violation $(\bar{s}, \bar{\bar{s}})$ that will be useful in the subsequent proof:

Claim 1. Let $s, s' \in S^W$ be salary vectors with $s' \geq s \geq \bar{s}$. If for each $w \notin D^\vee_f(s')$ we have $s_w = s'_w$, then $D^\vee_f(s) = D^\vee_f(s')$.

If s and s' differ only in the salary for one worker $w \in W$, then the claim is equivalent to the statement that

$$w \in D^\vee_f(s') \Rightarrow D^\vee_f(s) = D^\vee_f(s').$$

The proof of this statement, however, is (literally) the same as the proof of Lemma 2. To see that the same proof applies, observe that we use the fact that $(\bar{s}, \bar{\bar{s}})$ is a maximal violation and therefore for salary vectors above $\bar{s}$, workers are virtual gross substitutes for f. Thus, along the sequence $(s'(t))_{t=0,\dots,T'}$, virtual gross substitutability is satisfied. (Along the sequence $(s(t))_{t=0,\dots,T}$, virtual gross substitutability might be violated. Observe, however that the proof of Lemma 2 only requires virtual gross substitutability along the sequence $(s'(t))_{t=0,\dots,T'}$ but not along the sequence $(s(t))_{t=0,\dots,T})$.

Repeated application of the special case, where salaries differ only for one worker, yields the general result where salaries differ for multiple workers.

Claim 2. We have $D_f(\bar{s}) = D_f^\vee(\bar{s})$.

Suppose $D_f^\vee(\bar{s}) \neq D_f(\bar{s})$. Recall that $\bar{s}^\vee$ is the terminal offer vector for f in the salary adjustment process with reservation salaries $\bar{s}$. We have two cases: Let $w \in W$ be the worker with $\bar{s}_w \neq \bar{\bar{s}}_w$. Then either $\bar{s}_w^\vee \geq \bar{\bar{s}}_w = \bar{s}_w + 1$ or $\bar{s}_w^\vee = \bar{s}_w$. In the first case, note that $\bar{s}^\vee \geq \bar{\bar{s}}$. Moreover, since $\bar{s}_w^\vee > \bar{s}_w$, Lemma 9 in Appendix B implies that $w \in D_f^\vee(\bar{s}^\vee)$. For $\tilde{w} \neq w$ such that $\tilde{w} \notin D_f^\vee(\bar{s}) = D_f^\vee(\bar{s}^\vee)$, Lemma 9 in Appendix B implies $\bar{s}_{\tilde{w}}^\vee = \bar{s}_{\tilde{w}} = \bar{\bar{s}}_{\tilde{w}}$. Thus, by Claim 1, $D_f^\vee(\bar{s}^\vee) = D_f^\vee(\bar{\bar{s}})$. But by Lemma 9 in Appendix B, $D_f^\vee(\bar{s}^\vee) = D_f^\vee(\bar{s})$ contradicting the fact that $D_f^\vee(\bar{s}) \neq D_f^\vee(\bar{\bar{s}})$.

In the second case, consider the vector $\tilde{s} := (\bar{s}_{-w}^\vee, \bar{\bar{s}}_w) \geq \bar{\bar{s}}$. We show that for $\tilde{w} \notin D_f^\vee(\tilde{s})$ we have $\tilde{s}_{\tilde{w}} = \bar{\bar{s}}_{\tilde{w}}$: Clearly this is the case for $\tilde{w} = w$. For $\tilde{w} \neq w$, suppose that $\tilde{s}_{\tilde{w}} > \bar{s}_{\tilde{w}} = \bar{\bar{s}}_{\tilde{w}}$ but $\tilde{w} \notin D_f^\vee(\tilde{s})$. Since $(\bar{s}, \bar{\bar{s}})$ is a maximal violation and $\bar{s}^\vee \neq \bar{s}$, the pair $(\bar{s}^\vee, \tilde{s})$ is not a violation. Thus, virtual gross substitutability implies that $\tilde{w} \notin D_f^\vee(\bar{s}^\vee)$. However, since $\tilde{s}_{\tilde{w}} > \bar{s}_{\tilde{w}}$, Lemma 9 in Appendix B implies $\tilde{w} \in D_f^\vee(\bar{s}^\vee)$. Thus, we have a contradiction.

With the two claims we can finish the proof of the first case: There are $w, w' \in W$ with $w \neq w'$ such that $\bar{s}_{-w'} = \bar{s}_{-w'}$ and $w \in D_f^\vee(\bar{s})$ but $w \notin D_f^\vee(\bar{\bar{s}})$. Let $f' \in F \setminus \{f\}$ be one of the other firms. For each $s \in S^W$, we define $D_{f'}$ by

$$D_{f'}(s) = \begin{cases} \{w\}, & \text{if } s_w < \bar{\sigma}, s_w \leq s_{w'}, \\ \{w'\}, & \text{if } s_{w'} < \bar{\sigma}, s_{w'} < s_w, \\ \emptyset, & \text{else.} \end{cases} \tag{2}$$

Note that $D_{f'}$ is a unit demand function. In particular, it satisfies gross substitutability and the law of aggregate demand. Thus, by Proposition 2, demand and virtual demand for f' agree, $D_{f'} = D_{f'}^\vee$. All other firms $f'' \in F \setminus \{f, f'\}$, have the trivial demand function $D_{f''}(s) = \emptyset$ for each $s \in S^W$. We show that there is no D-stable, strategy-proof mechanism. We define the profile $\succeq \in \mathcal{R}^W$ by

$$(f, \bar{\sigma}) \succ_w \dots \succ_w (f, \bar{s}_w + 1) \succ_w (f', \bar{\sigma}) \succ_w (f, \bar{s}_w) \succ_w (f', \bar{\sigma} - 1) \succ_w \emptyset \succ_w \dots \tag{3}$$

$$(f, \bar{\sigma}) \succ_{w'} \dots \succ_{w'} (f, \bar{\bar{s}}_{w'} + 1) \succ_{w'} (f', \bar{\sigma}) \succ_{w'} (f, \bar{\bar{s}}_{w'}) \succ_{w'} (f', \bar{\sigma} - 1) \succ_{w'} (f, \bar{s}_{w'}) \succ_{w'} \emptyset \succ_{w'} \dots \tag{4}$$

$$(f, \bar{\sigma}) \succ_{w''} \dots \succ_{w''} (f, \bar{s}_{w''}) \succ_{w''} \emptyset \succ_{w''} \dots \quad \text{for } w'' \neq w, w'. \tag{5}$$

Consider the market $(D, \succeq)$. First we show that in the corresponding virtual market there is no stable allocation.

Claim 3. There is no stable allocation in $(D^\vee, \succeq)$.

Suppose there is a $(\mu, s) \in \mathcal{S}(D^\vee, \succeq)$. Consider the minimal potential blocking vector $\tilde{s}_f$ for allocation (μ, s), firm f and profile $\succeq$. By the definition of stability, we have $\mu(f) = D^\vee_f(\tilde{s}_f)$. (Note that this is a direct consequence of the definition of stability and does not require, as in Lemma 1, that the virtual demand function $D^\vee_f$ satisfies IRC). We consider two cases: either $\tilde{s}_{fw'} = \bar{s}_{w'}$ or $\tilde{s}_{fw'} \geq \bar{\bar{s}}_{w'} > \bar{s}_{w'}$. In the first case, $\tilde{s}_f \geq \bar{s}$ with $\tilde{s}_{fw''} = \bar{s}_{w''}$ for $w'' \notin \mu(f) = D^\vee_f(\tilde{s}_f)$. By Claim 1, this implies $\mu(f) = D^\vee_f(\tilde{s}_f) = D^\vee_f(\bar{s})$. Thus, $w, w' \in \mu(f)$ and therefore $\mu(f') = \emptyset$. Moreover, w' is matched to f under salary $s_{w'} = \bar{s}_{w'}$. Hence f' and w' can block the allocation (μ, s) with salary $\bar{\sigma} - 1$, contradicting the stability of (μ, s).

In the second case, $\tilde{s}_f \geq \bar{\bar{s}}$ with $\tilde{s}_{fw''} = \bar{\bar{s}}_{w''}$ for $w'' \notin \mu(f) = D^\vee_f(\tilde{s}_f)$. By Claim 1, this implies $\mu(f) = D^\vee_f(\tilde{s}_f) = D^\vee_f(\bar{\bar{s}})$. Thus, $w, w' \notin \mu(f)$ and $\mu(f') = \{w'\}$ with $s_{w'} = \bar{\sigma} - 1$. But then f' and w can block the allocation (μ, s) with salary $\bar{\sigma} - 1$ contradicting the stability of (μ, s).

Now we show that there is no D-stable, strategy-proof mechanism. Suppose there is a D-stable, strategy-proof mechanism $\mathcal{M}$ and let $(\mu, s) := \mathcal{M}(\succeq)$. We consider two cases: Either there is a $\tilde{w} \in W$ with $(\mu, s) \succ_{\tilde{w}} (f, \bar{s}_{\tilde{w}})$ or not. In the first case, truncate $\tilde{w}$'s preferences after $(\mu(\tilde{w}), s_{\tilde{w}})$. If $\tilde{w}$ submits the truncated preferences $\succeq'_{\tilde{w}}$ and everyone else submits preferences $\succeq_{-\tilde{w}}$, then - by strategy-proofness of $\mathcal{M}$ - $\tilde{w}$ will receive the same assignment $(\mu(\tilde{w}), s_{\tilde{w}})$ in $\mathcal{M}(\succeq)$ and $\mathcal{M}(\succeq'_{\tilde{w}}, \succeq_{-\tilde{w}})$. By Lemma 3, $\mathcal{M}(\succeq'_{\tilde{w}}, \succeq_{-\tilde{w}})$ is the worker-optimal stable allocation in the virtual market $(D^\vee, \succeq'_{\tilde{w}}, \succeq_{-\tilde{w}})$. Since $\tilde{w}$ is matched in $\mathcal{M}(\succeq'_{\tilde{w}}, \succeq_{-\tilde{w}})$, the allocation is also stable in the un-truncated virtual market $(D^\vee, \succeq)$. But this contradicts Claim 3.

In the second case, by Claim 2, $\mu(f) = D_f(\bar{s}) = D^\vee_f(\bar{s})$ and therefore $w, w' \in \mu(f)$. Hence $\mu(f') = \emptyset$. But then f' and w' can block (μ, s) with salary $\bar{\sigma} - 1$ contradicting the stability of (μ, s).

Case 2. Violation of the virtual law of aggregate demand.

We may assume that workers are virtual gross substitutes since otherwise we are back to Case 1. We now consider a violation of the virtual law of aggregate demand, i.e. two salary vectors $\bar{s}, \bar{\bar{s}} \in S^W$ such that there is a worker $w' \in W$ with $\bar{s}_{-w'} = \bar{\bar{s}}_{-w'}$, $\bar{\bar{s}}_{w'} = \bar{s}_{w'} + 1$ and $|D^\vee_f(\bar{\bar{s}})| > |D^\vee(\bar{s})|$. Since $D^\vee(\bar{s}) \neq D^\vee_f(\bar{\bar{s}})$, the IRC condition for $D^\vee_f$ (which holds by Lemma 2) implies that $w' \notin D^\vee_f(\bar{\bar{s}})$ but $w' \in D^\vee_f(\bar{s})$. Thus, there are two other workers $w_1, w_2 \in W \setminus \{w'\}$ with $w_1, w_2 \in D^\vee_f(\bar{\bar{s}})$ but $w_1, w_2 \notin D^\vee_f(\bar{s})$.

We define $D_{f'}$ by

$$D_{f'}(s) = \begin{cases} \{w_1\}, & \text{if } s_{w_1} < \bar{\sigma}, \\ \{w_2\}, & \text{if } s_{w_1} = \bar{\sigma} \text{ and } s_{w_2} < \bar{\sigma}, \\ \{w'\}, & \text{if } s_{w_1} = \bar{\sigma} \text{ and } s_{w_2} = \bar{\sigma} \text{ and } s_{w'} < \bar{\sigma}, \\ \emptyset, & \text{else.} \end{cases}$$

All other firms $f'' \in F \setminus \{f, f'\}$, have the trivial demand function $D_{f''}(s) = \emptyset$ for each $s \in S^W$. Suppose there is a D-stable, strategy-proof mechanism $\mathcal{M}$. Consider the profile $\succeq \in \mathcal{R}^W$ defined by

$$
\begin{aligned}
(f, \bar{\sigma}) \succ_{w'} \ldots \succ_{w'} (f, \bar{\bar{s}}_{w'}) \succ_{w'} (f', \bar{\sigma}) \succ_{w'} (f', \bar{\sigma} - 1) \succ_{w'} (f, \bar{s}_{w'}) \succ_{w'} \emptyset \succ_{w'} \ldots \\
(f, \bar{\sigma}) \succ_{w_1} \ldots \succ_{w_1} (f, \bar{s}_{w_1}) \succ_{w_1} (f', \bar{\sigma}) \succ_{w_1} (f', \bar{\sigma} - 1) \succ_{w_1} \emptyset \succ_{w_1} \ldots \\
(f, \bar{\sigma}) \succ_{w_2} \ldots \succ_{w_2} (f, \bar{s}_{w_2} + 1) \succ_{w_2} (f', \bar{\sigma}) \succ_{w_2} (f', \bar{\sigma} - 1) \succ_{w_2} (f, \bar{s}_{w_2}) \succ_{w_2} \emptyset \succ_{w_2} \ldots \\
(f, \bar{\sigma}) \succ_{w''} \ldots \succ_{w''} (f, \bar{s}_{w''}) \succ_{w''} \emptyset \succ_{w''} \ldots \text{ for } w'' \neq w', w_1, w_2
\end{aligned}
$$

Let $(\mu, s) := \mathcal{M}(\succeq)$. Let s_f be the minimal potential blocking vector of allocation (μ, s) for f under $\succeq$.

First consider the case that $\mu(f') = \{w'\}$ with $s_{w'} = \bar{\sigma} - 1$. Then we have $s_f \geq \bar{\bar{s}}$ with $s_{fw} = \bar{\bar{s}}_w$ for $w \notin D_f^{\vee}(s_f)$. Thus, by repeated application of Lemma 2, $\mu(f) = D_f^{\vee}(s_f) = D_f^{\vee}(\bar{s})$. But $w_2 \notin D_f^{\vee}(\bar{\bar{s}})$. Therefore f' and w_2 can block with salary $\bar{\sigma} - 1$ contradicting stability.

Second consider the case that $\mu(f') = \{w_1\}$ with $s_{w_1} = \bar{\sigma} - 1$. Then we have $s_f \geq \bar{s}$ with $s_{fw} = \bar{s}_w$ for $w \notin D_f^{\vee}(s_f)$. Thus, by repeated application of Lemma 2, $\mu(f) = D_f^{\vee}(s_f) = D_f^{\vee}(\bar{s})$. In particular, w_2 is unmatched under μ. Now suppose that w_2 changes his preferences to

$$
(f, \bar{\sigma}) \succ'_{w_2} \ldots \succ'_{w_2} (f, \bar{s}_{w_2}) \succ'_{w_2} \emptyset \succ'_{w_2} (f', \bar{\sigma}) \succ'_{w_2} \ldots .
$$

Let $(\mu', s') := \mathcal{M}(\succeq'_{w_2}, \succeq_{-w_2})$. We show that w_2 receives a better assignment in (μ', s') than in (μ, s) contradicting the strategy-proofness of $\mathcal{M}$. To show this, consider a third allocation (μ'', s'') defined by

$$
\mu''(f) = D_f^{\vee}(\bar{\bar{s}}), \quad \mu''(f') = \{w'\}, \quad s''_w = \begin{cases} \bar{\bar{s}}_w, & \text{if } w \in \mu''(f), \\ \bar{\sigma} - 1, & \text{if } w = w'. \end{cases}
$$

First we show that (μ'', s'') is stable in $(D^{\vee}, \succeq'_{w_2}, \succeq_{-w_2})$: For f, the minimal potential blocking vector for (μ'', s'') under $(\succeq'_{w_2}, \succeq_{-w_2})$ is $\bar{\bar{s}}$. For f' the minimal potential blocking vector for (μ'', s'') under $(\succeq'_{w_2}, \succeq_{-w_2})$ is $s_{f'} \in S^W$ defined by

$$
s_{f'w} = \begin{cases} \bar{\sigma} - 1, & \text{if } w = w', \\ \bar{\sigma}, & \text{if } w \neq w'. \end{cases}
$$

Now note that $D_f^{\vee}(\bar{\bar{s}}) = \mu''(f)$ and $D_{f'}^{\vee}(s_{f'}) = D_{f'}(s_{f'}) = \mu''(f')$. Thus, by Lemma 1, (note that we use the fact that $D_f^{\vee}$ and $D_{f'}^{\vee}$ satisfy IRC by Lemma 2), (μ'', s'') is stable in $(D^{\vee}, \succeq'_{w_2}, \succeq_{-w_2})$. By Lemma 3, (μ', s') is the worker-optimal stable allocation in $(D^{\vee}, \succeq'_{w_2}, \succeq_{-w_2})$. Thus, we have $(\mu', s') \succeq_{w'} (\mu''(w'), s''_{w'}) = (f', \bar{\sigma} - 1)$. Let s'_f be the minimal potential blocking vector of (μ', s') for f under $(\succeq'_{w_2}, \succeq_{-w_2})$. Since $(\mu', s') \succeq_{w'} (f', \bar{\sigma} - 1)$, we have $s'_f \geq \bar{\bar{s}}$ with $s'_{fw} = \bar{\bar{s}}_w$ for $w \notin D_f^{\vee}(s'_f)$. Thus, by repeated application of Lemma 2, $\mu'(f) = D_f^{\vee}(s'_f) = D_f^{\vee}(\bar{\bar{s}})$. Therefore, we have $w_2 \in \mu'(f)$ with $s'_{w_2} \geq \bar{\bar{s}}_{w_2}$. Hence $(\mu', s') \succ_{w_2} \mu(w_2) = \emptyset$. We have obtained the desired contradiction.

Third consider the case that $\mu(f') = \{w_2\}$ with $s_{w_2} = \bar{\sigma} - 1$. Then, we have $s_f \geq (\bar{s}_{-w_2}, \bar{s}_{w_2} + 1)$ with $s_{fw} = \bar{s}_{\tilde{w}}$ for $w \notin D^{\vee}_f(s_f) \setminus \{w_2\}$ and $s_{fw_2} = \bar{\bar{s}}_{w_2} + 1$ if $w_2 \notin D^{\vee}_f(s_f)$. Thus, by repeated application of Lemma 2, $\mu(f) = D^{\vee}_f(\bar{s}_{-w_2}, \bar{s}_{w_2} + 1)$. Since $D^{\vee}_f$ satisfies IRC by Lemma 2 and $w_2 \notin D^{\vee}_f(\bar{s})$ we have $D^{\vee}_f(\bar{s}_{-w_2}, \bar{s}_{w_2} + 1) = D^{\vee}_f(\bar{s})$. But then $w_1 \notin \mu(f)$. Therefore f' and w_1 can block with salary $\bar{\sigma} - 1$ contradicting stability.

Finally consider the case that $\mu(f') = \emptyset$. Then, we have $s_f \geq \bar{s}$ with $s_{fw} = \bar{s}_{\tilde{w}}$ for $w \notin D^{\vee}_f(s_f)$. Thus, by repeated application of Lemma 2, $\mu(f) = D^{\vee}_f(\bar{s})$. But then $w_1 \notin \mu(f)$. Therefore f' and w_1 can block with salary $\bar{\sigma} - 1$.

□

G Proof of Lemma 4

Proof. First we show that an allocation (μ, s) is individually rational in (y, u, σ_0) if and only if it is individually rational in $(D, \succeq)$. For each $w \in W$ we have $u_w(\mu, s) \geq u_w(\emptyset) \Leftrightarrow (\mu, s) \succeq_w \emptyset$. Moreover, $D_f(s_f) = \mu(f)$ implies that $\pi_f(\mu(f), s_f) > \pi_f(W', s_f)$ for each $W' \subseteq \mu(f)$. On the other hand, let $\pi_f(\mu(f), s_f) > \pi_f(W', s_f)$ for each $W' \subseteq \mu(f)$. Let $\tilde{s} \in S^W$, be the vector defined by $\tilde{s}_w = s_w$ for $w \in \mu(f)$ and $\tilde{s}_w = \bar{\sigma}$. Since $0 > \pi_f(W', \tilde{s})$ for each $W' \subseteq W$ with $W' \setminus \mu(f) \neq \emptyset$, we have $D_f(s_f) = D_f(\tilde{s}) = \mu(f)$.

Next we show that (μ, s) is blocked in (y, u, σ_0) if and only if it is blocked in $(D, \succeq)$. First suppose (μ, s) is blocked in (y, u, σ_0) by f and W' under salaries s'. It is no loss of generality to assume that for each $w \in W' \cap \mu(f)$ we have $s'_w = s_w$, since otherwise we can lower for each $w \in W' \cap \mu(f)$ the salary from s'_w to s_w while f and W' still block the allocation. Since the market (y, u, σ_0) has no ties, we have $\pi_f(W', s') > \pi_f(\mu, s)$. Thus, $\mu(f) \neq D_f(s')$. Thus, f and $D_f(s')$ block (μ, s) in $(D, \succeq)$ with salaries $(s'_w)_{w \in D_f(s')}$.

Next suppose (μ, s) is blocked in $(D, \succeq)$ by f and W' under salaries s'. Then $D_f(s') = W' \neq \mu(f)$. Therefore $\pi_f(W', s') > \pi_f(\mu(f), s') = \pi_f(\mu, s)$. Moreover $u_w(f, s'_w) \geq u_w(\mu, s)$ for each $w \in W$. Thus, f and W' also block (μ, s) with salaries $(s'_w)_{w \in W'}$ in (y, u, σ_0). □

H Proof of Lemma 5

Proof. For $(\mu, s) \in \mathcal{A}$, $f \in F$ and $u \in U^W$ define the **continuous minimal potential blocking vector** $\tilde{s}_f = (\tilde{s}_{fw})_{w \in W} \in \mathbb{R}^W_+$ as follows. Salary $\tilde{s}_{fw}$ is the salary that makes w indifferent between working for f and (μ, s), i.e.

$$u_w(f, \tilde{s}_{fw}) = u_w(\mu, s).$$

Since u_w is continuous and increasing in salaries such a salary is well defined, unless $u_w(f, \sigma) > u_w(\mu, s)$ for each $\sigma \in \mathbb{R}_{++}$ in which case we let $s_{fw} = 0$. Define a **surplus function** $\Phi : \mathcal{A} \to \mathbb{R}$ as follows. For $(\mu, s) \in \mathcal{A}$ and $f \in F$ let $\tilde{s}_f$ be the continuous minimal potential blocking vector of (μ, s). Then

$$\Phi(\mu, s) := \max_{f' \in F, W' \subseteq W} \pi_f(W', \tilde{s}_f) - \pi_f(\mu, s).$$

First we show that for $(\mu, s) \in \mathcal{A}$ with $\Phi(\mu, s) = 0$ we have $(\mu, s) \in \mathcal{C}(y, u)$. (The opposite direction is also true. This characterization of core allocations is due to Kelso and Crawford, 1982.) Suppose $(\mu, s) \notin \mathcal{C}(y, u)$. Then there is a firm f such that either (μ, s) is not individually rational for f, or f blocks (μ, s) together with a group of workers. In the first case there is a $W' \subseteq \mu(f)$ such that $\pi_f(W', \tilde{s}_f) > \pi_f(\mu, s)$, contradicting the assumption that $\Phi(\mu, s) = 0$. In the second case, there are salaries s' such that $\pi_f(W', s') > \pi_f(\mu, s)$ and $u_w(f, s'_w) > u_w(\mu, s)$. Let $\tilde{s}_f$ be the continuous minimal potential vector for f and (μ, s) under u. By the definition of $\tilde{s}_f$, we have $s' \geq \tilde{s}_f|_{W'}$. Thus $\pi_f(W', \tilde{s}_f) \geq \pi_f(W', s') > \pi_f(\mu, s)$. Therefore $\pi_f(W', \tilde{s}_f) - \pi_f(\mu, s) > 0$ contradicting the assumption that $\Phi(\mu, s) = 0$.

Now let $(\sigma_0(t))_{t=0,1,\dots}$ and $\{(\mu^t, s^t)\}_{t=0,1,\dots}$ be as in the statement of the lemma. First we show that for each t we have $\Phi(\mu^t, s^t) \leq \sigma_0(t) \cdot |W|$. Suppose not. Then there exists a $f \in F$ and $W' \subseteq W$ such that

$$\pi_f(W', \tilde{s}_f) > \pi_f(\mu, s) + \sigma_0(t) \cdot |W|.$$

For each $w \in W'$ let s'_w be the salary obtained by rounding $\tilde{s}_{fw}$ up to the nearest integer multiple of $\sigma_0(t)$. For each $w \in W'$ we have $u_w(f, s'_w) \geq u_w(\mu, s)$ and for f we have

$$\pi_f(W', s') \geq \pi_f(W', \tilde{s}_f) - \sigma_0(t) \cdot |W'| > \pi_f(\mu, s).$$

Thus f and W' block (μ^t, s^t) contradicting $(\mu^t, s^t) \in \mathcal{C}(y, u, \sigma_0(t))$.

As $\{(\mu^t, s^t)\}_{t=0,1,\dots}$ converges to allocation (μ, s), there is a T such that for $t > T$ we have $\mu^t = \mu$. The function Φ is continuous in salaries as profit functions and utility functions are continuous in salaries. Thus

$$\Phi(\mu, s) = \Phi(\mu, \lim_{t\to\infty} s^t) = \lim_{t\to\infty} \Phi(\mu, s^t) = \lim_{t\to\infty} \Phi(\mu^t, s^t) = 0,$$

where for the last equality we used that for each t we have $\Phi(\mu^t, s^t) \leq \sigma_0(t) \cdot |W|$. Therefore, $(\mu, s) \in \mathcal{C}(y, u)$ as desired. □

I Proof of Lemma 6

Proof. By Proposition 2, it suffices to show that D_f satisfies the law of demand invariance. Let $s, s' \in S^W$ with $s_{-w} = s'_{-w}$ and $s_w < s'_w$ and consider a salary change from s' to s: For each $W' \subseteq W$ with $w \in W'$ the profit increases by the same amount $\pi_f(W', s') - \pi_f(W', s) = s'_w - s_w$ and for each $W' \subseteq W$ with $w \notin W'$ the profit remains unchanged $\pi_f(W', s) = \pi_f(W', s')$. Thus, if $w \in D_f(s')$, then $D_f(s')$ is also profit maximizing under s, i.e. $D_f(s) = D_f(s')$. □

J Proof of Theorem 2

Proof. The first part follows from Proposition 1 and Lemma 4.

For the second part, Let y_f be a production function and $\sigma_0 \in \mathbb{R}_{++}$ a salary increment such that π_f has no ties on $\{k{\cdot}\sigma_0 : k \in \mathbb{N}\}$ and workers are not gross substitutes under the induced demand $D_f : \{k{\cdot}\sigma_0 : k \in \mathbb{N}\}^W \to 2^W$. Without loss of generality, we may assume $\sigma_0 = 1$. The proof of the general case is the same modulo multiplying all relevant salaries by the smallest salary increment. We choose an integer $\bar{\sigma} > \max_{f \in F, W' \subseteq W} y_f(W')$ and define $S := \{1, 2, \ldots, \bar{\sigma}\}$. Let $\bar{s}, \bar{\bar{s}} \in S^W$ and $w' \in W$ with $\bar{\bar{s}}_{-w} = \bar{s}_{-w}$ and $\bar{\bar{s}}_{w'} = \bar{s}_{w'} + 1$ such that there is a $w \neq w'$ with $w \in D_f(\bar{s})$ but $w \in D_f(\bar{\bar{s}})$. We define production functions y_{-f} as follows. Let $f' \in F \setminus \{f\}$ be one of the other firms. We define $y_{f'} : 2^W \to \mathbb{R}$ as follows. Choose

$$\bar{\sigma} - 1 < y_{f'}(\{w'\}) < y_{f'}(\{w\}) < \bar{\sigma}.$$

Let $y_{f'}(\{w''\}) = 0$ for $w'' \neq w, w'$. Define

$$y_{f'}(W') := \max_{\tilde{w} \in W'} y_{f'}(\{\tilde{w}\}) \text{ for each } W' \subseteq W.$$

For all other firms $f'' \in F \setminus \{f, f'\}$, we let $y_{f''} : 2^W \to \mathbb{R}$ be the trivial production function, defined by $y_{f''}(W') = 0$ for each $W' \subseteq W$. We define a utility profile u by

$$\begin{aligned}
u_{\tilde{w}}(f, \sigma) &:= \sigma - \bar{s}_{\tilde{w}} + \epsilon_1, && \text{for each } \tilde{w} \in W,\\
u_{\tilde{w}}(\emptyset) &:= 0, && \text{for each } \tilde{w} \in W,\\
u_w(f', \sigma) &:= \sigma - (\bar{\sigma} - 1) + \epsilon_2,\\
u_{w'}(f', \sigma) &:= \sigma - (\bar{\sigma} - 1) + \epsilon_3,
\end{aligned}$$

and arbitrarily for each other firm-salary pair. The parameters $\epsilon_1, \epsilon_2, \epsilon_3$ are chosen such that $0 < \epsilon_2 < \epsilon_1 < \epsilon_3 < \sigma_0 = 1$. For all other firm-salary pairs, the utility functions can be defined arbitrarily. One readily checks that the ordinal market $(D, \succeq)$ corresponding to $(y, u, 1)$ is the market defined by equations (2)-(5) in the proof of Theorem 1. By Lemma 6 and Claim 3 from the proof of Theorem 1 there is no stable allocation in $(D, \succeq)$. By Lemma 4 there is no strict core allocation in $(y, u, 1)$. □

K Proof of Theorem 3

The following lemma (see also Chambers and Echenique, 2016) will be useful in some of the subsequent proofs.

Lemma 10. *Let the demand correspondence D_f be induced by a quasi-linear profit function. For each $s \in \mathbb{R}^W_{++}$ there exists a neighborhood $U \subseteq \mathbb{R}^W_{++}$ of s such that for each $s' \in U$ we have $D_f(s') \subseteq D_f(s)$.*

Proof. Let $s \in \mathbb{R}^W_{++}$. We choose $\epsilon > 0$ such that

$$\epsilon < \frac{\min_{W' \in D_f(s), W'' \notin D_f(s)} \pi_f(W', s) - \pi_f(W'', s)}{\sqrt{|W|}}.$$

Let $U := \{s' \in \mathbb{R}^W_{++} : |s - s'| \leq \epsilon\}$, where $|\cdot|$ is the Euclidean norm. Let $s' \in U$, $W' \in D_f(s)$ and $W'' \notin D_f(s)$. We have

$$\begin{aligned}\pi_f(W', s') - \pi_f(W'', s') &\geq \pi_f(W', s) - \pi_f(W'', s) - \sum_{w \in W} |s_w - s'_w| \\ &> \sqrt{|W|} \cdot \epsilon - \sum_{w \in W} |s_w - s'_w| \geq 0.\end{aligned}$$

Thus, each bundle of workers that is not profit maximizing under s is also not profit maximizing under s'. We have $D_f(s') \subseteq D_f(s)$. □

Now we prove Theorem 3.

Proof. The first part is Theorem 2 of Kelso and Crawford (1982).

For the second part, let y_f be a production function such that workers are not gross substitutes under the demand D_f induced by y_f. First we show that there is a gross substitutes violation at salary vectors where the demand is single-valued.

Claim. There are $\bar{s}, \bar{\bar{s}} \in \mathbb{R}^W_{++}$ and $w' \in W$ with $\bar{\bar{s}}_{-w'} = \bar{s}_{-w'}$ and $\bar{\bar{s}}_{w'} > \bar{s}_{w'}$ such that $D_f(\bar{s}) = \{W'\}$, $D_f(\bar{\bar{s}}) = \{W''\}$ and there is a $w \neq w'$ with $w \in W'$ but $w \notin W''$.

Let $s, s' \in \mathbb{R}^W_{++}$ and $w' \in W$ with $s'_{-w'} = s_{-w'}$ and $s'_{w'} > s_{w'}$ such that there is a $W' \in D_f(s)$ such that for each $W'' \in D_f(s')$ there is a $w \neq w'$ with $w \in W'$ but $w \notin W''$. If D_f is single-valued at s and s' we are finished. Otherwise, by Lemma 10, we can find an $\epsilon' > 0$ such that for all $\tilde{s}$ in the ϵ'-neighborhood of s' we have $D_f(\tilde{s}) \subseteq D_f(s')$. Let $\epsilon = (\epsilon_w)_{w \in W}$ be defined by $\epsilon_w = \frac{\epsilon'}{\sqrt{|W'|}}$ for $w \in W'$ and $\epsilon_w = 0$ for $w \notin W'$. By construction $D_f(s - \epsilon) = \{W'\}$. Moreover, by construction, $s' - \epsilon$ is in the ϵ'-neighborhood of s' and therefore $D_f(s' - \epsilon) \subseteq D_f(s')$. If D_f is single-valued at $s' - \epsilon$ we can choose $\bar{s} = s - \epsilon$ and $\bar{\bar{s}} = s' - \epsilon$. Otherwise, by Lemma 10, we can find an $\epsilon'' > 0$ such that for all $\tilde{s}$ in the ϵ-neighborhood of $s - \epsilon$ we have $D_f(\tilde{s}) = D_f(s - \epsilon) = \{W'\}$. Choose an arbitrary $W'' \in D_f(s' - \epsilon) \subseteq D_f(s')$. Let $\tilde{\epsilon} = (\tilde{\epsilon}_w)_{w \in W}$ be defined by $\tilde{\epsilon}_w = \frac{\epsilon''}{\sqrt{|W''|}}$ for $w \in W''$ and $\epsilon_w = 0$ for $w \notin W''$. By construction, $D_f(s' - \epsilon - \tilde{\epsilon}) = \{W''\}$. Moreover, by construction, $s - \epsilon - \tilde{\epsilon}$ is in the ϵ''-neighborhood of $s - \epsilon$ and therefore $D_f(s - \epsilon - \tilde{\epsilon}) = D_f(s - \epsilon) = \{W'\}$. The claim holds for the salary vectors $\bar{s} = s - \epsilon - \tilde{\epsilon}$ and $\bar{\bar{s}} = s' - \epsilon - \tilde{\epsilon}$.

With the claim, we can finish the proof. Let $\bar{s}, \bar{\bar{s}} \in \mathbb{R}^W_{++}$, $w, w' \in W$ and $W', W'' \subseteq W$ be as in the claim. First, we define production functions y_{-f} for the other firms. Let $f' \in F \setminus \{f\}$ be one of the other firms. We define $y_{f'} : 2^W \to \mathbb{R}$ as follows. Choose $y_{f'}(\{w\}), y_{f'}(\{w'\}) > 0$ and $y_{f'}(\{w''\}) = 0$ for $w'' \neq w, w'$. Define

$$y_{f'}(W') := \max_{\tilde{w} \in W'} y_{f'}(\{\tilde{w}\}) \qquad \text{for each } W' \subseteq W.$$

For all other firms $f'' \in F \setminus \{f, f'\}$, we let $y_{f''} : 2^W \to \mathbb{R}$ be the trivial production function, defined by $y_{f''}(W') = 0$ for each $W' \subseteq W$. We define a utility profile u by

$$\begin{aligned} u_{\tilde{w}}(f,\sigma) &:= \sigma - \bar{s}_{\tilde{w}} + \epsilon_1, && \text{for each } \tilde{w} \in W, \\ u_{\tilde{w}}(\emptyset) &:= 0, && \text{for each } \tilde{w} \in W, \\ u_w(f',\sigma) &:= \sigma - y_{f'}(\{w\}) + \epsilon_2, \\ u_{w'}(f',\sigma) &:= \sigma - y_{f'}(\{w'\}) + \epsilon_3, \end{aligned}$$

and arbitrarily for each other firm-salary pair. The parameters $\epsilon_1, \epsilon_2, \epsilon_3$ are chosen as follows: By Lemma 10, we can choose $\epsilon_1 > 0$ such that for $\tilde{s} := (\bar{s}_{\tilde{w}} - \epsilon_1)_{\tilde{w} \in W}$, we have $D_f(\tilde{s}) = D_f(\bar{s}) = \{W'\}$. Choose $0 < \epsilon_2 < \epsilon_1$ and

$$\pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s}) < \epsilon_3 < \pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s}) + \epsilon_2.$$

We show that no (strict) core allocation in (y, u) exists. Suppose for the sake of contradiction that $(\mu, s) \in \mathcal{C}(y, u)$. We consider three cases: either $\mu(f') = \emptyset$ or $\mu(f') = \{w\}$ or $\mu(f') = \{w'\}$. In the first case, we have $\mu(f) = W'$ and $s_{w'} \leq \tilde{s}_{w'} + \pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s})$, since otherwise either the allocation would not be individually rational for f or f and W'' could block. Thus, $s_{w'} < \tilde{s}_{w'} + \epsilon_3$ and therefore

$$u_{w'}(\mu, s) = s_{w'} - \bar{s}_{w'} + \epsilon_1 < \tilde{s}_{w'} - \bar{s}_{w'} + \epsilon_1 + \epsilon_3 = \epsilon_3.$$

But now f' and w' can block (μ, s) with salary $\sigma = y_{f'}(\{w'\})$.

In the second case, by individual rationality, we have $s_w \leq y_{f'}(\{w\})$ and therefore $u_w(\mu, s) \leq \epsilon_2$. But then f and W' can block the allocation with salaries $s' \in \mathbb{R}^{W'}_{++}$ such that $s'_w = \bar{s}_w$ and $s'_{\tilde{w}} = s_{\tilde{w}}$ for $\tilde{w} \in W' \setminus \{w\}$.

In the third case, we have $u_{w'}(\mu, s) \geq \pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s})$, since otherwise f and W' can block the allocation with salaries $s' \in \mathbb{R}^{W'}_{++}$ such that $s'_{w'} = \tilde{s}_{w'} + u_{w'}(\mu, s)$ and $s'_{\tilde{w}} = s_{\tilde{w}}$ for $\tilde{w} \in W' \setminus \{w'\}$. Since $u_{w'}(\mu, s) \geq \pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s})$, we have

$$s_{w'} \geq y_{f'}(\{w'\}) + \pi_f(W',\tilde{s}) - \pi_f(W'',\tilde{s}) - \epsilon_3 > y_{f'}(\{w'\}) - \epsilon_2.$$

Thus, $\pi_{f'}(\mu, s) < \epsilon_2$. But then f' and w can block with salary $\sigma = y_{f'}(\{w\}) - \epsilon_2$ contradicting the assumption that $(\mu, s) \in \mathcal{C}(y, u)$. □

L Proof of Corollary 1

To prove the corollary, we use the Bolzano-Weierstrass theorem from real analysis. For future reference, we provide a version of the theorem for our set-up.

Lemma 11. *Any sequence of individually rational allocations $\{(\mu^t, s^t)\}_{t=0,1,2,\dots}$ in a continuous market has a converging subsequence $\{(\mu^{t_j}, s^{t_j})\}_{j=0,1,2,\dots}$. The subsequence can be chosen such that each allocation uses the same matching, $\mu^{t_j} = \mu$ for $j = 0, 1, \dots$.*

Proof. The set of individually rational allocations in (y, u) where allocations are understood, as above, as matrices in $\mathbb{R}^{F \times W}$ is bounded. Thus the Bolzano-Weierstrass theorem implies that there is a subsequence $\{(\mu^{t_j}, s^{t_j})\}_{j=0,1,\dots}$ converging to an allocation (μ, s). Since there are only finitely many matchings, we may choose the subsequence such that each allocation in the subsequence uses the same matching μ. □

With this lemma we can prove the corollary.

Proof. The first part is Theorem 1 of Kelso and Crawford (1982).

For the second part, let (y, u) be a market as in the second part of Theorem 3. We show that there is a σ_0' such that for $\sigma_0 < \sigma_0'$ we have $\mathcal{C}(y, u, \sigma_0) = \emptyset$. Suppose not. Then there exists a sequence of salary increments $(\sigma_0(t))_{t=0,1,\ldots}$ with $\lim_{t\to\infty} \sigma_0(t) = 0$ and allocations $\{(\mu^t, s^t)\}_{t=0,1,\ldots}$ with $(\mu^t, s^t) \in \mathcal{C}(y, u, \sigma_0(t))$. By Lemma 11, we can find a converging subsequence $\{(\mu^{t_j}, s^{t_j})\}_{j=0,1,\ldots}$ By Lemma 5, we have $\lim_{j\to\infty}(\mu^{t_j}, s^{t_j}) \in \mathcal{C}(y, u)$ contradicting $\mathcal{C}(y, u) = \emptyset$.

□

M Proof of Lemma 7

Proof. By Lemma 10 in Appendix K, there exists an $\epsilon' > 0$ such that for each $s \in S^W$ we have $D_f(s') \subseteq D_f(s)$ for each $|s - s'| \leq \epsilon'$. Choose for each $w \in W$ independently and uniformly at random, numbers $\epsilon_w \in (0, \frac{\epsilon'}{\sqrt{|W|}})$. Define a modified production function $\tilde{y}_f : 2^W \to \mathbb{R}$ by

$$\tilde{y}_f(W') := y_f(W') + \sum_{w \in W'} \epsilon_w. \tag{6}$$

The profit function $\tilde{\pi}_f$ induced by $\tilde{y}_f$ has with probability 1 no ties for salaries in S. In particular, there exist numbers $\epsilon = (\epsilon_w)_{w\in W} \in (0, \frac{\epsilon'}{\sqrt{|W|}})^W$ such that the profit function defined by Equation (6) induces a profit function without ties for salaries in S. We consider the induced demand function $\tilde{D}_f : S^W \to 2^W$. Note that for each $s \in S^W$ we have $\tilde{D}_f(s) = D_f(s - \epsilon)$. Thus gross substitutability and the law of aggregate demand for $\tilde{D}_f$ follow from the gross substitutability and the law of aggregate demand for D_f. Moreover,

$$|s - \epsilon - s| = |\epsilon| = \sqrt{\sum_{w \in W} \epsilon_w^2} \leq \epsilon',$$

and therefore $\tilde{D}_f(s) \in D_f(s)$.

□

N Proof of Proposition 5

Proof. We choose a $\bar{\sigma} > \max_{f\in F, W'\subseteq W} y_f(W')$ which is an integer multiple of σ_0 and let $S := \{\sigma_0, 2 \cdot \sigma_0, \ldots, \bar{\sigma}\}$. Consider a profile $D^{\sigma_0} = (D_f^{\sigma_0} : S^W \to 2^W)_{f\in F}$ of well-behaved demand functions as constructed in Lemma 7. Let $\mathcal{M} : \mathcal{R}^W \to \mathcal{A}$ be the salary adjustment mechanism for the profile D^{σ_0}. Let $\rhd$ be an ordering of the firms F. We use the ordering to break ties. For each $u \in U^W$, we define a $\succeq^{u,\sigma_0} \in \mathcal{R}^W$ by

$$(f, \sigma) \succ_w^{u,\sigma_0} (f', \sigma') :\Leftrightarrow (u_w(f, \sigma) > u_w(f', \sigma') \text{ or } (u_w(f, \sigma) = u_w(f', \sigma') \text{ and } f \rhd f')).$$

Indifferences with the outside option are broken in favor of employment, i.e.

$$(f,\sigma) \succ_w^{u,\sigma_0} \emptyset \Leftrightarrow u_w(f,\sigma) \geq u_w(\emptyset)$$

and

$$\emptyset \succ_w^{u,\sigma_0} (f,\sigma) \Leftrightarrow u_w(\emptyset) > u_w(f,\sigma).$$

We define a mechanism $\mathcal{M}_{\sigma_0} : U^W \to \mathcal{A}$ by

$$\mathcal{M}_{\sigma_0}(u) = \mathcal{M}(\succeq^{u,\sigma_0}).$$

We check that $\mathcal{M}_{\sigma_0}$ is strategy-proof. Suppose for the sake of contradiction that

$$u_w(\mathcal{M}_{\sigma_0}(u'_w, u_{-w})) > u_w(\mathcal{M}_{\sigma_0}(u)).$$

Then $\mathcal{M}(\succeq^{u',\sigma_0}) \succ_w^{u,\sigma_0} \mathcal{M}(\succeq^{u,\sigma_0})$. But this contradicts the strategy-proofness of $\mathcal{M}$.

Next we show that for each $u \in U^W$ the allocation $\mathcal{M}_{\sigma_0}(u)$ is in the core of (y,u,σ_0). Since $\mathcal{M}$ is D^{σ_0}-stable it suffices to show that if an allocation (μ,s) is stable in $(D^{\sigma_0}, \succeq^{u,\sigma_0}$), then it is in the core of (y,u,σ_0). Let $(\mu,s) \in \mathcal{S}(D^{\sigma_0}, \succeq^{u,\sigma_0})$. Since (μ,s) is individually rational in $(D^{\sigma_0}, \succeq^{u,\sigma_0})$ we have $D_f^{\sigma_0}(s_f) = \mu(f)$ for each $f \in F$. Since $D_f^{\sigma_0}$ is a selection from the demand correspondence induced by y_f, we have $\pi_f(\mu(f), s_f) \geq \pi_f(W', s_f)$ for each $W' \subseteq \mu(f)$. Moreover, since (μ,s) is individually rational in $(D^{\sigma_0}, \succeq^{u,\sigma_0})$ we have $(\mu,s) \succ_w^{u,\sigma_0} \emptyset$ for each matched w. Thus, we have $u_w(\mu,s) \geq u_w(\emptyset)$ for each matched w. Hence (μ,s) is individually rational in (y,u,σ_0).

Next, let $f \in F$, $W' \subseteq W$ and $s' \in S^{W'}$ such that $u_w(f, s'_w) > u_w(\mu,s)$ for each $w \in W'$. We show that $\pi_f(W', s') \leq \pi_f(\mu,s)$ and thus there is no strict blocking coalition for (μ,s) in (y,u,σ_0). Define a vector $\tilde{s} \in S^W$ by

$$\tilde{s}_w = \begin{cases} s_w & \text{for } w \in \mu(f), \\ s'_w & \text{for } w \in W' \setminus \mu(f), \\ \bar{\sigma} & \text{else.} \end{cases}$$

Since $\tilde{s}_w = s_w \leq s'_w$ for $w \in \mu(f) \cap W'$, we have $\pi_f(W', \tilde{s}) \geq \pi_f(W', s')$. Moreover, $(f, \tilde{s}_w) \succeq_w^{u,\sigma_0} (\mu,s)$ for each $w \in W$ (for $w \in \mu(f) \cap W'$ we have $(f, \tilde{s}_w) = (\mu(w), s_w)$ and for all other w we have $(f, s_w) \succ_w^{u,\sigma_0} (\mu,s)$). By stability of (μ,s) in $(D^{\sigma_0}, \succeq^{u,\sigma_0})$ this implies that $D_f^{\sigma_0}(\tilde{s}) = \mu(f)$. Therefore, $\pi_f(\mu,s) = \pi_f(\mu(f), \tilde{s}) \geq \pi_f(W', \tilde{s}) \geq \pi_f(W', s')$ as desired. □

O Proof of Theorem 4

Proof. The main step in the proof is to show that each two core allocations $(\mu^1, s^1), (\mu^2, s^2) \in \mathcal{C}(y,u)$ have a joint maximum with respect to workers' utility functions, i.e. there is a $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y,u)$ such that for each $w \in W$ we have

$$u_w(\bar{\mu}, \bar{s}) \geq \max\{u_w(\mu^1, s^1), u_w(\mu^2, s^2)\}.$$

We establish this result by using the existence of a worker-optimal stable allocation in discrete ordinal markets (Proposition **??**) and a limit argument. To illustrate the argument, we first give an informal description of it. Suppose for the moment that all of the salaries in the schedules s_1 and s_2 are rational numbers and no agent is indifferent between (μ^1, s^1) and (μ^2, s^2) unless he gets the same assignment in both allocations. These assumptions are for the sake of exposition. We will later provide a fully general argument. Under the simplifying assumptions, we can find a $\sigma_0 > 0$ such that both salary schedules s^1 and s^2 contain salaries that are integer-multiples of σ_0. Let $S = \{\sigma_0, 2 \cdot \sigma_0, \ldots, \bar{\sigma}\}$ where $\bar{\sigma} > \max_{f\in F, W'\subseteq W} y_f(W')$ is an integer-multiple of σ_0. Moreover, we can find for each $w \in W$ strict preferences $\succeq_w$ over $F \times S \cup \{\emptyset\}$ consistent with u_w in the sense that $u_w(\mu, s) > u_w(\mu', s') \Rightarrow (\mu, s) \succ_w (\mu', s')$, and for each $f \in F$ a well-behaved $\tilde{D}_f : S^W \to 2^W$ selecting from D_f such that both allocations (μ^1, s^1) and (μ^2, s^2) are stable in the ordinal market $(\tilde{D}, \succeq)$. By Proposition **??**, there is a worker optimal stable allocation $(\bar{\mu}, \bar{s})$ in the ordinal market $(\tilde{D}, \succeq)$. In an analog way, we can define ordinal markets for salary increments $\sigma_0/2, \sigma_0/3, \ldots$ such that (μ^1, s^1) and (μ^2, s^2) are Pareto dominated for workers by the worker-optimal stable allocations in the ordinal markets. By a continuity argument, we find in the limit an allocation $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y, u)$, Pareto dominating for workers the allocations (μ^1, s^1) and (μ^2, s^2) in the continuous market.

The general argument proceeds along similar lines. However, we have to make two modifications. Since the salary schedules can in general contain salaries that are irrational numbers, we have to augment the salary grid S by salaries in the schedules s_1 and s_2 to make the allocations (μ^1, s^1) and (μ^2, s^2) feasible. Moreover, as workers or firms could be indifferent between (μ^1, s^1) and (μ^2, s^2) without obtaining the same assignment in the two allocations, we have to deal with tie-breaking. In general, one of the two allocations can become unstable through the tie-breaking. However, we can show that in this case, the allocation will be Pareto dominated for workers by a stable allocation with respect to the preferences with broken ties.

The general argument is contained in the following claim that we prove next.

Claim. Let $(\mu^1, s^1), (\mu^2, s^2) \in \mathcal{C}(y, u)$. For each $\sigma_0 > 0$, there is an ordinal market $(D^{\sigma_0}, \succeq^{\sigma_0})$ such that the following holds. Let $\bar{\sigma} > \max_{f\in F, W'\subseteq W} y_f(W')$ be an integer multiple of σ_0 and define a finite set $S \subseteq \mathbb{R}_{++}$ of salaries by

$$S = \{\sigma_0, 2\sigma_0, \ldots, \bar{\sigma}\} \cup \{s^1_w : w \in \mu^1(F)\} \cup \{s^2_w : w \in \mu^2(F)\}.$$

1. For each $f \in F$ the demand is a function $D^{\sigma_0}_f : S^W \to 2^W$ that is a single-valued selection from D_f, i.e. for each $s \in S^W$ we have $D^{\sigma_0}_f(s) \in D_f(s)$,

2. under D^{σ_0} workers are gross substitutes and the law of aggregate demand holds for each firm,

3. for each $w \in W$, the preferences $\succeq^{\sigma_0}_w$ are defined over $F \times S \cup \{\emptyset\}$ and obtained from u_w by tie-breaking, i.e. we have

$$(\mu, s) \succ^{\sigma_0}_w (\mu', s') \Rightarrow u_w(\mu, s) \geq u_w(\mu', s'),$$

4. we have $(\mu^1, s^1) \in \mathcal{S}(D^{\sigma_0}, \succeq^{\sigma_0})$,

5. there is a $(\mu', s') \in \mathcal{S}(D^{\sigma_0}, \succeq^{\sigma_0})$ such that $u_w(\mu', s') \geq u_w(\mu^2, s^2)$ for each $w \in W$.

To prove the claim, we define for each $f \in F$ a demand function $D_f^{\sigma_0} : S \to 2^W$ as follows. By Lemma 10 in Appendix K, there exists an $\epsilon' > 0$ such that for each $s \in S^W$ we have $D_f(s') \subseteq D_f(s)$ for each $|s - s'| \leq \epsilon'$. We define a modified production function, $\tilde{y}_f : 2^W \to \mathbb{R}$ by Equation (6) in Appendix **??** where the vector $\epsilon \in (0, \frac{\epsilon'}{\sqrt{|W|}})^W$ is now defined such that

$$w \notin \mu^2(f), w' \in \mu^2(f) \Rightarrow \epsilon_w < \epsilon_{w'}. \tag{7}$$

By a perturbation argument as in the proof of Lemma 7, we may choose $(\epsilon_w)_{w \in W}$ such that the profit function $\tilde{\pi}_f$ induced by $\tilde{y}_f$ has no ties if salaries are in S. (We can achieve this, e.g. by partitioning the interval $(0, \frac{\epsilon'}{\sqrt{|W|}})$ in two non-empty intervals $(0, \epsilon'')$ and $(\epsilon'', \frac{\epsilon'}{\sqrt{|W|}})$ with $0 < \epsilon'' < \frac{\epsilon'}{\sqrt{|W|}}$ and choose independently and uniformly at random for each $w \notin \mu^2(f)$ a number $\epsilon_w \in (0, \epsilon'')$ and for each $w \in \mu^2(f)$ a number $\epsilon_w \in (\epsilon'', \frac{\epsilon'}{\sqrt{|W|}})$. With probability 1, the selected ϵ will perturb π_f such that it has no ties). The same argument as in the proof of Lemma 7 shows that $\tilde{\pi}_f$ induces a demand function $\tilde{D}_f$ that is a selection from D_f and that under $\tilde{D}_f$ workers are gross substitutes and the law of aggregate demand holds. We let $D_f^{\sigma_0} = \tilde{D}_f$. Note that (7) implies that for each $s \in S^W$, if $\mu^2(f) \in D_f(s)$, then $\mu^2(f) \subseteq D_f^{\sigma_0}(s)$. We will use this fact later in the proof.

For each $w \in W$, we define $\succeq_w^{\sigma}$ such that

$$u_w(\mu, s) > u_w(\mu', s') \Rightarrow (\mu, s) \succ_w^{\sigma_0} (\mu', s')$$

and ties with allocation (μ^1, s^1) are broken in favor of (μ^1, s^1), i.e. for $\mu^1(w) \neq \mu(w)$, we let

$$u_w(\mu^1, s^1) = u_w(\mu, s) \Rightarrow (\mu^1, s^1) \succ_w^{\sigma_0} (\mu, s).$$

Otherwise we break ties arbitrarily.

We show that $(\mu^1, s^1) \in \mathcal{S}(D^{\sigma_0}, \succeq^{\sigma_0})$. Since ties in workers' preferences are broken in favor of (μ^1, s^1) and (μ^1, s^1) is individually rational with respect to u, the allocation (μ^1, s^1) is individually rational with respect to $\succeq^{\sigma_0}$. Moreover, by construction of $\tilde{y}$, for each $f \in F$, $s \in S^W$, and $W'' \subseteq W' \subseteq W$ we have that $\pi_f(W', s) \geq \pi_f(W'', s)$ implies $\tilde{\pi}_f(W', s) > \tilde{\pi}_f(W'', s)$. Thus (μ^1, s^1) is individually rational for firms according to the demand induced by $(\tilde{\pi}_f)_{f \in F}$ which is D^{σ_0}. Now suppose f and W' block (μ^1, s^1) with salaries s' in the market $(D^{\sigma_0}, \succeq^{\sigma_0})$. We show that f and W' would also block (μ^1, s^1) in (y, u). For each $w \in W'$, we have $(f, s'_w) \succeq_w^{\sigma_0} (\mu^1, s^1)$ and therefore $u_w(f, s'_w) \geq u_w(\mu^1, s^1)$. Moreover, there is at least one $w \in W' \setminus \mu(f)$. For this worker w, we have $(f, s'_w) \succ_w^{\sigma_0} (\mu^1, s^1)$. Since ties are broken in favor of (μ^1, s^1) this implies $u_w(f, s'_w) > u_w(\mu^1, s^1)$. Moreover, since $D_f^{\sigma_0}$ is a selection from D_f and $D_f^{\sigma_0}(s') = W'$, we have $\pi_f(W', s') \geq \pi_f(\mu^1, s^1)$. Thus f and W' block (μ^1, s^1) in (y, u) with salaries s', a contradiction.

Next we show that there is a stable allocation in $(D^{\sigma_0}, \succeq^{\sigma_0})$ that every worker likes as least as much as (μ^2, s^2) according to u. We consider a modified preference profile $\succeq'$ that is obtained from $\succeq^{\sigma_0}$ as follows. For each $w \in W$ with $(\mu^2, s^2) \succ_w^{\sigma_0} \emptyset$, we truncate w's preferences after $(\mu^2(w), s_w^2)$. For each $w \in W$ with $\emptyset \succeq_w^{\sigma_0} (\mu^2, s^2)$ we leave preferences unchanged. Let (μ', s') be the worker-optimal stable allocation in the truncated market $(D^{\sigma_0}, \succeq')$. By construction, we have for each $w \in W$ that $u_w(\mu', s') \geq u_w(\mu^2, s^2)$. We show that (μ', s') is also stable in the untruncated market $(D^{\sigma_0}, \succeq^{\sigma_0})$. This will conclude the proof of the claim. Suppose for the sake of contradiction that (μ', s') is blocked in $(D^{\sigma_0}, \succeq^{\sigma_0})$. Since $\succeq'$ was obtained from $\succeq^{\sigma_0}$ by truncating below (μ^2, s^2) and (μ', s') is stable in $(D^{\sigma_0}, \succeq')$ but not in $(D^{\sigma_0}, \succeq^{\sigma_0})$, there exists a $\tilde{w} \in W$ such that $\mu'(\tilde{w}) = \emptyset$ and $(\mu^2, s^2) \succ_{\tilde{w}}^{\sigma_0} \emptyset$. Let $f = \mu^2(\tilde{w})$ and let s_f be the minimal potential blocking vector for f and (μ^2, s^2) under $\succeq^{\sigma_0}$. We have $\mu^2(f) \in D_f(s_f)$. As observed above, this implies $\mu^2(f) \subseteq D_f^{\sigma_0}(s_f)$. Define a salary vector $\tilde{s} \in S^W$ by

$$\tilde{s}_w = \begin{cases} s'_w, & \text{for } w \neq \tilde{w}, \\ s_w^2, & \text{for } w = \tilde{w}, \\ \bar{\sigma}, & \text{else.} \end{cases}$$

We have $\tilde{s} \geq s_f$ and $\tilde{s}_{\tilde{w}} = s_{f\tilde{w}}$. As workers are gross substitutes under $D_f^{\sigma_0}$, we have that $\tilde{w} \in \mu^2(f) \subseteq D_f^{\sigma_0}(s_f)$ implies $\tilde{w} \in D_f^{\sigma_0}(\tilde{s})$. As $\tilde{w} \notin \mu'(f)$ this implies $D_f^{\sigma_0}(\tilde{s}) \neq \mu'(f)$. Hence f and $W' = D_f^{\sigma_0}(\tilde{s})$ block (μ', s') via salaries $\tilde{s}$ in $(D^{\sigma_0}, \succeq')$, a contradiction. Moreover, for each $w \in W$ we have $(f, \tilde{s}_w) \succeq'_w (\mu', s')$.

With the claim, we can prove the theorem. First we prove the first part. Define a vector $\bar{u} \in \mathbb{R}^W$ by

$$\bar{u}_w := \sup_{(\mu,s) \in \mathcal{C}(y,u)} u_w(\mu, s).$$

We show that for each $w \in W$ there is a $(\mu, s) \in \mathcal{C}(y, u)$ such that $u_w(\mu, s) = \bar{u}_w$. Pick a sequence $(\mu^t, s^t)_{t=0,1,\ldots} \subseteq \mathcal{C}(y, u)$ with

$$\lim_{t \to \infty} u_w(\mu^t, s^t) = \bar{u}_w.$$

By Lemma 11, we can find a subsequence $\{(\mu^{t_j}, s^{t_j})\}_{j=0,1,\ldots}$ converging to an allocation (μ, s). By Lemma 5, $(\mu, s) \in \mathcal{C}(y, u)$. The subsequence can be chosen such that $\mu^{t_j} = \mu$ for each j. Moreover, u_w is continuous in salaries. Therefore,

$$u_w(\mu, s) = u_w(\mu, \lim_{j \to \infty} s^{t_j}) = \lim_{j \to \infty} u_w(\mu, s^{t_j}) = \bar{u}_w.$$

As for each $w \in W$ we have a core allocation that yields utility $\bar{u}_w$ and the set of workers is finite, it now suffices to show that any two core allocations have a supremum, i.e. to show that for each $(\mu^1, s^1), (\mu^2, s^2) \in \mathcal{C}(y, u)$ there exists a $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y, u)$ such that for each $w \in W$ we have $u_w(\bar{\mu}, \bar{s}) \geq \max\{u_w(\mu^1, s^1), u_w(\mu^2, s^2)\}$. This will follow from the claim and Proposition **??** by a limit argument. By Proposition **??** there is for each $\sigma_0 > 0$ a worker-optimal allocation $(\mu^{\sigma_0}, s^{\sigma_0})$ in $(D^{\sigma_0}, \succeq^{\sigma_0})$. For each $\sigma_0 > 0$, since allocation $(\mu^{\sigma_0}, s^{\sigma_0})$ is individually rational in $(D^{\sigma_0}, \succeq^{\sigma_0})$, allocation $(\mu^{\sigma_0}, s^{\sigma_0})$ is individually rational in (y, u). By parts 4 and 5 of the claim, we have for each $\sigma_0 > 0$ and each $w \in W$ that

$$u_w(\mu^{\sigma_0}, s^{\sigma_0}) \geq \max\{u_w(\mu^1, s^1), u_w(\mu^2, s^2)\}.$$

By Lemma 11, there exists a sequence $(\sigma_0(t))_{t=0,1,\dots}$ with $\lim_{t\to\infty} \sigma_0(t) = 0$ such that $(\mu^{\sigma_0(t)}, s^{\sigma_0(t)})_{t=0,1,\dots}$ converges to an individually rational allocation $(\bar{\mu}, \bar{s})$. The sequence can be chosen such that for each t we have $\mu^{\sigma_0(t)} = \bar{\mu}$. By Lemma 5, $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y, u)$. Moreover, as utility functions are continuous in salaries, for each $w \in W$ we have

$$\begin{aligned} u_w(\bar{\mu}, \bar{s}) &= u_w(\bar{\mu}, \lim_{t\to\infty} s^{\sigma_0(t)}) = \lim_{t\to\infty} u_w(\bar{\mu}, s^{\sigma_0(t)}) = \lim_{t\to\infty} u_w(\mu^{\sigma_0(t)}, s^{\sigma_0(t)}) \\ &\geq \lim_{t\to\infty} \max\{u_w(\mu^1, s^1), u_w(\mu^2, s^2)\} = \max\{u_w(\mu^1, s^1), u_w(\mu^2, s^2)\}. \end{aligned}$$

Next we show the second part. Let $(\mu, s), (\mu', s') \in \mathcal{C}(y, u)$ and suppose that for $w \in W$ we have $\mu(w) = \emptyset$. We show that $u_w(\mu', s') = u_w(\emptyset)$. By the first part of the theorem there is a worker-optimal core allocation $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y, u)$. Let $\sigma_0 > 0$. By the claim, with $(\mu^1, s^1) = (\mu, s)$ and $(\mu^2, s^2) = (\bar{\mu}, \bar{s})$, there is an ordinal market $(D^{\sigma_0}, \succeq^{\sigma_0})$ with $(\mu, s) \in \mathcal{S}(D^{\sigma_0}, \succeq^{\sigma_0})$ such that for each $w \in W$ we have

$$u_w(\mu^{\sigma_0}, s^{\sigma_0}) \geq \max\{u_w(\mu, s), u_w(\bar{\mu}, \bar{s})\}.$$

For each $f \in F$, workers are gross substitutes and the law of aggregate demand holds under $D_f^{\sigma_0}$. Thus, part 2.(a) of Proposition **??** implies $\mu^{\sigma_0}(w) = \mu(w) = \emptyset$. Therefore $u_w(\bar{\mu}, \bar{s}) \leq u_w(\mu^{\sigma_0}, s^{\sigma_0}) = u_w(\emptyset)$, where the weak inequality is an equality by the individual rationality of $(\bar{m}u, \bar{s})$. Since $(\bar{\mu}, \bar{s})$ is a worker-optimal core allocation in (y, u) we have $u_w(\mu', s') \leq u_w(\bar{\mu}, \bar{s}) = u_w(\emptyset)$, where the weak inequality is an equality by the individual rationality of (μ', s').

□

P Proof of Lemma 8

Proof. Let $u \in U^W$ and suppose $\mathcal{M}_t(u)$ converges to (μ, s) as $t \to \infty$. By Lemma 5, $(\mu, s) \in \mathcal{C}(y, u)$. By the first part of Theorem 4, there is a worker optimal allocation $(\bar{\mu}, \bar{s}) \in \mathcal{C}(y, u)$. We show that for each $w \in W$ we have $u_w(\mu, s) = u_w(\bar{\mu}, \bar{s})$. Suppose for the sake of contradiction that there is a $w \in W$ such that $u_w(\bar{\mu}, \bar{s}) > u_w(\mu, s)$. Let $u_w(\bar{\mu}, \bar{s}) > K > u_w(\mu, s)$ and define $u'_w \in U$ by $u'_w(f, \sigma) := u_w(f, \sigma)$ for each $(f, \sigma) \in F \times \mathbb{R}_{++}$ and $u'_w(\emptyset) := K$. Note that $(\bar{\mu}, \bar{s})$ is still a worker-optimal core allocation in (y, u'_w, u_{-w}). Consider the sequence of allocations $\{\mathcal{M}_t(u'_w, u_{-w})\}_{t=1,2,\dots}$. By Lemma 11, there is a subsequence $\{(\tilde{\mu}^j, s^j)\}_{j=0,1,\dots} := \{\mathcal{M}_{t_j}(u'_w, u_{-w})\}_{j=0,1,\dots}$ that converges to an allocation $(\tilde{\mu}, \tilde{s})$. The subsequence can be chosen such that for each $j = 0, 1, \dots$ we have $\tilde{\mu}^j = \tilde{\mu}$. By Lemma 5, $(\tilde{\mu}, \tilde{s}) \in \mathcal{C}(y, u'_w, u_{-w})$. By the second part of Theorem 4, we have $\tilde{\mu}(w) \neq \emptyset$. Thus $u_w(\tilde{\mu}, \tilde{s}) > u_w(\mu, s)$. By continuity of u_w in salaries, we can find a j such that $u_w(\tilde{\mu}, \tilde{s}^{t_j}) > u_w(\mu, s)$ contradicting the strategy-proofness of $\mathcal{M}_{t_j}$. □

Q Proof of Theorem 5

Proof. Let $(\mathcal{M}_{\sigma_0})_{\sigma_0>0}$ be a family of (y, σ_0)-stable and strategy-proof mechanisms. Such a family exists by Proposition 5. Let $u \in U^W$. Since the allocations $(\mathcal{M}_{\sigma_0}(u))_{\sigma_0>0}$ are individually rational in (y, u) there exists by Lemma 11, a sequence $(\sigma_0(t))_{t=0,1,\ldots}$ of salary increments with $\lim_{t\to\infty} \sigma_0(t) = 0$ such that $\mathcal{M}_{\sigma_0(t)}(u)$ converges to an allocation (μ, s) as $t \to \infty$. By Lemma 8, (μ, s) is a worker-optimal core allocation in (y, u). We define $\mathcal{M}(u) := (\mu, s)$.

We show that $\mathcal{M}$ is strategy-proof. Let $u_{-w} \in U^{W\setminus\{w\}}$ and $u_w, u'_w \in W$. Define $(\mu, s) := \mathcal{M}(u)$ and $(\mu', s') := \mathcal{M}(u'_w, u_{-w})$. Now suppose for the sake of contradiction that $u_w(\mu', s') > u_w(\mu, s)$. We show that this implies that there is a $\sigma_0 \in \mathbb{R}_{++}$ such that $u_w(\mathcal{M}_{\sigma_0}(u'_w, u_{-w})) > u_w(\mathcal{M}_{\sigma_0}(u))$ contradicting the strategy-proofness of $\mathcal{M}_{\sigma_0}$.

Let $(\sigma_0(t))_{t=0,1,2,\ldots}$ be the sequence of salary increments that we used to define $\mathcal{M}(u)$. It is not necessarily the case that the sequence $\{\mathcal{M}_{\sigma_0(t)}(u'_w, u_{-w})\}_{t=1,2,\ldots}$ converges to (μ', s'), since we might have used different decreasing sequences of salary increments to define $\mathcal{M}(u)$ and $\mathcal{M}(u'_w, u_{-w})$. However, by Lemma 11, there is a subsequence $(\sigma_0(t_j))_{j=0,1,\ldots}$ such that $\{\mathcal{M}_{\sigma_0(t_j)}(u'_w, u_{-w})\}_{j=1,2,\ldots}$ converges to an allocation (μ'', s'') (possibly $\mu'' \neq \mu'$). By Lemma 8, both (μ', s') and (μ'', s'') are worker-optimal core allocations in (y, u). Thus $u_w(\mu', s') = u_w(\mu'', s'')$. Define $\epsilon := u_w(\mu', s') - u_w(\mu, s)$.

By Lemma 11, it is no loss of generality to assume that for each t we have $\mathcal{M}_{\sigma_0(t)}(u) = (\mu, s^t)$ for a schedule $s^t \in \mathbb{R}_{++}^{\mu(F)}$. Since u_w is continuous in salaries, we can find a T such that for $t > T$ we have

$$|u_w(\mu, s^t) - u_w(\mu, s)| < \epsilon/2.$$

By Lemma 11, it is no loss of generality to assume that for each j we have $\mathcal{M}_{\sigma_0(t_j)}(u'_w, u_{-w}) = (\mu'', \tilde{s}^j)$ for a schedule $\tilde{s}^j \in \mathbb{R}_{++}^{\mu''(F)}$. Since u_w is continuous in salaries, we can find a J such that for $j > J$ we have

$$|u_w(\mu'', \tilde{s}^j) - u_w(\mu'', s'')| < \epsilon/2.$$

Choose $j > J$ such that $t_j > T$. For $\sigma_0 := \sigma_{t_j}$ we have

$$u_w(\mathcal{M}_{\sigma_0}(u'_w, u_{-w})) > u_w(\mu'', s'') - \epsilon/2 = u_w(\mu', s') - \epsilon/2 > u_w(\mu, s) + \epsilon/2 > u_w(\mathcal{M}_{\sigma_0}(u)).$$

This contradicts the strategy-proofness of $\mathcal{M}_{\sigma_0}$.

□