Solidarity Properties of Choice Correspondences*

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Abstract

We consider the problem of choosing a set of locations of a public good on the real line $\mathbb{R}$. Similarly to Klaus and Storcken (2002), we ordinally extend the agents' preferences over compact subsets of $\mathbb{R}$, and extend the results of Ching and Thomson (1996), Vohra (1999), and Klaus (2001) to choice correspondences. Specifically, we show that *Pareto-efficiency* and either *population-monotonicity* or *one-sided replacement-domination* characterize the class of target set correspondences on the domains of single-peaked preferences and symmetric single-peaked preferences.

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1 Introduction

We study the social choice problem where a non-empty and compact set (of points) is chosen on the real line $\mathbb{R}$. We consider this (chosen) set to represent a public good such that each point in the set represents an option for the public good together with its location. We assume that agents have single-peaked preferences, that is, an agent's welfare is strictly increasing up to a certain point, his "peak", and is strictly decreasing beyond this point. Given a non-empty and compact set (of points) that represents the public good's options and their locations, an agent is unable to compute his chance of obtaining the public good at

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a particular location, e.g., in the case of parking spaces along a street, an agent might know that he will (eventually) find a parking spot but he does not know where this will be. We therefore assume that agents, when comparing sets, only consider their best (most favorite) point(s) and their worst (least favorite) point(s) in each set. Finally, we assume that the set has adequate capacity to accommodate all agents, that is, all agents have access to the public good but possibly at different locations.

More specifically, we look into the situation where the social planner wishes to make a choice by providing the public good in a way that is *Pareto-efficient*, according to the agents' preferences, and that satisfies some notion of solidarity between agents towards changes in circumstances. Loosely speaking, solidarity requires that all agents not responsible for the change should be affected in the same direction. The changes in circumstances we study in this paper are changes in the agents' population, by considering the property of *population-monotonicity*, and changes in some agents' preferences, by considering the property of *replacement-domination*. *Population-monotonicity*, introduced by Thomson (1983a,b) in the context of bargaining, applies to a model with a variable population of agents and requires that if additional agents join a population, then the agents who were initially present should all be made at least as well off, as they were initially, or they should all be made at most as well off. *Replacement-domination*, introduced by Moulin (1987) in the context of quasi-linear binary public decision, applies to a model with a fixed population of agents and requires that if the preferences of an agent change, then the other agents whose preferences remained unchanged should all be made at least as well off, as they were initially, or they should all be made at most as well off.

Further to the parking zone example, already briefly mentioned and further explained in Section 2, another example of the described situation could be the following. A social planner drafts an 'if-needed' list of candidate locations to build a public hospital according to the agents' preferences. She does so in an effort to narrow down future construction scenarios while at the same time respecting (in a *Pareto- efficient* sense) the agents' preferences and adhering to some notion of solidarity, as described above. Then, if at some future time the need to build a hospital materializes, each location in this list is scrutinized and one of them is chosen for the hospital to be built at, with this final verdict assumed unpredictable at the time when the list is drafted.

Many more social choice problems can be phrased as problems of providing a public good by choosing the location of it on the real line $\mathbb{R}$ or an interval of it, or more generally, on a tree network[1], when agents have single-peaked preferences. In these types of problems, it is very natural for changes in the population (e.g., through a change in the birth or migration rate) or changes in the agents preferences (e.g., through the influence of public media or social networks) to arise. Hence, the properties of *population-monotonicity* and *replacement-domination* have been studied, together or individually, in a variety of contexts. For the special case where the tree network is a closed interval, the problem coincides with the problem of providing a public good by choosing its level when agents have single-peaked preferences (Moulin, 1980). Apart from the provision of public parking or the provision of a hospital by choosing an 'if-needed' list of locations, further examples of providing a public good in one or more locations include the provision of (one or more) schools, parks, or libraries on a tree network that represents an infrastructure, e.g., the network of roads in a neighborhood.

For choice functions, providing a public good on an interval, or on a tree network respectively, Thomson (1993), Ching and Thomson (1996), and Vohra (1999) consider the solidarity properties *population-monotonicity* and *replacement-domination*, respectively. For the location problem on an interval (on a tree network), the class of choice functions satisfying *Pareto-efficiency* and *population-monotonicity* is the class of "target point functions" (Ching and Thomson, 1996; Thomson, 1993).[2] Each target point function is determined by its target point: if the target point is *Pareto-efficient*, then the target point is chosen; if it is not *Pareto-efficient*, then the closest *Pareto-efficient* point to the target point is chosen. For the location problem on a tree network, Vohra (1999) proves that if the set of agents is fixed, contains at least three agents, and each agent has symmetric single-peaked preferences, then the class of choice functions satisfying *Pareto-efficiency* and *replacement-domination* is the class of target point functions. This result also holds on the larger domain of single-peaked preferences. Klaus (2001) shows that *Pareto-efficiency* and *population-monotonicity* imply *replacement-domination* and she extends the characterization of Ching and Thomson (1996) to the smaller domain of symmetric single-peaked preferences and to tree networks. For the location problem on an interval, Harless (2015) weakens the property of *replacement-domination* to *ϵ-replacement-domination*[3] and proves that if the set of agents contains at

[1]A tree network is a connected graph that contains no cycles.

[2]Target point functions are sometimes called status quo rules or status quo solutions.

[3]Agents' solidarity is only required if the change in an agent's preferences are below a certain threshold.

least three agents and each agents has single-peaked preferences, then the class of choice functions satisfying *Pareto-efficiency* and *ϵ-replacement-domination* is the class of target point functions. Gordon (2007), studying the problem of providing a public good on a circle, shows that if the agents have symmetric single-peaked preferences and at least four agents exist, no choice function satisfies *Pareto-efficiency* and either *replacement-domination* or *population-monotonicity*.

Regarding choice correspondences, the case of providing a public good at exactly two locations has been studied under different settings. Miyagawa (2001) studies the case where agents have single-peaked preferences and compare pairs of locations using the max-extension.[4] He proves that for an interval in $\mathbb{R}$, if the set of agents is fixed and contains at least four agents with at least two distinct peaks, then the class of choice functions satisfying *Pareto-efficiency* and *replacement-domination* are the "left-peaks choice function" and the "right-peaks choice function".[5] Umezawa (2012) extends this model to tree networks and shows that if agents have single-peaked preferences and at least four agents exist, then no choice function satisfies *Pareto-efficiency* and *replacement-domination*. For the problem of providing a public good at exactly two locations on an interval, Ehlers (2002) also studies single-peaked preferences but, in contrast to Miyagawa (2001), agents compare pairs of locations using the leximin-extension.[6] He proves that if the set of agents contains at least three agents, then the class of choice functions satisfying *Pareto-efficiency* and *replacement-domination* is the class of "single-peaked preference choice functions".[7]

All the above mentioned work analyzes solidarity properties where at most two locations to provide some public good are chosen at each preference profile. In this paper we study a class of problems where more than two locations to provide a public good might be chosen,

[4]Under the max-extension, an agent prefers set X to set Y if and only if he prefers his best point(s) in set X to his best point(s) in set Y.

[5]The left (right) peaks choice function chooses the two unique left-most (right-most) peaks.

[6]Under the leximin-extension, in the case of sets containing exactly two points, an agent prefers set X to set Y if and only if he either [prefers his best point(s) in set X to his best point(s) in set Y] or [he is indifferent between his best point in set X and his best point in set Y and prefers his second best point in set X to his second best point in set Y].

[7]Each single-peaked preference choice function is determined by fixed single-peaked preferences R_0: if the peak of R_0 lies within the convex hull of the agents' peaks, then according to R_0, the choice function chooses the best of the agents' peaks and its indifferent point; if the peak of R_0 lies left (right) of the smallest (largest) peak, then the smallest (largest) peak is chosen.

as considered in Klaus and Storcken (2002), albeit in a median voter context. The main reason in Klaus and Storcken (2002) to extend the standard choice function setup to choice correspondences is that for an even number of agents or voters, a set of median voter locations exists, hence choosing the median implies choosing a set of median points. To capture the full spirit of this median voter result, choice correspondences are introduced. Our motivation for extending choice from one or two locations to a set of locations is that we study situations in which the public good is usually provided through 'larger' sets of options, e.g., the assignment of neighborhood parking spots along a street.

On the domain of single-peaked preferences as well as the smaller domain of symmetric single-peaked preferences, we show that the class of choice correspondences satisfying *Pareto-efficiency* and either *one-sided replacement-domination*[8] or *population-monotonicity*, is the class of target set correspondences (Theorems 1 and 2). Each target set correspondence is determined by a target set $[a, b]$: if this set is *Pareto-efficient*, then it is chosen; if it is not *Pareto-efficient*, then its largest *Pareto-efficient* subset is chosen, if it exists; otherwise, the closest *Pareto-efficient* point to the target set is chosen. We also show that *Pareto-efficiency* and *replacement-domination* characterize the sub-class of target set correspondences where $a = b$, i.e., we obtain the class of target point functions (Corollary 3). Hence, we obtain corresponding results to Thomson (1993), Ching and Thomson (1996), and Vohra (1999).

Our results are parallel to Ehlers and Klaus (2001) who study probabilistic choice functions and extend the results of Thomson (1993) and Ching and Thomson (1996) to the case where the public good is provided via a lottery over locations on an interval. They characterize probabilistic target choice functions on the basis of *Pareto-efficiency* and either *one-sided replacement-domination* or *population-monotonicity*.

The paper proceeds as follows. Section 2 explains the model and shows some preliminary results. Section 3 contains the definition of target set correspondences. Section 4 contains the solidarity properties and further preliminary results. Section 5 presents characterizations of target set correspondences. Finally, all proofs not shown in the main text are shown in the Appendices.

[8] *One-sided replacement-domination* is weaker than *replacement-domination*: solidarity is not required when the preferences of the agent with the unique smallest peak are changed such that he becomes the agent with the unique largest peak, and vice-versa.

2 The Model

Denote the set of natural numbers by $\mathbb{N}$. There is a population of 'potential' agents, indexed by $\mathbb{P} \subseteq \mathbb{N}$, where $\mathbb{P}$ contains at least 3 agents, $|\mathbb{P}| \geq 3$. We denote the class of non-empty and finite subsets of $\mathbb{P}$ by $\mathcal{P}$. A set of agents $N \in \mathcal{P}$ is called a *coalition*.

All agents $i \in \mathbb{P}$ are equipped with 'single-peaked' *preferences* R_i, defined on the real line $\mathbb{R}$, that are *complete*, *transitive*, and *reflexive*. As usual, $x\, R_i\, y$ is interpreted as 'x is at least as good as y', $x\, P_i\, y$ as 'x is preferred to y', and $x\, I_i\, y$ as 'x is indifferent to y'. *Single-peakedness* of R_i means that there exists a point $p(R_i) \in \mathbb{R}$, called the *peak (level) of agent* i, with the following property: for each pair x, $y \in \mathbb{R}$ such that either $y < x \leq p(R_i)$, or $y > x \geq p(R_i)$, we have $x\, P_i\, y$. We denote the *domain of all single-peaked preferences* on $\mathbb{R}$ by $\mathcal{R}$. An agent i's preferences $R_i \in \mathcal{R}$ are *symmetric* if $x \in \mathbb{R}$ is preferred to $y \in \mathbb{R}$ if and only if x is closer to the peak $p(R_i)$ than y, that is, for each pair $x, y \in \mathbb{R}$, $x\, P_i\, y$ if and only if $|x - p(R_i)| < |y - p(R_i)|$. We denote the *domain of all symmetric (and single-peaked) preferences* on $\mathbb{R}$ by $\mathcal{S}$. Note that $\mathcal{S}$ is a sub-domain of $\mathcal{R}$, $\mathcal{S} \subsetneq \mathcal{R}$.

For any coalition $N \in \mathcal{P}$, we denote the set of *(preference) profiles* $R = (R_i)_{i\in N}$ such that for each $i \in N$, $R_i \in \mathcal{R}$ by $\mathcal{R}^N$. Similarly, we denote the set of (preference) profiles $R = (R_i)_{i\in N}$ such that for each $i \in N$, $R_i \in \mathcal{S}$ by $\mathcal{S}^N$. For each pair of coalitions $N, M \in \mathcal{P}$, with $N \subseteq M$, we denote the restriction $(R_i)_{i\in N} \in \mathcal{R}^N$ of profile $R \in \mathcal{R}^M$ to coalition N by R_N. Given profile $R \in \mathcal{R}^N$, for each pair of agents $i, j \in N$ we also use the notation $R_{-i} = R_{N\setminus\{i\}}$ and $R_{-i,j} = R_{N\setminus\{i,j\}}$.

In the sequel, all notation and definitions presented refer to single-peaked preferences but also apply to symmetric single-peaked preferences, i.e., we could use $\mathcal{S}$ instead of $\mathcal{R}$.

Given a coalition $N \in \mathcal{P}$ and a profile $R \in \mathcal{R}^N$, we denote the *set of peaks* of profile R as $p(R) = \{p(R_i)\}_{i\in N}$, with *smallest peak* $\underline{p}(R) = \min\{p(R_i)\}_{i\in N}$, *largest peak* $\bar{p}(R) = \max\{p(R_i)\}_{i\in N}$, and the *convex hull* of the (set of) peaks $\mathrm{Conv}(R) = [\underline{p}(R), \bar{p}(R)]$.

Denote the class of non-empty and compact subsets of $\mathbb{R}$ by $\mathcal{C}$. Given a set $X \in \mathcal{C}$, we denote the *minimum (point)* of X by $\underline{X} = \min X$ and the *maximum (point)* of X by $\bar{X} = \max X$. Given a set $X \in \mathcal{C}$ and preferences $R_i \in \mathcal{R}$, the most preferred or *best point(s)* of agent i in set X is defined by $b_X(R_i) := \{x \in X : \text{for each } y \in X,\, x\, R_i\, y\}$. Similarly, the least preferred or *worst point(s)* of agent i in set X is defined by $w_X(R_i) := \{x \in X : \text{for each } y \in X,\, y R_i x\}$.

Note that by single-peakedness of preferences R_i, the sets $b_X(R_i)$ and $w_X(R_i)$ can contain one or two elements, and agent i is indifferent between each alternative contained within each set. Hence, with some abuse of notation, we treat sets $b_X(R_i)$ and $w_X(R_i)$ as if they are points and for each $x \in X$, we write $b_X(R_i)\; R_i\; x\; R_i\; w_X(R_i)$.

Before describing in detail the 'best-worst' extension of preferences over sets that is used in our model, we first introduce the properties of *weak-dominance* and *weak-independence* that characterize it (Barbera et al., 1984), and then, in the succeeding examples, we show why these properties are reasonable to assume in our model. In the definitions following, we denote preferences defined over $\mathcal{C}$ by $R^{\mathcal{C}}$.

Weak-Dominance. Let points $x, y \in \mathbb{R}$. If $x\; P^{\mathcal{C}}\; y$, then $\{x\}\; P^{\mathcal{C}}\; \{x, y\}\; P^{\mathcal{C}}\; \{y\}$.

Weak-Independence. Let sets $X, Y, Z \in \mathcal{C}$ such that $[X \cap Z = \emptyset$ and $Y \cap Z = \emptyset]$. If $X\; P^{\mathcal{C}}\; Y$, then $[X \cup Z]\; R^{\mathcal{C}}\; [Y \cup Z]$.

The following two examples illustrate the properties of *weak-independence* and *weak-dominance* in our model. Both examples consider a (linear) city whose residents own exactly one car each and have single-peaked preferences on where to park it.

Example 1 (*Weak-Dominance*)**.** All public parking is located in two car parks x and y located at points $x, y \in \mathbb{R}$ with $x \neq y$. Neither car park's capacity can accommodate all residents but the joint capacity is sufficient. Initially, a two-zone scheme is adopted, such that the residents of zone 1 (zone 2) are only allowed to use car park x (car park y), which has the capacity to accommodate them. Later, a single-zone scheme is adopted, i.e., each resident can now use either of the two car parks. Consider a resident i of zone 1 who prefers using car park x to car park y. Under the two-zone scheme he always uses car park x, while under the single-zone scheme he sometimes uses car park y (whenever car park x is full). We would expect resident i to be worse off under the two-zone scheme, that is, if $x\; P_i\; y$, then $\{x\}\; P_i\; \{x, y\}\; P_i\; \{y\}$. In other words, we would expect the property of *weak-dominance* to hold. □

Example 2 (*Weak-Independence*)**.** Two (single-zone parking) schemes, X and Y, are being considered for adoption. Before a final decision is reached, and following a development project on some previously unused land, (parking) space Z now becomes available. Now assume that instead of schemes X and Y, two new schemes are being considered for adoption,

$X \cup Z$ and $Y \cup Z$. Let resident i initially prefer scheme X to scheme Y. Since space Z was unavailable under initial schemes X and Y and is now available under both new schemes, $X \cup Z$ and $Y \cup Z$, we expect resident i to find scheme $X \cup Z$ at least as good as scheme $Y \cup Z$. That is, if $X \cap Z = \emptyset$, $Y \cap Z = \emptyset$, and $X \; P_i \; Y$, then $[X \cup Z] \; R_i \; [Y \cup Z]$. In other words, we would expect the property of *weak-independence* to hold. □

Under the best-worst extension of preferences over sets, when comparing two sets, an agent only considers his best and his worst point(s) in them. Loosely speaking, given two sets $X, Y \in \mathcal{C}$, an agent prefers X to Y if he prefers his best point(s) in X to his best point(s) in Y and his worst point(s) in X to his worst point(s) in Y.

With some abuse of notation, we use the same symbols to denote preferences over points and preferences over sets.

Best-Worst Extension of Preferences over Sets. For each agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and each pair of sets $X, Y \in \mathcal{C}$, we have

$$X \; R_i \; Y \text{ if and only if } \begin{cases} b_X(R_i) \; R_i \; b_Y(R_i) \\ \qquad\quad \text{and} \\ w_X(R_i) \; R_i \; w_Y(R_i) \end{cases}$$

and

$$X \; P_i \; Y \text{ if and only if } X \; R_i \; Y \text{ and } \begin{cases} b_X(R_i) \; P_i \; b_Y(R_i) \\ \qquad\quad \text{or} \\ w_X(R_i) \; P_i \; w_Y(R_i). \end{cases}$$

This extension of preferences is *transitive*, i.e., for each triple $X, Y, Z \in \mathcal{C}$, if $X \; R_i \; Y$ and $Y \; R_i \; Z$, then $X \; R_i \; Z$. However, this extension of preferences does not achieve *completeness* because two sets may not be *comparable*, i.e., there exist sets $X, Y \in \mathcal{C}$ such that neither $X \; R_i \; Y$ nor $Y \; R_i \; X$. To be more precise, we now define the property of *comparability*.

Comparability. Sets $X, Y \in \mathcal{C}$ are *comparable* by agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ if and only if $[b_X(R_i) \; P_i \; b_Y(R_i)$ implies $w_X(R_i) \; R_i \; w_Y(R_i)]$ and $[w_X(R_i) \; P_i \; w_Y(R_i)$ implies $b_X(R_i) \; R_i \; b_Y(R_i)]$.

Regarding the best-worst extension of preferences over sets, we now define *Pareto-efficiency* and *Pareto-indifference*, and then characterize *Pareto-efficient* sets.

Pareto-Efficiency and Pareto-Equivalence (Sets). Let coalition $N \in \mathcal{P}$ and profile $R \in \mathcal{R}^N$. Set $X \in \mathcal{C}$ is *Pareto-efficient* if and only if there is no set $Y \in \mathcal{C}$ such that for each agent $i \in N$, $Y\ R_i\ X$, and for at least one agent $j \in N$, $Y\ P_j\ X$. We denote the set containing all *Pareto-efficient* sets for $R \in \mathcal{R}^N$ by $\text{PE}(R)$. For each pair of *Pareto-efficient* sets $X, Y \in \text{PE}(R)$, if each agent $i \in N$ is indifferent between the two sets, then X and Y are *Pareto-equivalent*.

Proposition 1 (*Pareto-Efficiency*)**.** *For each coalition $N \in \mathcal{P}$ and each profile $R \in \mathcal{R}^N$, a set $X \in \mathcal{C}$ is Pareto-efficient if and only if the following two conditions hold.*

(i) *Set X is a subset of the convex hull of the agents' peaks. That is,*

$$X \subseteq \text{Conv}(R).$$

(ii) *All of the agents' peaks that lie in the convex hull of set X are included in X. That is,*

$$\text{Conv}\,(X) \cap p(R) \subseteq X.$$

We prove Proposition 1 in Appendix A and demonstrate it in Figure 1.

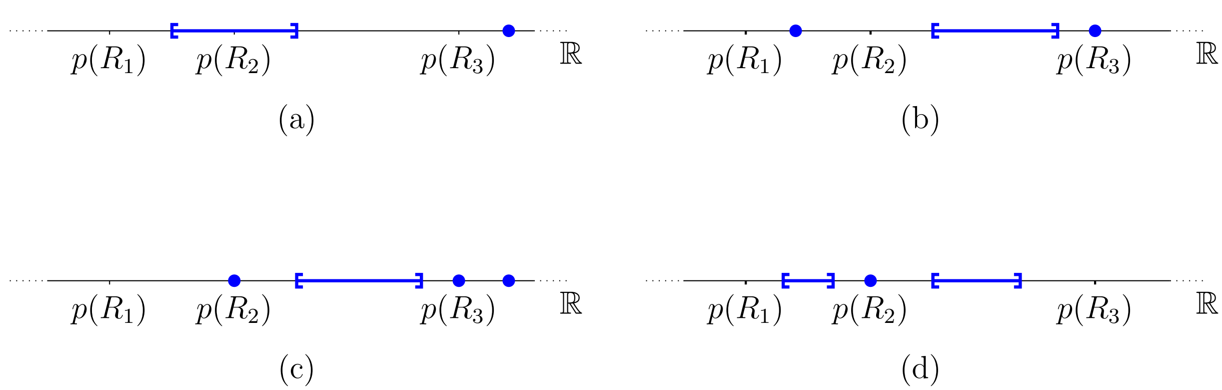

Figure 1: Let $N = \{1, 2, 3\}$ with $R \in \mathcal{R}^N$ and $p(R) = \{p(R_1), p(R_2), p(R_3)\}$. Sets under consideration are shown in blue. The set in Case (a) satisfies neither condition (i) nor condition (ii). The set in Case (b) satisfies condition (i) but it does not satisfy condition (ii). The set in Case (c) does not satisfy condition (i) but it satisfies condition (ii). The set in Case (d) satisfies both conditions (i) and (ii), hence shows a *Pareto-efficient* set.

Remark 1 (Convex *Pareto-Efficient* Sets)**.** For each coalition $N \in \mathcal{P}$, each profile $R \in \mathcal{R}^N$, and each convex set $X = \text{Conv}(X) \in \mathcal{C}$, $X \in \text{PE}(R)$ if and only if $X \subseteq \text{Conv}(R)$.

Further consequences of Proposition 1 are provided by Corollaries 1 and 2. Essentially, Corollary 1 shows that given a coalition M with profile R, if a set $X \in \mathcal{C}$ is *Pareto-efficient*, then it is also *Pareto-efficient* for each coalition $N \subsetneq M$ such that the convex hull of coalition N's peaks at profile R_N, and that of coalition M's peaks at profile R, is the same.

Corollary 1 (*Pareto-Efficiency for Fewer Agents*)**.** *Let coalition* $M \in \mathcal{P}$*, profile* $R \in \mathcal{R}^M$*, and set* $X \in \mathrm{PE}(R)$*. Then, for each coalition* $N \in \mathcal{P}$ *such that* $N \subsetneq M$ *and* $\mathrm{Conv}(R_N) = \mathrm{Conv}(R)$*,* $X \in \mathrm{PE}(R_N)$*.*

***Proof*.** Let coalitions $N, M \in \mathcal{P}$ such that $N \subsetneq M$, profile $R \in \mathcal{R}^M$, and set $X \in \mathrm{PE}(R)$. By Proposition 1 (i), $X \subseteq \mathrm{Conv}(R)$. Since, $\mathrm{Conv}(R) = \mathrm{Conv}(R_N)$, $X \subseteq \mathrm{Conv}(R_N)$. By Proposition 1 (ii), $X \cap p(R) \subseteq X$. Since, $p(R_N) \subsetneq p(R)$, $X \cap p(R_N) \subseteq X$. By Proposition 1, $X \in \mathrm{PE}(R_N)$. □

Corollary 2 shows that if a set $X \in \mathcal{C}$ is *Pareto-efficient*, then the convex hull of X is the unique *maximal Pareto-equivalent* set of X.

Corollary 2 (*Pareto-Equivalent* Sets)**.** *Let coalition* $N \in \mathcal{P}$*, profile* $R \in \mathcal{R}^N$*, and set* $X \in \mathrm{PE}(R)$*. Then, the convex hull of set* X*,* $\mathrm{Conv}(X)$*, is a Pareto-equivalent set of set* X*. Moreover, if a set* Y *is a Pareto-equivalent set of set* X*, then* $\mathrm{Conv}(Y) = \mathrm{Conv}(X)$*.*

We prove Corollary 2 in Appendix A.

To simplify notation, in the sequel we always represent any *Pareto-efficient* set by its convex hull.

3 Choice Correspondences

A *choice correspondence* Φ assigns to each coalition $N \in \mathcal{P}$ and every profile $R \in \mathcal{R}^N$ a set $\Phi(R) \in \mathcal{C}$, i.e., $\Phi\colon \bigcup_{N \in \mathcal{P}} \mathcal{R}^N \to \mathcal{C}$. We denote the set of choice correspondences Φ by Ω.

In the sequel, when the properties of *replacement-domination* and *one-sided replacement-domination* (defined in Section 4) are considered, the coalition of agents does not change. For this reason, we introduce *fixed-coalition choice correspondences*, henceforth *fc-choice correspondences*.

For a fixed coalition $N \in \mathcal{P}$, a *fc-choice correspondence* Φ assigns to every profile $R \in \mathcal{R}^N$ a set $\Phi(R) \in \mathcal{C}$, i.e., $\Phi\colon \mathcal{R}^N \to \mathcal{C}$. For each coalition $N \in \mathcal{P}$, we denote the family of fc-choice correspondences Φ by Ω^N. Hence, a choice correspondence is a set of fc-choice correspondences, one for each coalition $N \in \mathcal{P}$.

Remark 2 (Choice Functions)**.** Given a fixed coalition $N \in \mathcal{P}$, if an fc-choice correspondence assigns to every profile $R \in \mathcal{R}^N$ a set containing a single point, it is essentially an fc-choice function. Similarly, if a choice correspondence assigns to every coalition $N \in \mathcal{P}$ and every profile $R \in \mathcal{R}^N$ a set containing a single point, it is essentially a choice function.

We now define the property of *Pareto-efficiency* for fc-choice correspondences and choice correspondences, a property that is considered in all of our results in the sequel.

Pareto-Efficiency (Choice Correspondences).

(a) Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each profile $R \in \mathcal{R}^N$, $\Phi(R) \in \mathrm{PE}(R)$.

(b) Let choice correspondence $\Phi \in \Omega$. For each coalition $N \in \mathcal{P}$ and each profile $R \in \mathcal{R}^N$, $\Phi(R) \in \mathrm{PE}(R)$.

The following classes of 'target correspondences' and 'fc-target correspondences' play an important role in the sequel.

Any *fc-target point correspondence* is determined by its fixed coalition and its target point. Similarly, any *target point correspondence* is determined by its target point. In both cases: if the target point is *Pareto-efficient*, then it is chosen. If the target point is not *Pareto-efficient*, then the (unique) closest *Pareto-efficient* point to it is chosen.

Target Point Correspondences. Let point $a \in \mathbb{R} \cup \{-\infty, \infty\}$. We define:

(a) for fixed coalition $N \in \mathcal{P}$, the *fc-target point correspondence* with target point a, $\varphi^a \in \Omega^N$, is such that for each profile $R \in \mathcal{R}^N$,

(b) the *target point correspondence* with target point a, $\varphi^a \in \Omega$, is such that for each coalition $N \in \mathcal{P}$ and each profile $R \in \mathcal{R}^N$,

$$\varphi^a(R) = \begin{cases} \{\underline{p}(R)\} & \text{if } a < \underline{p}(R) \\ \{\bar{p}(R)\} & \text{if } a > \bar{p}(R) \\ \{a\} & \text{otherwise.} \end{cases}$$

A (fc-)target point correspondence φ^a is essentially a *(fc-)target point function*.[9]

Any *target set correspondence* is determined by its non-empty, closed, and convex target set. Similarly, any *fc-target set correspondence* is determined by its coalition and its non-empty, closed, and convex target set. In both cases: if the target set is *Pareto-efficient*, then it is chosen. If the target set is not *Pareto-efficient*, then the (unique) maximal *Pareto-efficient* subset of the target set is chosen, if one exists; otherwise, the (unique) closest *Pareto-efficient* point to the target set is chosen.

Target Set Correspondences. Let closed interval $[a,b] \subseteq \mathbb{R} \cup \{-\infty, \infty\}$. We define:

(a) for fixed coalition $N \in \mathcal{P}$, the *fc-target set correspondence* with target set $[a,b]$, $\Phi^{a,b} \in \Omega^N$, is such that for each profile $R \in \mathcal{R}^N$,

(b) the *target set correspondence* with target set $[a,b]$, $\Phi^{a,b} \in \Omega$, is such that for each coalition $N \in \mathcal{P}$ and each profile $R \in \mathcal{R}^N$,

$$\Phi^{a,b}(R) = \begin{cases} \{\underline{p}(R)\} & \text{if } b < \underline{p}(R) \\ \{\bar{p}(R)\} & \text{if } a > \bar{p}(R) \\ [a,b] \cap \mathrm{Conv}(R) & \text{otherwise.} \end{cases}$$

Each target set correspondence is a set of fc-target set correspondences, one for each coalition $N \in \mathcal{P}$, where the target set is constant and independent of the coalition. Also, each (fc-)target set correspondence with a target set $[a,b] \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ such that $a = b$, is a (fc-)target point correspondence.

By Proposition 1, it follows that each (fc-)target set correspondence satisfies *Pareto-efficiency*.

We demonstrate fc-target set correspondences in Figure 2. Since each target set correspondence is a set of fc-target set correspondences a similar example for target set correspondences can be easily obtained if in Figure 2 we allow the coalition of agents to change.

Remark 3 (Fixed Coalition versus Variable Coalition)**.** In the sequel (Section 4), we introduce some properties of fc-choice correspondences. However, since a choice correspondence is a set of fc-choice correspondences, these properties easily extend to choice correspondences.

[9]The difference is that a (fc-)target point correspondence φ^a only assigns singleton sets while the corresponding (fc-)target point function assigns the points in these sets.

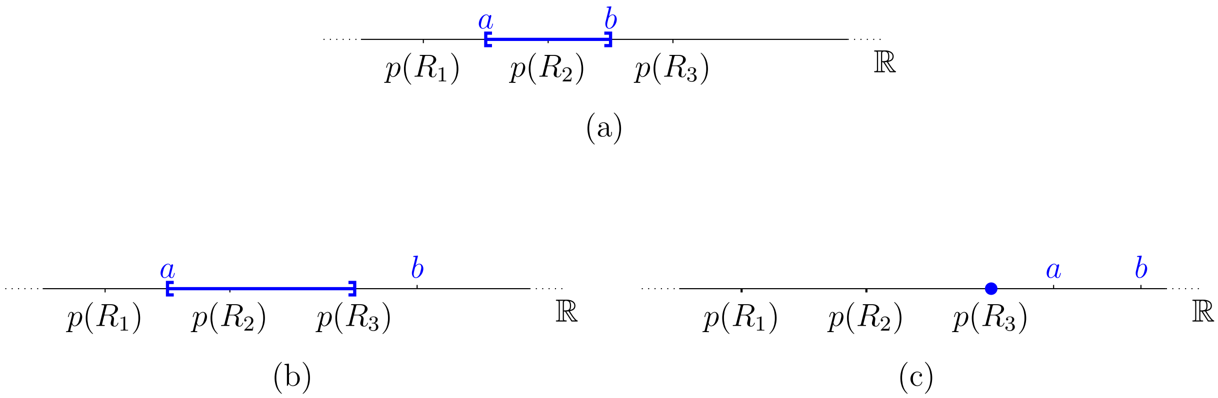

Figure 2: Let $N = \{1, 2, 3\}$ with $R \in \mathcal{R}^N$ and $p(R) = \{p(R_1), p(R_2), p(R_3)\}$. Let $\Phi^{a,b} \in \Omega^N$. Points a, b and the chosen sets in each case are shown in blue. The target set in Case (a) is *Pareto-efficient* and is chosen. The target set in Case (b) is not *Pareto-efficient* but the maximal *Pareto-efficient* subset exists and is chosen. The target set in Case (c) is not *Pareto-efficient* and no maximal *Pareto-efficient* subset exists, hence the closest *Pareto-efficient* point is chosen.

4 Properties of Choice Correspondences

In the sequel, all properties and results presented refer to single-peaked preferences but also apply to symmetric single-peaked preferences, i.e., we could use $\mathcal{S}$ instead of $\mathcal{R}$.

We consider two desirable solidarity properties of choice correspondences. The first solidarity property, introduced by Thomson (1983a,b), expresses the solidarity of agents towards changes in the population. If agents are added to the coalition then agents who were initially present should all be made at least as well off, as they were initially, or they should all be made at most as well off.

Population-Monotonicity. Let choice correspondence $\Phi \in \Omega$. For each pair of coalitions $N, M \in \mathcal{P}$ such that $N \subseteq M$ and each profile $R \in \mathcal{R}^M$ the following holds. For all $i \in N$, $\Phi(R_N)\ R_i\ \Phi(R)$ or for all $i \in N$, $\Phi(R)\ R_i\ \Phi(R_N)$. In particular, for all agents $i \in N$, sets $\Phi(R)$ and $\Phi(R_N)$ are *comparable*.

The next result shows that if a choice correspondence satisfies *Pareto-efficiency* and *population-monotonicity*, then if agents are added to the coalition, all agents who were initially present are at most as well off.

Lemma 1 (*Population-Monotonicity*)**.** *Let choice correspondence $\varPhi \in \Omega$ satisfy* Pareto-efficiency *and* population-monotonicity*. Then, for each pair of coalitions $N, M \in \mathcal{P}$ such that $N \subseteq M$, each profile $R \in \mathcal{R}^M$, and all agents $i \in N$, $\varPhi(R_N)\ R_i\ \varPhi(R)$. In particular, if $\text{Conv}(R_N) = \text{Conv}(R)$, then $\varPhi(R_N) = \varPhi(R)$.*

Proof. Let choice correspondence $\varPhi \in \Omega$ satisfy *Pareto-efficiency* and *population-monotonicity*. Let coalitions $N, M \in \mathcal{P}$ such that $N \subseteq M$. Let profile $R \in \mathcal{R}^M$.

By *Pareto-efficiency*, $\varPhi(R) \in \text{PE}(R)$ and $\varPhi(R_N) \in \text{PE}(R_N)$. By *population-monotonicity*, for all agents $i \in N$, $\varPhi(R)\ R_i\ \varPhi(R_N)$ or for all agents $i \in N$, $\varPhi(R_N)\ R_i\ \varPhi(R)$. If for all agents $i \in N$, $\varPhi(R)\ R_i\ \varPhi(R_N)$, since $\varPhi(R_N) \in \text{PE}(R_N)$, it must be that for all agents $i \in N$, $\varPhi(R_N)\ I_i\ \varPhi(R)$. Therefore, for all agents $i \in N$, $\varPhi(R_N)\ R_i\ \varPhi(R)$.

In particular, if $\text{Conv}(R_N) = \text{Conv}(R)$, then by $\varPhi(R) \in \text{PE}(R)$ and Corollary 1, $\varPhi(R) \in \text{PE}(R_N)$. Since for all agents $i \in N$, $\varPhi(R_N)\ R_i\ \varPhi(R)$, and moreover, [$\varPhi(R) \in \text{PE}(R_N)$ and $\varPhi(R_N) \in \text{PE}(R_N)$], for all agents $i \in N$, $\varPhi(R_N)\ I_i\ \varPhi(R)$. By Corollary 2, $\text{Conv}(\varPhi(R_N)) = \text{Conv}(\varPhi(R))$, and since we always represent any *Pareto-efficient* set by its convex hull, $\varPhi(R_N) = \varPhi(R)$. □

Proposition 2 ($\varPhi^{a,b}$ satisfies *Population-Monotonicity*)**.** *Each target set correspondence satisfies* population-monotonicity*.*

Proof. Let target set correspondence $\varPhi^{a,b} \in \Omega$. Let coalition $N \in \mathcal{P}$ such that $|N| \geq 2$ and profile $R \in \mathcal{R}^N$. We prove *population-monotonicity* of $\varPhi^{a,b}$ by showing that if an agent $j \in N$ leaves the coalition, then all remaining agents are at least as well of, i.e., for all $i \in N \setminus \{j\}$, $\varPhi^{a,b}(R_{-j})\ R_i\ \varPhi^{a,b}(R)$.

Case 1. Let $\text{Conv}(R_{-j}) = \text{Conv}(R)$. Then, the set chosen remains the same, $\varPhi^{a,b}(R_{-j}) = \varPhi^{a,b}(R)$.

Case 2. Let $\text{Conv}(R_{-j}) \neq \text{Conv}(R)$. Then, agent j has either the unique smallest peak at profile R or the unique largest peak at profile R. By symmetry of arguments, assume that agent j has the unique smallest peak at profile R, $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(R_{-j})$.

Let $a, b < \underline{p}(R_{-j})$. Then, $\varPhi^{a,b}(R_{-j}) = \underline{p}(R_{-j})$. Furthermore, if $b \leq p(R_j)$, then $\varPhi^{a,b}(R) = p(R_j)$; if $a \leq p(R_j)$ and $b > p(R_j)$, then $\varPhi^{a,b}(R) = [p(R_j), b]$; and if $a > p(R_j)$, then $\varPhi^{a,b}(R) = [a, b]$. Hence, for all agents $i \in N \setminus \{j\}$, $b_{\varPhi^{a,b}(R_{-j})}(R_i) = w_{\varPhi^{a,b}(R_{-j})}(R_i) = \underline{p}(R_{-j})$,

$b_{\Phi^{a,b}(R)}(R_i) \in \{p(R_j), b\}$, and $w_{\Phi^{a,b}(R)}(R_i) \in \{p(R_j), a\}$. Thus, for all agents $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(R)}(R_i) < b_{\Phi^{a,b}(R_{-j})}(R_i) \leq p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(R_{-j})}(R_i) \leq p(R_i)$. By single-peakedness, for all agents $i \in N\setminus\{j\}$, best and worst points are improved. Hence, $\Phi^{a,b}(R_{-j}) P_i$ $\Phi^{a,b}(R)$.

Let $a < \underline{p}(R_{-j})$ and $b \geq \underline{p}(R_{-j})$. Then, for the minima $\underline{\Phi}^{a,b}(R)$ and $\underline{\Phi}^{a,b}(R_{-j})$ we have $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(R_{-j}) = \underline{p}(R_{-j})$ and for the maxima $\bar{\Phi}^{a,b}(R)$ and $\bar{\Phi}^{a,b}(R_{-j})$ we have $\bar{\Phi}^{a,b}(R) = \bar{\Phi}^{a,b}(R_{-j})$. Thus, for all agents $i \in N \setminus \{j\}$, minimum $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(R_{-j}) \leq p(R_i)$. If maximum $\bar{\Phi}^{a,b}(R_{-j}) < p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(R_{-j})}(R_i) < p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(R_{-j})}(R_i) \leq p(R_i)$. Hence, by single-peakedness, agent i's best point is at least as good and his worst point is improved. If maximum $\bar{\Phi}^{a,b}(R_{-j}) \geq p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(R_{-j})}(R_i) = p(R_i)$ and $w_{\Phi^{a,b}(R_{-j})}(R_i) \in \Phi^{a,b}(R_{-j}) \subseteq \Phi^{a,b}(R)$. Thus, agent i's best and worst points are at least as good. Hence, for all agents $i \in N \setminus \{j\}$, $\Phi^{a,b}(R_{-j})\, R_i\, \Phi^{a,b}(R)$.

Let $a, b \geq \underline{p}(R_{-j})$. Then, the set chosen remains the same, $\Phi^{a,b}(R_{-j}) = \Phi^{a,b}(R)$. □

The second solidarity property, introduced by Moulin (1987), expresses the solidarity of agents towards changes in preferences. Specifically, if the preferences of an agent change, then the other agents whose preferences remained unchanged should all be made at least as well off, as they were initially, or they should all be made at most as well off. We formulate *replacement-domination* for fc-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Replacement-Domination. Let coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each agent $j \in N$, and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ the following holds. For all $i \in N \setminus \{j\}$, $\Phi(R)\, R_i\, \Phi(\bar{R})$ or for all $i \in N \setminus \{j\}$, $\Phi(\bar{R})\, R_i\, \Phi(R)$. In particular, for all agents $i \in N \setminus \{j\}$ sets $\Phi(R)$ and $\Phi(\bar{R})$ are *comparable*.

Note that for a coalition $N \in \mathcal{P}$ such that $|N| \leq 2$, *replacement-domination* imposes no restriction on fc-choice correspondences $\Phi \in \Omega^N$. Hence, for each fixed coalition $N \in \mathcal{P}$ such that $|N| \leq 2$, each fc-target set correspondence $\Phi^{a,b} \in \Omega^N$ satisfies *replacement-domination*.

Proposition 3 (φ^a satisfies *Replacement-Domination*)**.** *If a fixed coalition consists of at least 3 agents, then an associated fc-target set correspondence satisfies* replacement-domination *if and only if it is an fc-target point correspondence.*

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-target set correspondence $\Phi^{a,b} \in \Omega^N$.

First, if $a = b$, we prove *replacement-domination* of φ^a ($\Phi^{a,b}$, $a = b$) by showing that for any profile $R \in \mathcal{R}^N$, if the preferences of an agent $j \in N$ change, such that $\bar{R} \in \mathcal{R}^N$ and $R_{-j} = \bar{R}_{-j}$, then the remaining agents are all at least as well off or are all at most as well off, i.e., [for all $i \in N \setminus \{j\}$, $\varphi^a(R)\, R_i\, \varphi^a(\bar{R})$] or [for all $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, R_i\, \varphi^a(R)$].

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

Case 2.1. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Then, agent j has either the unique smallest peak at profile R or the unique largest peak at profile R. By symmetry of arguments, assume that agent j has the unique smallest peak at profile R, $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) = \bar{p}(\bar{R})$.

Let $a < \underline{p}(\bar{R})$. Then, $\varphi^a(\bar{R}) = \{\underline{p}(\bar{R})\}$. Furthermore, if $a \leq \underline{p}(R)$, then $\varphi^a(R) = \{\underline{p}(R)\}$ and if $a > \underline{p}(R)$, then $\varphi^a(R) = \{a\}$. Hence, for all agents $i \in N \setminus \{j\}$, $\varphi^a(R) < \varphi^a(\bar{R}) \leq p(\bar{R}_i)$. Hence, by single-peakedness, for all agents $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, P_i\, \varphi^a(R)$.

Let $a \geq \underline{p}(\bar{R})$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

Case 2.2. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Then, by Case 2.1 (with the roles of R and $\bar{R}$ reversed), for all agents $i \in N \setminus \{j\}$, $\varphi^a(R)\, R_i\, \varphi^a(\bar{R})$.

Case 3. Let $\text{Conv}(\bar{R}) \not\subseteq \text{Conv}(R)$ and $\text{Conv}(\bar{R}) \not\supseteq \text{Conv}(R)$. Then, agent j has either [the unique smallest peak at profile R and the unique largest peak at profile $\bar{R}$] or [the unique largest peak at profile R and the unique smallest peak at profile $\bar{R}$]. By symmetry of arguments, assume that agent j has the unique smallest peak at profile R and the unique largest peak at profile $\bar{R}$, $p(R_j) = \underline{p}(R)$ and $p(\bar{R}_j) = \bar{p}(\bar{R})$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) < \bar{p}(\bar{R})$.

Let $a < \underline{p}(\bar{R})$. Then, as shown in Case 2.1, for all agents $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, P_i\, \varphi^a(R)$.

Let $\underline{p}(\bar{R}) \leq a \leq \bar{p}(R)$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

Let $a > \bar{p}(R)$. Then, $\varphi^a(R) = \{\bar{p}(R)\}$. Furthermore, if $a \geq \bar{p}(\bar{R})$, then $\varphi^a(\bar{R}) = \{\bar{p}(\bar{R})\}$ and if $a < \bar{p}(\bar{R})$, then $\varphi^a(\bar{R}) = \{a\}$. Hence, for all agents $i \in N \setminus \{j\}$, $p(\bar{R}_i) \leq \varphi^a(R) < \varphi^a(\bar{R})$. Hence, by single-peakedness, for all agents $i \in N \setminus \{j\}$, $\varphi^a(R)\, P_i\, \varphi^a(\bar{R})$.

Second, we prove that if $a < b$, then $\Phi^{a,b}$ does not satisfy *replacement-domination*. Without loss of generality, assume that $1, 2, 3 \in N$.

If $a = -\infty$, then choose a point $\bar{a} \in \mathbb{R}$ such that $\bar{a} < b$, otherwise, set $\bar{a} = a$. If $b = \infty$, then choose a point $\bar{b} \in \mathbb{R}$ such that $\bar{b} > \bar{a}$, otherwise, set $\bar{b} = b$. Hence, $[\bar{a}, \bar{b}] \subseteq [a, b]$. We divide the interval $[\bar{a}, \bar{b}]$ into three equal parts and use the four points $a_1 = \bar{a}$, $a_2 = \left(\bar{a} + \frac{1}{3}(\bar{b} - \bar{a})\right)$, $a_3 = \left(\bar{a} + \frac{2}{3}(\bar{b} - \bar{a})\right)$, and $a_4 = \bar{b}$ to construct symmetric profiles $R, \bar{R} \in \mathcal{S}^N$ such that $p(R_1) = a_1$, $p(R_2) = p(\bar{R}_2) = a_2$, $p(R_3) = p(\bar{R}_3) = a_3$, $p(\bar{R}_1) = a_4$, and for all $i \in N \backslash \{1, 2, 3\}$, $p(R_i) = p(\bar{R}_i) = a_2$. Note that $R_{-1} = \bar{R}_{-1}$.

By the definition of $\Phi^{a,b}$, we have $\Phi^{a,b}(R) = [a_1, a_3]$ and $\Phi^{a,b}(\bar{R}) = [a_2, a_4]$. Under both profiles R and $\bar{R}$, the best points of agents 2 and 3 remain the same, $b_{\Phi^{a,b}(R)}(R_2) = b_{\Phi^{a,b}(\bar{R})}(R_2) = p(R_2)$ and $b_{\Phi^{a,b}(R)}(R_3) = b_{\Phi^{a,b}(\bar{R})}(R_3) = p(R_3)$. However, the worst points of agent 2 and 3 change as follows. For agent 2, $w_{\Phi^{a,b}(R)}(R_2) = \{a_1, a_3\}$ and $w_{\Phi^{a,b}(\bar{R})}(R_1) = \{a_4\}$. Since $p(R_2) = a_2 < a_3 < a_4$, single-peakedness implies $\Phi^{a,b}(R) \; P_2 \; \Phi^{a,b}(\bar{R})$. For agent 3, $w_{\Phi^{a,b}(R)}(R_3) = \{a_1\}$ and $w_{\Phi^{a,b}(\bar{R})}(R_3) = \{a_2, a_4\}$. Since $a_1 < a_2 < a_3 = p(R_3)$, single-peakedness implies $\Phi^{a,b}(\bar{R}) \; P_3 \; \Phi^{a,b}(R)$. This contradicts *replacement-domination*. □

We next introduce a weaker *replacement-domination* property, namely *one-sided replacement-domination*, which does not require solidarity when the preferences of the agent with the unique smallest peak are changed such that he becomes the agent with the unique largest peak, or vice-versa. We formulate *one-sided replacement-domination* for fc-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

One-Sided Replacement-Domination. Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\mathrm{Conv}(R) \subseteq \mathrm{Conv}(\bar{R})$ or $\mathrm{Conv}(R) \supseteq \mathrm{Conv}(\bar{R})$ the following holds. For all $i \in N \setminus \{j\}$, $\Phi(R) \; R_i \; \Phi(\bar{R})$ or for all $i \in N \setminus \{j\}$, $\Phi(\bar{R}) \; R_i \; \Phi(R)$. In particular, for all agents $i \in N \setminus \{j\}$ sets $\Phi(R)$ and $\Phi(\bar{R})$ are *comparable*.

Replacement-domination implies *one-sided replacement-domination*.

The next result shows that given a fixed coalition of at least three agents and an associated fc-choice correspondence satisfying *Pareto-efficiency* and *one-sided replacement-domination*, if the preferences of an agent change such that the new set of peaks is a subset of the initial one, all other agents are at least as well off.

Lemma 2 (*One-Sided Replacement-Domination*)**.** *Let fixed coalition* $N \in \mathcal{P}$ *such that* $|N| \geq 3$ *and fc-choice correspondence* $\Phi \in \Omega^N$ *satisfy Pareto-efficiency and one-sided replacement-domination. Then, for each agent* $j \in N$*, each pair of profiles* $R, \bar{R} \in \mathcal{R}^N$ *such that* $[R_{-j} = \bar{R}_{-j}$ *and* $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)]$*, and all agents* $i \in N \setminus \{j\}$*,* $\Phi(\bar{R})\ R_i\ \Phi(R)$*. In particular, if* $\text{Conv}(\bar{R}) = \text{Conv}(R)$*, then* $\Phi(\bar{R}) = \Phi(R)$*.*

We prove Lemma 2 in Appendix B.

Recall that for a fixed coalition $N \in \mathcal{P}$ such that $|N| \leq 2$, *(one-sided) replacement-domination* imposes no restriction on fc-choice correspondences $\Phi \in \Omega^N$. The following example shows why Lemma 2 does not hold for a fixed coalition $N \in \mathcal{P}$ such that $|N| = 2$.

Example 3. Let fixed coalition $N \in \mathcal{P}$ such that $N = \{1, 2\}$ and fc-choice correspondence $\Phi \in \Omega^N$ such that

$$\Phi(R) = \begin{cases} p(R_2) & \text{if } p(R_2) = 1 \\ p(R_1) & \text{otherwise.} \end{cases}$$

Hence, Φ satisfies *Pareto-efficiency*, and since $|N| = 2$, it trivially satisfies *(one-sided) replacement-domination*. Let $R, \bar{R} \in \mathcal{R}^N$ such that $p(R_1) = p(\bar{R}_1) = 0$, $p(R_2) = 2$, and $p(\bar{R}_2) = 1$. Hence, $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. It follows, that $\Phi(R) = 0$ and $\Phi(\bar{R}) = 1$. Hence, agent 1's peak $p(R_1) = \Phi(R) < \Phi(\bar{R})$. By single-peakedness, $\Phi(R)\ P_1\ \Phi(\bar{R})$. □

Proposition 4 ($\Phi^{a,b}$ satisfies *One-Sided Replacement-Domination*)**.** *Each fc-target set correspondence satisfies one-sided replacement-domination.*

Proof. Let fixed coalition $N \in \mathcal{P}$ and fc-target set correspondence $\Phi^{a,b} \in \Omega^N$. Since for $|N| \leq 2$, (*one-sided*) *replacement-domination* imposes no restriction on fc-choice correspondence $\Phi^{a,b}$, let $|N| \geq 3$.

We prove that $\Phi^{a,b}$ satisfies *one-sided replacement-domination*, i.e., we show that for any profile $R \in \mathcal{R}^N$, if the preferences of an agent $j \in N$ change, such that $\bar{R} \in \mathcal{R}^N$, $R_{-j} = \bar{R}_{-j}$, and $\text{Conv}(R) \subseteq \text{Conv}(\bar{R})$ or $\text{Conv}(R) \supseteq \text{Conv}(\bar{R})$, then the other agents are all at least as well off, or are all at most as well off, i.e., [for all $i \in N \setminus \{j\}$, $\Phi^{a,b}(R)\ R_i\ \Phi^{a,b}(\bar{R})$] or [for all $i \in N \setminus \{j\}$, $\Phi^{a,b}(\bar{R})\ R_i\ \Phi^{a,b}(R)$].

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Then, the set chosen remains the same, $\Phi^{a,b}(\bar{R}) = \Phi^{a,b}(R)$.

Case 2.1. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Then, agent j has either the unique smallest peak at profile R or the unique largest peak at profile R. By symmetry of arguments, assume that agent j has the unique smallest peak at profile R, $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) = \bar{p}(\bar{R})$.

Let $a, b < \underline{p}(\bar{R})$. Then $\Phi^{a,b}(\bar{R}) = \underline{p}(\bar{R})$. Furthermore, if $a, b \leq \underline{p}(R)$, then $\Phi^{a,b}(R) = \underline{p}(R)$; if $a \leq \underline{p}(R)$ and $b > \underline{p}(R)$, then $\Phi^{a,b}(R) = [\underline{p}(R), b]$; and if $a, b > \underline{p}(R)$, then $\Phi^{a,b}(R) = [a, b]$. Hence, for all agents $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(\bar{R})}(R_i) = w_{\Phi^{a,b}(\bar{R})}(R_i) = \{\underline{p}(\bar{R})\}$, $b_{\Phi^{a,b}(R)}(R_i) \in \{\underline{p}(R), b\}$, and $w_{\Phi^{a,b}(R)}(R_i) \in \{\underline{p}(R), a\}$. Thus, for all agents $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(R)}(R_i) < b_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$. By single-peakedness, for all agents $i \in N \setminus \{j\}$, best and worst points improve. Hence, $\Phi^{a,b}(\bar{R}) \; P_i \; \Phi^{a,b}(R)$.

Let $a < \underline{p}(\bar{R})$ and $b \geq \underline{p}(\bar{R})$. Then, for the minima $\underline{\Phi}^{a,b}(R)$ and $\underline{\Phi}^{a,b}(\bar{R})$ we have $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(\bar{R}) = \underline{p}(\bar{R})$ and for the maxima $\bar{\Phi}(R)$ and $\bar{\Phi}(\bar{R})$ we have $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$. Thus, for all agents $i \in N \setminus \{j\}$, minimum $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(\bar{R}) \leq p(R_i)$. If maximum $\bar{\Phi}^{a,b}(\bar{R}) < p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(\bar{R})}(R_i) < p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$. Hence, by single-peakedness, agent i's best point is at least as good and his worst point improves. If maximum $\bar{\Phi}^{a,b}(\bar{R}) \geq p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(\bar{R})}(R_i) = p(R_i)$ and $w_{\Phi^{a,b}(\bar{R})}(R_i) \in \Phi^{a,b}(\bar{R}) \subseteq \Phi^{a,b}(R)$. Thus, agent i's best and worst points are at least as good. It follows, that for all agents $i \in N \setminus \{j\}$, $\Phi^{a,b}(\bar{R}) \; R_i \; \Phi^{a,b}(R)$.

Let $a, b \geq \underline{p}(\bar{R})$. Then, the set chosen remains the same, $\Phi^{a,b}(\bar{R}) = \Phi^{a,b}(R)$.

Case 2.2. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Then, by Case 2.1 (with the roles of R and $\bar{R}$ reversed), for all agents $i \in N \setminus \{j\}$, $\Phi^{a,b}(R) \; R_i \; \Phi^{a,b}(\bar{R})$. □

Proposition 5 (*Population-Monotonicity* Implies *One-Sided Replacement-Domination*)**.** *Each choice correspondence satisfying Pareto-efficiency and population-monotonicity also satisfies one-sided replacement-domination.*

We prove Proposition 5 in Appendix C.

5 Characterizing Target Set Correspondences

In the sequel, all results presented refer to single-peaked preferences but also apply to symmetric single-peaked preferences, i.e., we could use $\mathcal{S}$ instead of $\mathcal{R}$.

Theorem 1 (*One-Sided Replacement-Domination*: Characterization of $\Phi^{a,b}$)**.** *If a fixed coalition consists of at least 3 agents, then an associated fc-choice correspondence satisfies Pareto-efficiency and one-sided replacement-domination if and only if it is an fc-target choice correspondence.*

We prove Theorem 1 in Appendix D.

Corollary 3 that follows, is equivalent to a result by Thomson (1993).

Corollary 3 (*Replacement-Domination*: Characterization of φ^a)**.** *If a fixed coalition consists of at least 3 agents, then an associated fc-choice correspondence satisfies Pareto-efficiency and replacement-domination if and only if it is an fc-target point correspondence.*

Proof. *If part.* By Propositions 1 and 3, all fc-target point correspondences satisfy *Pareto-efficiency* and *replacement-domination*.

Only if part. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *replacement-domination*. Then, Φ satisfies *one-sided replacement-domination* and by Theorem 1 it is an fc-target set correspondence $\Phi^{a,b} \in \Omega^N$. By Proposition 3, $\Phi^{a,b}$ satisfies *replacement-domination* if and only if it is an fc-target point correspondence $\varphi^a \in \Omega^N$. □

We have formulated Theorem 1 and Corollary 3 for fc-choice correspondences where the fixed coalition of agents contains at least 3 agents. If instead we consider choice correspondences, then *Pareto-efficiency* and *one-sided replacement-domination* (*replacement-domination*) imply that for each coalition with at least 3 agents, a different target set or target point can be chosen, while for each coalition with at most 2 agents, the choice correspondence can equal any *Pareto-efficient* fc-choice correspondence.

Theorem 2 (*Population-Monotonicity*: Characterization of $\Phi^{a,b}$)**.** *A choice correspondence satisfies Pareto-efficiency and population-monotonicity if and only if it is a target choice correspondence.*

Proof. *If part.* By Propositions 1 and 2, all target set correspondences satisfy *Pareto-efficiency* and *population-monotonicity*.

Only if part. Let choice correspondence $\Phi \in \Omega$ satisfy *Pareto-efficiency* and *population-monotonicity*. By Proposition 5, Φ satisfies *one-sided replacement-domination*. Let coalition $M \in \mathcal{P}$ such that $|M| \geq 3$. By Theorem 1, for each profile $R \in \mathcal{R}^M$, $\Phi = \Phi^{a_M,b_M} \in \Omega^M$. Define points $a := a_M$ and $b := b_M$.

We show that for each coalition $N \in \mathcal{P}$ and each profile $\bar{R} \in \mathcal{R}^N$, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$. We do so by showing that for each coalition $N \in \mathcal{P}$, each profile $\bar{R} \in \mathcal{R}^N$, and each profile $R \in \mathcal{R}^M$, if $\text{Conv}(\bar{R}) = \text{Conv}(R)$, then $\Phi(\bar{R}) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$ (the latter equality follows by the definition of $\Phi^{a,b}$).

Let profiles $R \in \mathcal{R}^M$ and $\bar{R} \in \mathcal{R}^N$. Recall that $\Phi(R) = \Phi^{a,b}(R)$. Begin from profile $R \in \mathcal{R}^M$ and construct profile $R^1 \in \mathcal{R}^{M \cup N}$ by adding the set of agents $N \setminus M$ with profile $\bar{R}_{N \setminus M}$, i.e., $R^1 = (R, \bar{R}_{N \setminus M})$. Since $\text{Conv}(R^1) = \text{Conv}(R)$, by *population-monotonicity* and Lemma 1, $\Phi(R^1) = \Phi(R)$. Next, change the preferences of each agent $i \in N$ to $\bar{R}_i$ and denote the new profile $R^2 = (R^1_{M \setminus N}, \bar{R}) \in \mathcal{R}^{M \cup N}$. Since $\text{Conv}(R^2) = \text{Conv}(R^1)$, by *population-monotonicity* and Lemma 1, $\Phi(R^2) = \Phi(R^1)$. Finally, remove the set of agents $M \setminus N$ and notice that the new profile $R^2_N = \bar{R} \in \mathcal{R}^N$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R^2)$. Hence, $\Phi(\bar{R}) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$. □

Remark 4 (Independence of Properties)**.** Note that the properties in all our characterization results are independent. A constant choice correspondence that always chooses a fixed set $X \in \mathcal{C}$ satisfies *(one-sided) replacement-domination* and *population-monotonicity* but violates *Pareto-efficiency*. A choice correspondence that always chooses the peak of the agent with the lowest index satisfies *Pareto-efficiency*, but it violates *one-sided replacement-domination* and *population-monotonicity*.

Appendices

Throughout the Appendices we use the domain of single-peaked preferences $\mathcal{R}$, with the exception of Lemma 8 (Appendix D), where we use the domain of symmetric preferences $\mathcal{S}$. All results proven for $\mathcal{R}$ also hold on $\mathcal{S}$; however, for Lemma 8, the proof for $\mathcal{S}$ requires a different approach (and additional 'proof steps') that also holds on $\mathcal{R}$.

A Proofs of Proposition 1 and Corollaries 1 and 2

The following terms describe a set obtained by a truncation of a given set $X \in \mathcal{C}$ on one side at a specific point x, which is added to the new set to ensure that this new set is closed.

Left Truncaddition (of a Set at a Point). Let point $x \in \mathbb{R}$ and set $X \in \mathcal{C}$. Then, set $Y \in \mathcal{C}$ is a *left truncaddition* of X at x if $Y = [X \cap (x, \infty)] \cup \{x\}$.

Right Truncaddition (of a Set at a Point). Let point $x \in \mathbb{R}$ and set $X \in \mathcal{C}$. Then, set $Y \in \mathcal{C}$ is a *right truncaddition* of X at x if $Y = [X \cap (-\infty, x)] \cup \{x\}$.

Before proceeding with the proof of Proposition 1 we present two lemmas. First, we describe some cases where a truncaddition of a set at a point makes an agent weakly better off.

Lemma 3. *Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $X \in \mathcal{C}$.*

(i) *Let minimum $\underline{X} < p(R_i)$, point $\underline{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i)$, and set $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$ be a left truncaddition of set X at point $\underline{x}$. Then, $Y\, R_i\, X$. Moreover, if the unique worst point $w_X(R_i) = \underline{X}$, then $Y\, P_i\, X$.*

(ii) *Let maximum $\bar{X} > p(R_i)$, point $\bar{x} \in \mathbb{R}$ such that $\bar{X} > \bar{x} \geq p(R_i)$, and set $Y = [X \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$ be a right truncaddition of set X at point $\bar{x}$. Then, $Y\, R_i\, X$. Moreover, if the unique worst point $w_X(R_i) = \bar{X}$,then $Y\, P_i\, X$.*

(iii) *Let minimum $\underline{X} < p(R_i)$, maximum $\bar{X} > p(R_i)$, and points $\underline{x}, \bar{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i) \leq \bar{x} < \bar{X}$, set $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$ be a left truncaddition of set X at point $\underline{x}$, and set $Z = [Y \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$ be a right truncaddition of set Y at point $\bar{x}$. Then, $Z\, P_i\, X$.*

Proof. Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $X \in \mathcal{C}$.

(i) Let minimum $\underline{X} < p(R_i)$, point $\underline{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i)$, truncaddition $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$, and Z be the set of truncated points, $Z = X \setminus Y$. By single-peakedness, for all points $z \in Z$, agent i prefers $\underline{x}$ to z, $\underline{x}\, P_i\, z$. Hence, his best and worst points in Y are at least as good as his (respective) best and worst points in X. It follows, that $Y\, R_i\, X$. If additionally his worst point $w_X(R_i) = \underline{X} \notin Y$ is unique, then $\bar{X}\, P_i\, w_X(R_i)$ and $w_Y(R_i) = \{\underline{x}, \bar{X}\}$. Since by single-peakedness $\underline{x}\, P_i\, w_X(R_i)$, his worst point improves. It follows that $Y\, P_i\, X$.

(ii) Symmetric proof to (i).

(iii) Let minimum $\underline{X} < p(R_i)$, maximum $\bar{X} > p(R_i)$, points $\underline{x}, \bar{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i) \leq \bar{x} < \bar{X}$, truncaddition $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$, and truncaddition $Z = [Y \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$. By part (i), $Y\, R_i\, X$. By part (ii), $Z\, R_i\, Y$. Hence, by transitivity, $Z\, R_i\, X$. Moreover, by single-peakedness, his worst point(s) $w_X(R_i) \subseteq \{\underline{X}, \bar{X}\}$ and $w_Z(R_i) \subseteq \{\underline{x}, \bar{x}\}$. Since by single-peakedness $\underline{x}\ P_i\ w_X(R_i)$ and $\bar{x}\ P_i\ w_X(R_i)$, his worst point(s) improves. It follows that $Z\ P_i\ X$. □

Second, adding a closed interval to a set, without changing its convex hull, makes an agent indifferent, unless his best point improves, in which case he is better off. Furthermore, removing an open interval from a set, without changing its convex hull, makes an agent indifferent, unless his best point worsens, in which case he is worse off.

Lemma 4. *Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $X \in \mathcal{C}$.*

(i) *Let closed interval $[x, y] \subseteq \text{Conv}(X)$ and set $Y = X \cup [x, y]$. Then, $Y\, I_i\, X$ unless agent i's best point(s) improves, i.e., $b_Y(R_i)\ P_i\ b_X(R_i)$, in which case, $Y\ P_i\ X$.*

(ii) *Let open interval $(x, y) \subsetneq \text{Conv}(X)$ and set $Y = X \setminus (x, y)$. Then, $X\, I_i\, Y$ unless agent i's best point(s) worsens, i.e., $b_X(R_i)\ P_i\ b_Y(R_i)$, in which case, $X\ P_i\ Y$.*

***Proof*.** Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $Y \in \mathcal{C}$.

(i) Let $[x, y] \subseteq \text{Conv}(X)$ and $Y = X \cup [x, y]$. By single-peakedness, agent i's worst point(s) does not change, $w_X(R_i) = w_Y(R_i) \subseteq \{\underline{X}, \bar{X}\}$. If for his best point(s) we have $b_X(R_i)\ I_i\ b_Y(R_i)$, then $b_X(R_i) \subseteq b_Y(R_i)$ and $Y\ I_i\ X$. Otherwise, $b_X(R_i) \nsubseteq b_Y(R_i)$, his best point(s) improves, $b_Y(R_i)\ P_i\ b_X(R_i)$, and $Y\ P_i\ X$.

(ii) Let $(x, y) \subsetneq \text{Conv}(X)$ and $Y = X \setminus (x, y)$. By single-peakedness, agent i's worst point(s) does not change, $w_X(R_i) = w_Y(R_i) \subseteq \{\underline{X}, \bar{X}\}$. If for his best point(s) we have $b_X(R_i)\ I_i\ b_Y(R_i)$, then $b_X(R_i) \supseteq b_Y(R_i)$ and $Y\ I_i\ X$. Otherwise, $b_X(R_i) \nsupseteq b_Y(R_i)$, his best point(s) worsens, $b_X(R_i)\ P_i\ b_Y(R_i)$, and $X\ P_i\ Y$. □

***Proof of Proposition 1*.** Let coalition $N \in \mathcal{P}$, profile $R \in \mathcal{R}^N$, and set $X \in \mathcal{C}$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq \ldots \leq p(R_n) = \bar{p}(R)$.

The proof follows in three steps.

Step 1. We show that if set $X \in \mathrm{PE}(R)$ then condition (i) holds, that is, $X \subseteq \mathrm{Conv}(R)$.

Let set $X \in \mathrm{PE}(R)$. Assume by contradiction that $X \nsubseteq \mathrm{Conv}(R)$. Then, minimum $\underline{X} < p(R_1)$ or maximum $\bar{X} > p(R_n)$. By symmetry of arguments, assume that $\underline{X} < p(R_1)$.

Case 1. Let maximum $\bar{X} > p(R_n)$. Then, for each $i \in N$, minimum $\underline{X} < p(R_1) \leq p(R_i) \leq p(R_n) < \bar{X}$. Let $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$ be a left truncaddition of X at $p(R_1)$, and $Z = [Y \cap (-\infty, p(R_n))] \cup \{p(R_n)\}$ be a right truncaddition of Y at $p(R_n)$. Therefore, by Lemma 3 (iii), for each $i \in N$, $Z\ P_i\ X$. Hence, $X \notin \mathrm{PE}(R)$; a contradiction.

Case 2. Let maximum $\bar{X} \leq p(R_n)$. Then, for each $i \in N$, minimum $\underline{X} < p(R_1) \leq p(R_i)$. Let $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$ be a left truncaddition of X at $p(R_1)$. By Lemma 3 (i), for each $i \in N$, $Y\ R_i\ X$. Furthermore, agent n's worst point $w_X(R_n) = \underline{X}$ is unique. Therefore, by Lemma 3 (i), $Y\ P_n\ X$. Hence, $X \notin \mathrm{PE}(R)$; a contradiction.

Step 2. We show that if set $X \in \mathrm{PE}(R)$ then condition (ii) holds, that is, $(\mathrm{Conv}(X) \cap p(R)) \subseteq X$.

Let set $X \in \mathrm{PE}(R)$. By Step 1, $X \subseteq \mathrm{Conv}(R)$. Assume by contradiction that $(\mathrm{Conv}(X) \cap p(R)) \nsubseteq X$. Then, there exists agent $j \in N$ such that $p(R_j) \in \mathrm{Conv}(X)$ and $p(R_j) \notin X$.

Let set $Y = X \cup \{p(R_j)\}$. By Lemma 4 (i), for each $i \in N$, $Y\ R_i\ X$. Furthermore, agent j's best point $b_Y(R_j) = p(R_j)\ P_j\ b_X(R_j)$. Therefore, by Lemma 4 (i), $Y\ P_j\ X$. Hence, $X \notin \mathrm{PE}(R)$; a contradiction.

Step 3. We show that if conditions (i) and (ii) hold for set $X \in \mathcal{C}$, then $X \in \mathrm{PE}(R)$.

Let set $X \in \mathcal{C}$ such that $X \subseteq \mathrm{Conv}(R)$ and $(\mathrm{Conv}(X) \cap p(R)) \subseteq X$. Assume by contradiction that $X \notin \mathrm{PE}(R)$. Hence, there exists a set $Y \subseteq \mathbb{R}$ that *Pareto-dominates* set X, i.e., for all agents $i \in N$, $Y\ R_i\ X$, and for at least one agent $j \in N$, $Y\ P_j\ X$.

Case 1. Let agent j's peak $p(R_j) \in \mathrm{Conv}(X)$. By condition (ii), $p(R_j) \in X$. Agent j's best point $b_X(R_j) = p(R_j) \in X$ cannot be improved. By single-peakedness, agent j's worst point(s) $w_X(R_j) \subseteq \{\underline{X}, \bar{X}\}$; if his worst point(s) $w_Y(R_j)\ P_j\ w_X(R_j)$, by single-peakedness, minimum $\underline{X} < \underline{Y}$ or maximum $\bar{X} > \bar{Y}$. By symmetry of arguments, assume minimum $\underline{X} < \underline{Y}$. Consider agent 1; by condition (i), his peak $p(R_1) \leq \underline{X} < \underline{Y}$. By single-peakedness, his best point $b_X(R_1)\ P_1\ b_Y(R_1)$. It follows that for agent 1 set Y is not at least as good as set X. Hence, set Y does not *Pareto-dominate* set X; a contradiction.

Case 2. Let agent j's peak $p(R_j) \notin \text{Conv}(X)$. Then, either $p(R_j) < \underline{X}$ or $p(R_j) > \bar{X}$. By symmetry of arguments, assume that $p(R_j) > \bar{X}$. By single-peakedness, agent j's best point $b_X(R_j) = \bar{X}$ and agent j's worst point $w_X(R_j) = \underline{X}$. If his best point(s) $b_Y(R_j)\ P_j\ b_X(R_j)$, by single-peakedness, maximum $\bar{X} < \bar{Y}$. If his worst point(s) $w_Y(R_j)\ P_j\ w_X(R_j)$, by single-peakedness, minimum $\underline{X} < \underline{Y}$. Consider now agent 1. By condition (i), his peak $p(R_1) \leq \underline{X} \leq \bar{X}$. By single-peakedness, his best and worst point(s) are $b_X(R_1) = \underline{X}$ and $w_X(R_1) = \bar{X}$. If minimum $\underline{X} < \underline{Y}$, by single-peakedness, $b_X(R_1)\, P_1\, b_Y(R_1)$. If maximum $\bar{X} < \bar{Y}$, by single-peakedness, $w_X(R_1)\ P_1\ w_Y(R_1)$. It follows that for agent 1 set Y is not at least as good as set X. Hence, set Y does not *Pareto-dominate* set X; a contradiction. □

Proof of Corollary 2. Let coalition $N \in \mathcal{P}$, profile $R \in \mathcal{R}^N$, and set $X \in \text{PE}(R)$.

First, we show that $\text{Conv}(X)$ and X are *Pareto-equivalent* sets. By single-peakedness, for each agent $i \in N$ such that $p(R_i) \in \text{Conv}(X)$, the best point $b_{\text{Conv}(X)}(R_i) = p(R_i)$ and by Proposition 1 (ii), $(\text{Conv}(X) \cap p(R)) \subseteq X$. Hence, the best point $b_{\text{Conv}(X)}(R_i) = b_X(R_i)$. By single-peakedness, for each agent $i \in N$ such that $p(R_i) \notin \text{Conv}(X)$, the best point $b_{\text{Conv}(X)}(R_i) \in \{\underline{X}, \bar{X}\}$. Since $\{\underline{X}, \bar{X}\} \subseteq X$, the best point $b_{\text{Conv}(X)}(R_i) = b_X(R_i)$. Moreover, since $\text{Conv}(X)$ is a closed interval and (trivially) $\text{Conv}(X) = X \cup \text{Conv}(X)$, by Lemma 4 (i), for each agent $i \in N$, $\text{Conv}(X)\ I_i\ X$.

Second, we show that if X and Y are *Pareto-equivalent* sets, then $\text{Conv}(X) = \text{Conv}(Y)$. Let $Y \in \mathcal{C}$ be a Pareto-equivalent set to $X \in \text{PE}(R)$. Let agent $1 \in N$ have the smallest peak at profile R, $p(R_1) = \underline{p}(R)$. By Proposition 1 (i), $X, Y \subseteq \text{Conv}(R)$, hence, $p(R_1) \leq \underline{X} \leq \bar{X}$ and $p(R_1) \leq \underline{Y} \leq \bar{Y}$. By single-peakedness, for agent 1, [best points are $b_X(R_1) = \underline{X}$ and $b_Y(R_1) = \underline{Y}$] and [worst points are $w_X(R_1) = \bar{X}$ and $w_Y(R_1) = \bar{Y}$]. Since $X\ I_1\ Y$, $b_X(R_1) = b_Y(R_1)$ and $w_X(R_1) = w_Y(R_1)$. Therefore, $\text{Conv}(X) = \text{Conv}(Y)$. □

B Proof of Lemma 2

Before proceeding with the proof of Lemma 2, we first prove an implication of *Pareto-efficiency* and *(one-sided) replacement-domination*.

An fc-choice correspondence satisfies *extreme peaks-onliness* if the set chosen only depends on the convex hull of the peaks of the profile. We formulate *extreme peaks-onliness* for fc-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Extreme Peaks-Onliness. Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each pair of profiles $R, \bar{R} \in \mathcal{R}^N$, if $\text{Conv}(R) = \text{Conv}(\bar{R})$, then $\Phi(R) = \Phi(\bar{R})$.

Notice that *extreme peaks-onliness* not only implies the properties of *anonymity*[10] and *peaks-onliness*,[11] but since it only depends on the extreme agents' peaks, it is a much stronger property.

Lemma 5 (*Extreme Peaks-Onliness*)**.** *If a fixed coalition consists of at least 3 agents, then each associated fc-choice correspondence satisfying* Pareto-efficiency *and* one-sided replacement-domination *also satisfies* extreme peaks-onliness*.*

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. Let the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(R) = \text{Conv}(\bar{R})$. Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

We prove that $\Phi(R) = \Phi(\bar{R})$ in three steps.

Step 1. We show that if the preferences of one agent change and the convex hull of the peaks does not change, the chosen set does not change.

Case 1.1. The preferences of a middle agent at profile R change such that the convex hull of the peaks does not change. Let agent $k \in N$ be a middle agent at profile R and let profile $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}_{-k} = R_{-k}$, and $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Notice that agent k is also a middle agent at profile $\bar{R}$.[12]

By *Pareto-efficiency*, $\Phi(\bar{R}) \in PE(\bar{R})$ and $\Phi(R) \in PE(R)$. Since agent k is a middle agent at both profiles R and $\bar{R}$, $\text{Conv}(\bar{R}) = \text{Conv}(R) = \text{Conv}(R_{-k})$, and by Corollary 1, $\Phi(\bar{R}), \Phi(R) \in \text{PE}(R_{-k})$. Since $\bar{R}_{-k} = R_{-k}$, by *one-sided replacement-domination*, for all agents $i \in N \setminus \{k\}$, $\Phi(\bar{R}) \; R_i \; \Phi(R)$ or for all agents $i \in N \setminus \{k\}$, $\Phi(R) \; R_i \; \Phi(\bar{R})$. By *Pareto-efficiency* of both sets

[10] *Anonymity*: the identities of the agents do not affect the set chosen.

[11] *Peaks-Onliness*: only the peaks of the agents affect the set chosen.

[12] Note that if agent 1 (agent n) does not have the unique smallest (largest) peak, then he is a middle agent.

$\Phi(R)$ and $\Phi(\bar{R})$ at profile R_{-k}, for all agents $i \in N \setminus \{k\}$, $\Phi(R)\ I_i\ \Phi(\bar{R})$. By Corollary 2, $\text{Conv}(\Phi(\bar{R})) = \text{Conv}(\Phi(R))$ and since we always represent any *Pareto-efficient* set by its convex hull, $\Phi(\bar{R}) = \Phi(R)$.

Case 1.2. Either the preferences of the agent with the unique smallest peak at profiles R and $\bar{R}$ change (agent 1), or the preferences of the agent with the unique largest peak at profiles R and $\bar{R}$ change (agent n), such that the convex hull of the peaks does not change. By symmetry of arguments, assume that profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$ and $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Hence, $p(\bar{R}_1) = p(R_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing middle agent 2's preferences to $R^1_2 = R_1$, i.e., $R^1 = (R_{-2}, R^1_2)$ where $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change middle agent 1's preferences to $R^2_1 = \bar{R}_1$ such that the new profile is $R^2 = (R^1_{-1}, R^2_1)$ where $\text{Conv}(R^2) = \text{Conv}(R^1)$. By Case 1.1, $\Phi(R^2) = \Phi(R^1)$. Finally, change middle agent 2's preferences back to R_2 and notice that the new profile $(R^2_{-2}, R_2) = \bar{R}$ where $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$. By Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Step 2. We show that if two agents swap preferences, then the chosen set does not change.

Case 2.1. At least one of the swapping agents is a middle agent at profile R. Assume profile $\bar{R}$ is obtained from profile R by agents $j, k \in N$ swapping preferences, i.e., $\bar{R}_{-j,k} = R_{-j,k}$, $\bar{R}_j = R_k$, and $\bar{R}_k = R_j$. Let agent $k \in N$ be a middle agent at profile R. Begin from profile R and construct profile R^1 by changing agent k's preferences to $R^1_k = R_j$, i.e., $R^1 = (R_{-k}, R^1_k)$ where $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1.1, $\Phi(R^1) = \Phi(R)$. Finally, change agent j's preferences to $R^2_j = R_k$ and notice that the new profile $(R^1_{-j}, R^2_j) = \bar{R}$ where $\text{Conv}(\bar{R}) = \text{Conv}(R^1)$. By Case 1.1, $\Phi(\bar{R}) = \Phi(R^1)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 2.2. None of the swapping agents is a middle agent at profile R. Hence, $p(R_1) < p(R_2) \leq \ldots < p(R_n)$. Note that in this case, $\bar{R} \in \mathcal{R}^N$ is such that $\bar{R}_{-1,n} = R_{-1,n}$, $\bar{R}_1 = R_n$, and $\bar{R}_n = R_1$. Begin from profile R and construct profile R^1 by swapping middle agent 2's preferences with agent 1's preferences, denoting the new profile by R^1. By Case 2.1, $\Phi(R^1) = \Phi(R)$. Next, swap middle agent 1's preferences with agent n's preferences, denoting the new profile by R^2. By Case 2.1, $\Phi(R^2) = \Phi(R^1)$. Finally, swap middle agent n's preferences with agent 2's preferences and notice that the new profile is $\bar{R}$. By Case 2.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Step 3. We show how any profile $\bar{R}$, where $\text{Conv}(\bar{R}) = \text{Conv}(R)$, can be constructed from profile R by sequentially repeating the first two steps of the proof. Let profile $\bar{R}$ such that $\bar{R} = (R_{\bar{1}}, \ldots, R_{\bar{n}})$ and, without loss of generality, assume $\underline{p}(\bar{R}) = p(\bar{R}_{\bar{1}}) \leq \ldots \leq p(\bar{R}_{\bar{n}}) = \bar{p}(\bar{R})$. Notice that set $\{\bar{1}, \ldots, \bar{n}\}$ is a permutation of set $N = \{1, \ldots, n\}$.

Begin from profile R and construct profile R^1 by sequentially replacing each agent' preferences R_i with $\bar{R}_{\bar{i}}$, i.e., for each $i \in N$, $R^1_i = \bar{R}_{\bar{i}}$. Note that the stepwise change of agents' preferences never changes the convex hull of peaks and that $\text{Conv}(R^1) = \text{Conv}(R)$. By Step 1, $\Phi(R^1) = \Phi(R)$. Finally, permute the agents' preferences such that each agent $\bar{i}$ obtains the preferences of agent i, i.e., the new profile R^2 is such that for all $i \in N$, $R^2_{\bar{i}} = R^1_i$. Hence, for all $i \in N$, $R^2_{\bar{i}} = \bar{R}_{\bar{i}}$ and $R^2 = \bar{R}$. Since all permutations can be obtained via sequential pairwise swaps, by Step 2, $\Phi(\bar{R}) = \Phi(R)$. □

We use Lemma 5 in the proof of Lemma 2.

Proof of Lemma 2. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. By Lemma 5, Φ satisfies *extreme peaks-onliness*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$.

We show that if $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, then all remaining agents are at least as well off, i.e., for all $i \in N \setminus \{j\}$, $\Phi(\bar{R})\, R_i\, \Phi(R)$. Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme peaks-onliness*, $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, at profile R, either agent $j = 1$ has the unique smallest peak or agent $j = n$ has the unique largest peak. By symmetry of arguments, assume that $j = 1$ has the unique smallest peak and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$.

Case 2.1. Agent 1 is a middle agent at profile $\bar{R}$. Then, $\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$. By *Pareto-efficiency*, $\Phi(\bar{R}) \in PE(\bar{R})$ and $\Phi(R) \in PE(R)$. By Corollary 1, $\Phi(\bar{R}) \in PE(R_{-1})$.

Assume that $\Phi(R) \subseteq \text{Conv}(R_{-1})$. Since $\Phi(R) \in PE(R)$, by Proposition 1 (ii), $\text{Conv}(\Phi(R)) \cap p(R) \subseteq \Phi(R)$. Hence, $\text{Conv}(\Phi(R)) \cap p(R_{-1}) \subseteq \Phi(R)$ and by Proposition 1, $\Phi(R) \in \text{PE}(R_{-1})$. Since $\bar{R}_{-1} = R_{-1}$ and $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$, by *one-sided replacement-domination*, for all agents $i \in N \setminus \{1\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ or for all agents $i \in N \setminus \{1\}$, $\Phi(R)\ R_i\ \Phi(\bar{R})$. By *Pareto-efficiency* of both sets $\Phi(R)$ and $\Phi(\bar{R})$ at profile R_{-1}, for all agents $i \in N \setminus \{1\}$, $\Phi(R)\, I_i\, \Phi(\bar{R})$. By Corollary 2, $\text{Conv}(\Phi(\bar{R})) = \text{Conv}(\Phi(R))$, and since we always represent any *Pareto-efficient* set by its convex hull, $\Phi(\bar{R}) = \Phi(R)$.

Assume that $\Phi(R) \not\subseteq \text{Conv}(R_{-1})$. Then, minimum $\underline{\Phi}(R) < \underline{p}(R_{-1}) \leq \underline{\Phi}(\bar{R}) \leq p(R_n)$. Hence, agent n's worst points are $w_{\Phi(R)}(R_n) = \{\underline{\Phi}(R)\}$ and $w_{\Phi(\bar{R})}(R_n) = \{\underline{\Phi}(\bar{R})\}$. By single-peakedness, $w_{\Phi(\bar{R})}(R_n)\ P_n\ w_{\Phi(R)}(R_n)$. By *one-sided replacement-domination*, agent n must be better off, $\Phi(\bar{R})\ P_n\ \Phi(R)$. Hence, by *one-sided replacement-domination*, for all agents $i \in N \setminus \{1\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$.

Case 2.2. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and that agent 1 has the unique smallest peak at profile R. In addition, let agent 1 also have the unique smallest peak at profile $\bar{R}$. Then, $\text{Conv}(R_{-1}) \subsetneq \text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, $p(R_1) < p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing middle agent 2's preferences to $R^1_2 = \bar{R}_1$, i.e., $R^1 = (R_{-2}, R^1_2)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$, by *extreme peaks-onliness*, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R^2_1 = \bar{R}_1$ such that the new profile is $R^2 = (R^1_{-1}, R^2_1)$. Since agent 1 has the unique smallest peak at profile R^1 and is a middle agent at profile R^2, by Case 2.1, for all agents $i \in N \setminus \{1,2\}$, $\Phi(R^2)\, R_i\, \Phi(R^1)$. Finally, change middle agent 2's preferences back to R_2 and notice that the new profile $(R^2_{-2}, R_2) = \bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$, by *extreme peaks-onliness*, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, for all agents $i \in N \setminus \{1,2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$. In particular, $\Phi(\bar{R})\ R_n\ \Phi(R)$. Since agent n has the largest peak, *Pareto-efficiency* and single-peakedness imply $\underline{\Phi}(R) \leq \underline{\Phi}(\bar{R})$ and $\bar{\Phi}(R) \leq \bar{\Phi}(\bar{R})$. Hence, either $\Phi(\bar{R}) = \Phi(R)$ or $\Phi(\bar{R})\ P_n\ \Phi(R)$. Then, since $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and $\bar{R}_{-1} = R_{-1}$, by *one-sided replacement-domination*, for all agents $i \in N \setminus \{1\}$ (including agent 2 now), $\Phi(\bar{R})\ R_i\ \Phi(R)$. $\square$

C Proof of Proposition 5

Before proceeding with the proof of Proposition 5, we first prove an implication of *Pareto-efficiency* and *population-monotonicity*.

Lemma 6. *Let choice correspondence $\Phi \in \Omega$ satisfy Pareto-efficiency and population-monotonicity. Then, for each coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and each profile $R \in \mathcal{R}^N$, the following hold.*

(i) *Without loss of generality, let agents $1, 2 \in N$ where $p(R_1) = \underline{p}(R)$ and $p(R_2) = \underline{p}(R_{-1})$. If maximum $\bar{\Phi}(R) \in \text{Conv}(R_{-1})$ and maximum $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$, then maxima $\bar{\Phi}(R) = \bar{\Phi}(R_{-1})$. Moreover, if $\Phi(R) \subseteq \text{Conv}(R_{-1})$, then $\Phi(R) = \Phi(R_{-1})$.*

(ii) *Without loss of generality, let agents $n-1, n \in N$ where $p(R_n) = \bar{p}(R)$ and $p(R_{n-1}) = \bar{p}(R_{-n})$. If minimum $\underline{\Phi}(R) \in \text{Conv}(R_{-n})$ and minimum $\underline{\Phi}(R) \in w_{\Phi(R)}(R_{n-1})$, then minima $\underline{\Phi}(R) = \underline{\Phi}(R_{-n})$. Moreover, if $\Phi(R) \subseteq \text{Conv}(R_{-n})$, then $\Phi(R) = \Phi(R_{-n})$.*

Proof. Let choice correspondence $\Phi \in \Omega$ satisfy *Pareto-efficiency* and *population-monotonicity*. Let coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and profile $R \in \mathcal{R}^N$.

(i) Let agents $1, 2 \in N$ such that $p(R_1) = \underline{p}(R)$ and $p(R_2) = \underline{p}(R_{-1})$. Let maximum $\bar{\Phi}(R) \in \text{Conv}(R_{-1})$ and maximum $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$. Hence, $p(R_2) \leq \bar{\Phi}(R)$. By *population-monotonicity* and Lemma 1, for all agents $i \in N \setminus \{1\}$, $\Phi(R_{-1})\, R_i\, \Phi(R)$. Let agent $n \in N \setminus \{1, 2\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R) = \bar{p}(R_{-1})$. Since agent n has the largest peak at profiles R and R_{-1}, $\Phi(R_{-1})\, R_n\, \Phi(R)$ and *Pareto-efficiency* imply $\underline{\Phi}(R) \leq \underline{\Phi}(R_{-1}) \leq p(R_n)$ and $\bar{\Phi}(R) \leq \bar{\Phi}(R_{-1}) \leq p(R_n)$. Since agent 2 has the smallest peak at profile R_{-1}, $p(R_2) \leq \bar{\Phi}(R)$, and $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$, $\Phi(R_{-1})\, R_1\, \Phi(R)$ and *Pareto-efficiency* imply $p(R_2) \leq \bar{\Phi}(R_{-1}) \leq \bar{\Phi}(R)$. Therefore, maxima $\bar{\Phi}(R) = \bar{\Phi}(R_{-1})$.

Moreover, let $\Phi(R) \subseteq \text{Conv}(R_{-1})$. Hence, $p(R_2) \leq \underline{\Phi}(R)$. Since agent 2 has the smallest peak at profile R_{-1} and $p(R_2) \leq \underline{\Phi}(R)$, $\Phi(R_{-1})\, R_1\, \Phi(R)$ and *Pareto-efficiency* imply $p(R_2) \leq \underline{\Phi}(R_{-1}) \leq \underline{\Phi}(R)$. Therefore, minima $\underline{\Phi}(R) = \underline{\Phi}(R_{-1})$ and thus, $\Phi(R) = \Phi(R_{-n})$.

(ii) Symmetric proof to (i). □

Proof of Proposition 5. Let choice correspondence $\Phi \in \Omega$ satisfy *Pareto-efficiency* and *population-monotonicity*. Recall that for each coalition $N \in \mathcal{P}$, each choice correspondence $\Phi \in \Omega$ specifies an fc-choice correspondence $\Phi \in \Omega^N$. Since for each $N \in \mathcal{P}$ such that $|N| \leq 2$, (*one-sided*) *replacement-domination* imposes no restriction on fc-choice correspondence $\Phi \in \Omega^N$, let $N \in \mathcal{P}$ such that $|N| \geq 3$.

We show that for any profile $R \in \mathcal{R}^N$, if the preferences of an agent $j \in N$ change, such that $R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, then the other agents whose preferences remained unchanged all are at least as well off, as they were initially, or all are made at most as well off, i.e., [for all $i \in N \setminus \{j\}$, $\Phi(R)\, R_i\, \Phi(\bar{R})$] or [for all $i \in N \setminus \{j\}$, $\Phi(\bar{R})\, R_i\, \Phi(R)$]. Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$.

Case 1.1. Let agent j be a middle agent at both profiles R and $\bar{R}$. Then, $\text{Conv}(\bar{R}) = \text{Conv}(R) = \text{Conv}(R_{-j})$. Remove agent j from profile R to obtain profile R_{-j}. Since $\text{Conv}(R_{-j}) = \text{Conv}(R)$, by *population-monotonicity* and Lemma 1, $\Phi(R_{-j}) = \Phi(R)$. Next, add agent j with preferences $\bar{R}_j$ to obtain profile $\bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R_{-j})$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R_{-j})$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 1.2. Let agent j have the unique smallest (largest) peak at both profiles R and $\bar{R}$. Hence, either agent $j = 1$ has the unique smallest peak at both profiles R and $\bar{R}$ or agent $j = n$ has the unique largest peak at both profiles R and $\bar{R}$. By symmetry of arguments, assume that $j = 1$ and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$. Hence, $p(R_1) = p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing agent 2's preferences to $R^1_2 = R_1$, i.e., $R^1 = (R_{-2}, R^1_2)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$ and agent 2 is a middle agent at both profiles R^1 and R, by Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R^2_1 = \bar{R}_1$ such that the new profile is $R^2 = (R^1_{-1}, R^2_1)$. Since $\text{Conv}(R^2) = \text{Conv}(R^1)$ and agent 1 is a middle agent at both profiles R^2 and R^1, by Case 1.1, $\Phi(R^2) = \Phi(R^1)$. Finally, change agent 2's preferences back to R_2 and notice that the new profile $(R^2_{-2}, R_2) = \bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$ and agent 2 is a middle agent at both profiles $\bar{R}$ and R^2, by Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, either agent $j = 1$ has the unique smallest peak at profile R or agent $j = n$ has the unique largest peak at profile R. By symmetry of arguments, assume that $j = 1$ and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$.

Case 2.1. Let agent 1 be a middle agent at profile $\bar{R}$. Then, $\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$. Begin from profile R and remove agent 1 from profile R to obtain profile R_{-1}. By *population-monotonicity* and Lemma 1, for all agents $i \in N \backslash \{1\}$, $\Phi(R_{-1}) R_i \Phi(R)$. Next, add agent 1 with preferences $\bar{R}_1$ to obtain profile $\bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R_{-1})$. Therefore, for all agents $i \in N \setminus \{1\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$.

Case 2.2. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and let agent 1 have the unique smallest peak at profile R. In addition, let agent 1 also have the unique smallest peak at profile $\bar{R}$. Then, $\text{Conv}(R_{-1}) \subsetneq \text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, $p(R_1) < p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$. The proof of this case proceeds in two parts.

First, we show that for all agents $i \in N \setminus \{1, 2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ and $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$. Begin from profile R and construct profile R^1 by changing agent 2's preferences to $R^1_2 = \bar{R}_1$, i.e., $R^1 = (R_{-2}, R^1_2)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$ and agent 2 is a middle agent at both profiles R^1 and R, by Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R^2_1 = \bar{R}_1$ such that the new profile is $R^2 = (R^1_{-1}, R^2_1)$. Since agent 1 is a middle agent at profile R^2, by Case 2.1, for all agents $i \in N \setminus \{1\}$, $\Phi(R^2)\ R^1_i\ \Phi(R^1)$. Hence, for all agents $i \in N \setminus \{1, 2\}$, $\Phi(R^2)\ R_i\ \Phi(R^1)$ and $\Phi(R^2)\ \bar{R}_1\ \Phi(R^1)$. Finally, change agent 2's preferences back to R_2 and notice that the new profile $(R^2_{-2}, R_2) = \bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$ and agent 2 is a middle agent at both profiles $\bar{R}$ and R^2, by Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, for all agents $i \in N \setminus \{1, 2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ and $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$.

Second, we prove that $\Phi(\bar{R})\ R_2\ \Phi(R)$. Since agent n has the largest peak at both profiles R and $\bar{R}$, $\Phi(\bar{R}) R_n \Phi(R)$ and *Pareto-efficiency* imply $\underline{\Phi}(R) \leq \underline{\Phi}(\bar{R}) \leq p(R_n)$ and $\bar{\Phi}(R) \leq \bar{\Phi}(\bar{R}) \leq p(R_n)$. Hence, either $\Phi(\bar{R}) = \Phi(R)$ or $\Phi(\bar{R})\ P_n\ \Phi(R)$. If $\Phi(\bar{R}) = \Phi(R)$, then $\Phi(\bar{R})\ R_2\ \Phi(R)$. If $\Phi(\bar{R})\ P_n\ \Phi(R)$, then (a) $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) \leq p(R_n)$ or (b) $\bar{\Phi}(R) < \bar{\Phi}(\bar{R}) \leq p(R_n)$.

If $\underline{\Phi}(R) \geq p(R_2)$, then $\Phi(\bar{R}) \subseteq \text{Conv}(R_{-1})$ and by Lemma 6 (i), $\Phi(\bar{R}) = \Phi(R_{-1})$. Next, consider the change from profile R to R_{-1}. By *population-monotonicity* and Lemma 1, for all agents $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. Therefore, for all agents $i \in N \setminus \{1\}$ (including agent 2 now), $\Phi(\bar{R})\ R_i\ \Phi(R)$.

The remaining case is that $\underline{\Phi}(R) < p(R_2)$. Since agent 1 has the smallest peak at profile $\bar{R}$, *Pareto-efficiency* implies $p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R})$. If (a) $\underline{\Phi}(R) < \underline{\Phi}(\bar{R})$, then $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$ implies $\underline{\Phi}(R) < p(\bar{R}_1)$ and if (b) $\bar{\Phi}(R) < \bar{\Phi}(\bar{R})$, then $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$ implies $\bar{\Phi}(R) < p(\bar{R}_1)$ and thus, $\underline{\Phi}(R) < p(\bar{R}_1)$.

Hence, there are two cases (2.2.α) $\underline{\Phi}(R) < p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) < p(R_2)$ and $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ and (2.2.β) $\underline{\Phi}(R) \leq \bar{\Phi}(R) < p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) < p(R_2)$.

Case 2.2.α. If $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) \leq p(R_2)$, then $b_{\Phi(R)}(R_2) = \bar{\Phi}(R) = \bar{\Phi}(\bar{R}) = b_{\Phi(\bar{R})}(R_2) \leq p(R_2)$ and $w_{\Phi(R)}(R_2) = \underline{\Phi}(R) < \underline{\Phi}(\bar{R}) = w_{\Phi(\bar{R})}(R_2) < p(R_2)$. By single-peakedness, $\Phi(\bar{R})\ P_2\ \Phi(R)$.

If $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) > p(R_2)$, then $b_{\Phi(R)}(R_2) = b_{\Phi(\bar{R})}(R_2) = p(R_2)$, $w_{\Phi(R)}(R_2) \in \{\underline{\Phi}(R), \bar{\Phi}(R)\}$, and $w_{\Phi(\bar{R})}(R_2) \in \{\underline{\Phi}(\bar{R}), \bar{\Phi}(\bar{R})\}$. Then, $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) < p(R_2) < \bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ and single-peakedness imply $\Phi(\bar{R})\ R_2\ \Phi(R)$.

Case 2.2.β. Notice that $b_{\Phi(R)}(R_2) = \{\bar{\Phi}(R)\}$ and $w_{\Phi(R)}(R_2) = \{\underline{\Phi}(R)\}$.

If $\bar{\Phi}(\bar{R}) \leq p(R_2)$, then $\bar{\Phi}(\bar{R}) \in b_{\Phi(\bar{R})}(R_2)$ and $\underline{\Phi}(\bar{R}) \in w_{\Phi(\bar{R})}(R_2)$. Since then $\underline{\Phi}(R) \leq \bar{\Phi}(R) < \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R}) \leq p(R_2)$, by single-peakedness, $\Phi(\bar{R})\ P_2\ \Phi(R)$.

If $\bar{\Phi}(\bar{R}) > p(R_2)$, then $b_{\Phi(\bar{R})}(R_2) = \{p(R_2)\}$ and $w_{\Phi(\bar{R})}(R_2) \subseteq \{\underline{\Phi}(\bar{R}), \bar{\Phi}(\bar{R})\}$. Hence, $b_{\Phi(\bar{R})}(R_2)\ P_2\ b_{\Phi(R)}(R_2)$. Since $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) < p(R_2)$, by single-peakedness, $\underline{\Phi}(\bar{R})\ P_2\ \underline{\Phi}(R) = w_{\Phi(R)}(R_2)$.

If $\underline{\Phi}(\bar{R}) \in w_{\Phi(\bar{R})}(R_2)$, then $w_{\Phi(\bar{R})}(R_2)\ P_2\ w_{\Phi(R)}(R_2)$ and $\Phi(\bar{R})\ P_2\ \Phi(R)$.

Finally, if $\underline{\Phi}(\bar{R}) \notin w_{\Phi(\bar{R})}(R_2)$, then $w_{\Phi(\bar{R})}(R_2) = \{\bar{\Phi}(\bar{R})\}$. Note that $\bar{\Phi}(\bar{R}) \in \mathrm{Conv}(R_{-1})$. By Lemma 6 (i), $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R_{-1})$. Consider the change from profile R to R_{-1}. By *population-monotonicity* and Lemma 1, for all agents $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. In particular, $\Phi(R_{-1})\, R_2\, \Phi(R)$ and $w_{\Phi(R_{-1})}(R_2)\, R_2\, w_{\Phi(R)}(R_2)$. Since agent 2 has the smallest peak at profile R_{-1}, *Pareto-efficiency* and single-peakedness imply that $\bar{\Phi}(R_{-1}) \in w_{\Phi(R_{-1})}(R_2)$. Hence, $\bar{\Phi}(\bar{R}) \in w_{\Phi(R_{-1})}(R_2)$ and $w_{\Phi(\bar{R})}(R_2) = \bar{\Phi}(\bar{R})\ R_2\ w_{\Phi(R)}(R_2)$. Therefore, $\Phi(\bar{R})\ R_2\ \Phi(R)$.

Case 3. Let $\mathrm{Conv}(\bar{R}) \supsetneq \mathrm{Conv}(R)$. By Case 2 (with the roles of R and $\bar{R}$ reversed), for all agents $i \in N \setminus \{j\}$, $\Phi(R)\ R_j\ \Phi(\bar{R})$. □

D Proof of Theorem 1

Before proceeding with the proof of Theorem 1, we first prove some implications of *Pareto-efficiency* and *(one-sided) replacement-domination*. The first implication is *peak-monotonicity*, introduced by Ching (1994). The definition follows.

An fc-choice correspondence satisfies *peak-monotonicity* if whenever an agent's preferences change such that his peak moves to the left (right), the set chosen moves to the left (right). We formulate *peak-monotonicity* for fc-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Peak-Monotonicity. Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } p(\bar{R}_j) \leq p(R_j), \text{ then } \begin{cases} \text{minimum } \underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R) \\ \qquad\qquad \text{and} \\ \text{maximum } \bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R). \end{cases}$$

Lemma 7 (*Peak-Monotonicity*)**.** *If a fixed coalition consists of at least 3 agents, then an associated fc-choice correspondence that satisfies* Pareto-efficiency *and* one-sided replacement-domination *also satisfies* peak-monotonicity*.*

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $p(\bar{R}_j) \leq p(R_j)$. By *Pareto-efficiency*, $\Phi(R) \in \text{PE}(R)$ and $\Phi(\bar{R}) \in \text{PE}(\bar{R})$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let agent j be a middle agent or have the smallest peak at profile R. Hence, $\underline{p}(\bar{R}) \leq \underline{p}(R) \leq \bar{p}(\bar{R}) = \bar{p}(R)$ and $\text{Conv}(R) \subseteq \text{Conv}(\bar{R})$. By *one-sided replacement-domination* and Lemma 2, for all agents $i \in N \setminus \{j\}$, $\Phi(R)\, R_i\, \Phi(\bar{R})$. Finally, let agent $n \in N \setminus \{j\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R) = \bar{p}(\bar{R})$. By $\Phi(R)\, R_n\, \Phi(\bar{R})$ and *Pareto-efficiency*, $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R) \leq p(R_n)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R) \leq p(R_n)$.

Case 2. Let agent j have the unique largest peak at profile R.

Case 2.1. Let agent j have the unique largest peak at profile R and be a middle agent at profile $\bar{R}$. Hence, $\underline{p}(\bar{R}) = \underline{p}(R) \leq \bar{p}(\bar{R}) < \bar{p}(R)$. By the symmetric argument of Case 1 (with agent n being a middle agent at profile $\bar{R}$ instead of agent 1 being a middle agent at profile R, and with agent n's peak moving to the right instead of agent 1's peak moving to the left), $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R)$.

Case 2.2. Let agent j have the unique largest peak at profile R and the unique smallest peak at profile $\bar{R}$. Hence, $\underline{p}(\bar{R}) < \underline{p}(R) \leq \bar{p}(\bar{R}) < \bar{p}(R)$. Begin from profile R and construct profile R^1 by changing agent j's preferences to R^1_j such that his peak $p(R^1_j) = \underline{p}(R)$, i.e., $R^1 = (R_{-j}, R^1_j)$. Since agent j has the unique largest peak at profile R and is a middle agent at profile R^1, by Case 2.1, $\underline{\Phi}(R^1) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(R^1) \leq \bar{\Phi}(R)$. Finally, change agent j's preferences to $\bar{R}_j$ and notice that the new profile $(R^1_{-j}, \bar{R}_j) = \bar{R}$. Since agent j is a middle agent at profile R^1, by Case 1, $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R^1) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R^1) \leq \bar{\Phi}(R)$. □

The second implication of *Pareto-efficiency* and *(one-sided) replacement-domination* is *uncompromisingness*, introduced by Border and Jordan (1983). The definition follows.

Loosely speaking, an fc-choice correspondence satisfies *uncompromisingness* if whenever an agent's preferences change such that both his peaks, before and after this change, both lie on the same side of the minimum (maximum) point chosen, the minimum (maximum) point chosen does not change. We formulate *uncompromisingess* –and later *set-uncompromisigness*– for fc-choice correspondences but as discussed in Remark 3, they easily extend to choice correspondences.

Uncompromisigness. Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } \begin{cases} p(R_j) < \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \underline{\Phi}(R) \\ \qquad\qquad \text{or} \\ p(R_j) > \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \underline{\Phi}(R), \end{cases} \quad \text{then minima } \underline{\Phi}(R) = \underline{\Phi}(\bar{R})$$

and

$$\text{if } \begin{cases} p(R_j) > \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \bar{\Phi}(R) \\ \qquad\qquad \text{or} \\ p(R_j) < \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \bar{\Phi}(R), \end{cases} \quad \text{then maxima } \bar{\Phi}(R) = \bar{\Phi}(\bar{R}).$$

Uncompromisingness immediately implies the following notion of *set-uncompromisingness.*

Set-Uncompromisigness. Let fixed coalition $N \in \mathcal{P}$ and fc-choice correspondence $\Phi \in \Omega^N$.

For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } \begin{cases} p(R_j) < \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \underline{\Phi}(R) \\ \text{or} \\ p(R_j) > \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \bar{\Phi}(R), \end{cases} \quad \text{then } \Phi(R) = \Phi(\bar{R}).$$

Corollary 4 (*Uncompromisingness* implies *Set-Uncompromisingness*)**.** *Each fc-choice correspondence satisfying uncompromisingness also satisfies set-uncompromisingness.*

Before showing in Lemma 9 some conditions under which an fc-choice correspondence satisfies *uncompromisingness*, we first show a result for the domain of symmetric preferences $\mathcal{S}$ (Lemma 8). This is the only result where we have to change the proof technique when dealing with domain $\mathcal{S}$.[13] Specifically, we prove Lemma 8 using a so-called 'leapfrogging' argument. During each 'leapfrog' we right (left) extend the convex hull of the peaks by some distance and if this distance is not enough we repeat this argument as many (finite) times as necessary. Notice that Lemma 8 also holds on the domain of single-peaked preferences $\mathcal{R}$.

Lemma 8. *Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy Pareto-efficiency and one-sided replacement-domination. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(R) \subsetneq \text{Conv}(\bar{R})$,*

- **(i)** *if minimum $\underline{\Phi}(R) < \bar{p}(R) < p(\bar{R}_j)$, then minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$. Moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R)$, then $\Phi(\bar{R}) = \Phi(R)$,*
- **(ii)** *if maximum $\bar{\Phi}(R) > \underline{p}(R) > p(\bar{R}_j)$, then maxima $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R)$. Moreover, if also minimum $\underline{\Phi}(R) > \underline{p}(R)$, then $\Phi(\bar{R}) = \Phi(R)$,*

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. By Lemmas 5 (Appendix B) and 7, Φ satisfies *extreme peaks-onliness* and *peak-monotonicity*.

Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(R) \subsetneq \text{Conv}(\bar{R})$. By *Pareto-efficiency*, $\Phi(R) \in \text{PE}(R)$. By *extreme peaks-onliness*, it is without loss of generality to assume that both profiles R and $\bar{R}$ are symmetric, i.e., $R, \bar{R} \in \mathcal{S}^N$.[14] In

[13] Recall that all steps in all other proofs are for domain $\mathcal{R}$ but they automatically apply to domain $\mathcal{S}$.

[14] For each agent $i \in N$, we can replace preferences $R_i, \bar{R}_i \in \mathcal{R}$ by preferences $R'_i, \bar{R}'_i \in \mathcal{S}$ such that $p(R_i) = p(R'_i)$ and $p(\bar{R}_i) = p(\bar{R}'_i)$. Then, by *extreme peaks-onliness*, $\Phi(R) = \Phi(R')$ and $\Phi(\bar{R}) = \Phi(\bar{R}')$.

the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

(i) Let minimum $\underline{\Phi}(R) < \bar{p}(R) < p(\bar{R}_j)$. Since $\bar{p}(R) < p(\bar{R}_j)$ and $\Phi(R) \in \mathrm{PE}(R)$, by Proposition 1 (i), $\underline{p}(R) \leq \underline{\Phi}(R) \leq \bar{\Phi}(R) \leq \bar{p}(R) < p(\bar{R}_j)$. Since also $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(\bar{R})$, agent j either [is a middle agent at profile R and has the unique largest peak at profile $\bar{R}$] or [has the unique largest peak at both profiles R and $\bar{R}$].

Case 1. Let agent j be a middle agent at profile R and have the unique largest peak at profile $\bar{R}$. Let agent $n \in N \setminus \{j\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R)$. Hence, minimum $\underline{\Phi}(R) < \bar{p}(R)$ and *Pareto-efficiency* imply $\underline{\Phi}(R) < p(R_n)$ and $\bar{\Phi}(R) \leq p(R_n)$. By single-peakedness, $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$ and $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$.

Let the distance between minimum $\underline{\Phi}(R)$ and peak $p(R_n)$ be $\delta_0 = |\underline{\Phi}(R) - p(R_n)|$. Let point $x_1 \in \mathbb{R}$ be on the right side of peak $p(R_n)$, i.e., $x_1 > p(R_n) = \bar{p}(R)$, such that the distance between minimum $\underline{\Phi}(R)$ and point x_1 is $\delta_1 = |\underline{\Phi}(R) - x_1| = \frac{3}{2}\delta_0$. Hence, distance $|p(R_n) - x_1| = |\underline{\Phi}(R) - x_1| - |\underline{\Phi}(R) - p(R_n)| = \frac{1}{2}\delta_0 = \frac{1}{2}|\underline{\Phi}(R) - p(R_n)|$ and point x_1 is closer to peak $p(R_n)$ than minimum $\underline{\Phi}(R)$ is.

Step 1. Begin from profile R and construct profile R^1 by changing agent j's preferences to $R^1_j \in \mathcal{S}$ such that his peak

$$p(R^1_j) = \begin{cases} p(\bar{R}_j) & \text{if } p(\bar{R}_j) \leq x_1 \\ x_1 & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-j}, R^1_j)$. Hence, $R^1_{-j} = R_{-j}$. By *Pareto-efficiency* and Proposition 1 (i), $\underline{p}(R) = \underline{p}(R^1) \leq \underline{\Phi}(R^1) \leq \bar{\Phi}(R^1) \leq \bar{p}(R^1) = p(\bar{R}_j)$. Since $p(R^1_j) > p(R_j)$, by *peak-monotonicity*, minimum $\underline{\Phi}(R^1) \geq \underline{\Phi}(R)$ and maximum $\bar{\Phi}(R^1) \geq \bar{\Phi}(R)$. Hence, $\underline{\Phi}(R^1) \in [\underline{\Phi}(R), p(\bar{R}_j)]$ and $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(\bar{R}_j)]$. Since $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(R^1)$, by *one-sided replacement-domination* and Lemma 2, agent n is at most as well off, $\Phi(R)\, R_n\, \Phi(R^1)$. Hence, $b_{\Phi(R)}(R_n)\, R_n\, b_{\Phi(R^1)}(R_n)$ and $w_{\Phi(R)}(R_n)\, R_n\, w_{\Phi(R^1)}(R_n)$.

If $\underline{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$, then $w_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$. The distance of agent n's worst point $\bar{\Phi}(R^1)$ to peak $p(R_n)$ is $|p(R_n) - \bar{\Phi}(R^1)| \leq |p(R_n) - p(R^1_j)| \leq |p(R_n) - x_1| = \frac{1}{2}\delta_0 = \frac{1}{2}|\underline{\Phi}(R) - p(R_n)|$, which is smaller than the distance of minimum $\underline{\Phi}(R)$ to peak $p(R_n)$. By symmetric single-peakedness, agent n prefers $w_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$ to $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$; a contradiction. Hence, $\underline{\Phi}(R^1) \in [\underline{\Phi}(R), p(R_n))$ and $w_{\Phi(R^1)}(R_n) = \underline{\Phi}(R^1)$. Since $\underline{\Phi}(R^1) <$

$p(R_n)$, for agent n to find $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$ at least as good as $w_{\Phi(R^1)}(R_n) = \underline{\Phi}(R^1)$, it must be that minimum $\underline{\Phi}(R) \geq \underline{\Phi}(R^1)$. Hence, minima $\underline{\Phi}(R^1) = \underline{\Phi}(R)$.

Moreover, let maximum $\bar{\Phi}(R) < \bar{p}(R) = p(R_n)$. Then, $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$. Recall that $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(\bar{R}_j)]$. If $\bar{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$, then agent n prefers $b_{\Phi(R^1)}(R_n) = p(R_n)$ to $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$; a contradiction. Hence, $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(R_n))$ and $b_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$. Since $\bar{\Phi}(R^1) < p(R_n)$, for agent n to find $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$ at least as good as $b_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$, it must be that maximum $\bar{\Phi}(R) \geq \bar{\Phi}(R^1)$. Hence, maxima $\bar{\Phi}(R^1) = \bar{\Phi}(R)$ and $\Phi(R^1) = \Phi(R)$.

If $p(R^1_j) = p(\bar{R}_j)$, then $\text{Conv}(R^1) = \text{Conv}(\bar{R})$ and by *extreme peaks-onliness*, $\Phi(R^1) = \Phi(\bar{R})$ and we are done. If $p(R^1_j) \neq p(\bar{R}_j)$, then note that agent n is now a middle agent and agent j has the unique largest peak at profile R^1. We now explain the term 'leapfrogging' in order to explain the proof technique: in Step 1, the peak of agent j moves to the right of agent n's peak by figuratively 'leapfrogging' over agent n. In Step 2, the roles of agents j and n reverse, and agent n 'leapfrogs' over agent j to the right, etc.

Let point $x_2 \in \mathbb{R}$ be on the right side of peak $p(R^1_j)$, i.e., $x_2 > p(R^1_j) = \bar{p}(R^1)$, such that the distance between minimum $\underline{\Phi}(R)$ and point x_2 is $\delta_2 = |\underline{\Phi}(R) - x_2| = \frac{3}{2}\delta_1$. Hence, distance $|p(R^1_j) - x_2| = |\underline{\Phi}(R) - x_2| - |\underline{\Phi}(R) - p(R^1_j)| = \frac{1}{2}\delta_1 = \frac{1}{2}|\underline{\Phi}(R) - p(R^1_j)|$ and point x_2 is closer to peak $p(R^1_j)$ than minimum $\underline{\Phi}(R)$ is.

Step 2. Begin from profile R^1 and construct profile R^2 by changing agent n's preferences to $R^2_n \in \mathcal{S}$ such that his peak

$$p(R^2_n) = \begin{cases} p(\bar{R}_j) & \text{if } p(\bar{R}_j) \leq x_2 \\ x_2 & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R^1_{-n}, R^2_n)$. Hence, $R^2_{-n} = R^1_{-n}$. By the arguments described in the previous step (with profiles R and R^1 replaced by profiles R^1 and R^2 and with agent n in the role of agent j), minima $\underline{\Phi}(R^2) = \underline{\Phi}(R^1) = \underline{\Phi}(R)$.

Moreover, let maximum $\bar{\Phi}(R) < \bar{p}(R)$. Then, maximum $\bar{\Phi}(R) = \bar{\Phi}(R^1) < \bar{p}(R^1) = p(R^1_j)$ and by the arguments described in the previous step (with profiles R and R^1 replaced by profiles R^1 and R^2 and with agent n in the role of agent j), $\Phi(R^2) = \Phi(R^1) = \Phi(R)$.

If $p(R_n^2) = p(\bar{R}_j)$, then $\mathrm{Conv}(R^2) = \mathrm{Conv}(\bar{R})$ and by *extreme peaks-onliness*, $\Phi(R^2) = \Phi(\bar{R})$ and we are done. If $p(R_j^2) \neq p(\bar{R}_j)$. Then, according to the reasoning described below, repeat the 'leapfrogging' steps described above $\nu \in \mathbb{N}^+$ amount of times.

Recall that $\delta_1 = \frac{3}{2}\delta_0$ and $\delta_2 = \frac{3}{2}\delta_1$. Hence, $\delta_\nu = \frac{3}{2}\delta_{\nu-1} = \left(\frac{3}{2}\right)^\nu \delta_0$ and since $\delta_0 \neq 0$, in the limit, $\lim_{\nu\to\infty} \delta_\nu = \infty$. Thus, for each profile $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}_{-j} = R_{-j}$ and $p(\bar{R}_j) > p(R_j)$, there exists a finite $\nu \in \mathbb{N}^+$ such that the distance $\delta_\nu > |\underline{\Phi}(R) - p(\bar{R}_j)|$. Therefore, for each profile $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}_{-j} = R_{-j}$ and $p(\bar{R}_j) > p(R_j)$, there exists a profile R^ν such that $\mathrm{Conv}(R^\nu) = \mathrm{Conv}(\bar{R})$ and the following holds. If minimum $\underline{\Phi}(R) < \bar{p}(R) = p(R_n) < p(\bar{R}_j)$, then minima $\underline{\Phi}(R^\nu) = \underline{\Phi}(R)$ and moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R)$, then $\Phi(R^\nu) = \Phi(R)$. Since $\mathrm{Conv}(R^\nu) = \mathrm{Conv}(\bar{R})$, by *extreme peaks-onliness*, $\Phi(R^\nu) = \Phi(\bar{R})$ and we are done.

Case 2. Let agent $j = n$ have the unique largest peak at profiles R and $\bar{R}$. Let agent $k \in N \backslash \{j\}$ be a middle agent at profile R and construct profile R^1 by changing his preferences to R_k^1 such that his peak $p(R_k^1) = \bar{p}(R)$, i.e., $R^1 = (R_{-k}, R_k^1)$. Since $\mathrm{Conv}(R^1) = \mathrm{Conv}(R)$, by *extreme peaks-onliness*, $\Phi(R^1) = \Phi(R)$. Therefore, since minimum $\underline{\Phi}(R) < \bar{p}(R) = \bar{p}(R^1) = p(R_k^1) < p(\bar{R}_j)$, by Case 1, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R^1) = \underline{\Phi}(R)$ and moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R) = \bar{p}(R^1)$, by Case 1, $\Phi(\bar{R}) = \Phi(R^1) = \Phi(R)$.

(ii) Symmetric proof to (i). □

Lemma 9 (*Uncompromisigness*)**.** *If a fixed coalition consists of at least 3 agents, then an associated fc-choice correspondence that satisfies* Pareto-efficiency *and* one-sided replacement-domination *also satisfies* uncompromisingness.

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. By Lemmas 5 (Appendix B) and 7, Φ satisfies *extreme peaks-onliness* and *peak-monotonicity*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

(i) We show that if $[p(R_j) < \underline{\Phi}(R)$ and $p(\bar{R}_j) \leq \underline{\Phi}(R)]$ or $[p(R_j) > \underline{\Phi}(R)$ and $p(\bar{R}_j) \geq \underline{\Phi}(R)]$, then minima $\underline{\Phi}(R) = \underline{\Phi}(\bar{R})$. By *Pareto-efficiency*, $\Phi(R) \in \mathrm{PE}(R)$. Hence by Proposition 1 (i), $\Phi(R) \subseteq \mathrm{Conv}(R)$. Notice that $\mathrm{Conv}(\bar{R}) \subseteq \mathrm{Conv}(R)$ or $\mathrm{Conv}(\bar{R}) \supseteq \mathrm{Conv}(R)$.

Case 1. Let $p(R_j) < \underline{\Phi}(R)$ and $p(\bar{R}_j) \leq \underline{\Phi}(R)$. Hence, since $\Phi(R) \subseteq \mathrm{Conv}(R)$, $p(R_j) \neq \bar{p}(R)$.

Case 1.1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme peaks-onliness*, $\Phi(R) = \Phi(\bar{R})$.

Case 1.2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, agent j has the unique smallest peak at profile R and minimum $\underline{\Phi}(R) \geq p(\bar{R}_j) \geq \underline{p}(\bar{R}) > p(R_j)$. Begin from profile R and construct profile $\bar{R}$ by changing agent j's preferences to $\bar{R}_j$, i.e., $\bar{R} = (R_{-j}, \bar{R}_j)$. Since $p(\bar{R}_j) > p(R_j)$ and $\bar{R}_{-j} = R_{-j}$, by *peak-monotonicity*, minimum $\underline{\Phi}(\bar{R}) \geq \underline{\Phi}(R)$. If minimum $\underline{\Phi}(\bar{R}) > \underline{\Phi}(R) \geq p(\bar{R}_j)$, then $\Phi(\bar{R}) \neq \Phi(R)$ and minimum $\underline{\Phi}(\bar{R}) > \underline{p}(\bar{R}) > p(R_j)$. Since $\bar{R}_{-j} = R_{-j}$, by Lemma 8 (ii) (with the roles of R and $\bar{R}$ reversed), $\Phi(\bar{R}) = \Phi(R) \neq \Phi(\bar{R})$, a contradiction. Therefore, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

Case 1.3. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Hence, agent j has the unique smallest peak at profile $\bar{R}$ and minimum $\underline{\Phi}(R) > p(R_j) \geq \underline{p}(R) > p(\bar{R}_j)$. By Lemma 8 (ii), $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $p(R_j) > \underline{\Phi}(R)$ and $p(\bar{R}_j) \geq \underline{\Phi}(R)$. Hence, since $\Phi(R) \subseteq \text{Conv}(R)$, $p(R_j) \neq \underline{p}(R)$.

Case 2.1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme peaks-onliness*, $\Phi(R) = \Phi(\bar{R})$.

Case 2.2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, agent j has the unique largest peak at profile R and minimum $\underline{\Phi}(R) \leq p(\bar{R}_j) \leq \bar{p}(\bar{R}) < p(R_j)$. Begin from profile R and construct profile $\bar{R}$ by changing agent j's preferences to $\bar{R}_j$, i.e., $\bar{R} = (R_{-j}, \bar{R}_j)$. Since $p(\bar{R}_j) < p(R_j)$ and $\bar{R}_{-j} = R_{-j}$, by *peak-monotonicity*, minimum $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R)$. If minimum $\underline{\Phi}(\bar{R}) < \underline{\Phi}(R) \leq p(\bar{R}_j)$, then minimum $\underline{\Phi}(\bar{R}) < \bar{p}(\bar{R}) < p(R_j)$. Since $\bar{R}_{-j} = R_{-j}$, by Lemma 8 (i) (with the roles of R and $\bar{R}$ reversed), minimum $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R) \neq \underline{\Phi}(\bar{R})$, a contradiction. Therefore, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

Case 2.3. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Hence, agent j has the unique largest peak at profile $\bar{R}$ and minimum $\underline{\Phi}(R) < p(R_j) \leq \bar{p}(R) < p(\bar{R}_j)$. By Lemma 8 (i), minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

(ii) The proof that if $[p(R_j) > \bar{\Phi}(R)$ and $p(\bar{R}_j) \geq \bar{\Phi}(R)]$ or $[p(R_j) < \bar{\Phi}(R)$ and $p(\bar{R}_j) \leq \bar{\Phi}(R)]$, then maxima $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ is symmetric to the proof of (i). □

The next result is crucial in the proof of Theorem 1.

Lemma 10. *Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy* Pareto-efficiency *and* one-sided replacement-domination*. Let fc-target set correspondence $\Phi^{a,b} \in \Omega^N$. For each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that* $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$*, if $\Phi(R) = \Phi^{a,b}(R)$, then $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.*

Proof. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. Let fc-target set correspondence $\Phi^{a,b} \in \Omega^N$. By Propositions 1 and 4, $\Phi^{a,b}$ satisfies *Pareto-efficiency* and *one-sided replacement-domination*. By Lemma 5 (Appendix B), Lemma 9, and Corollary 4, Φ and $\Phi^{a,b}$ satisfy *extreme peaks-onliness*, *uncompromisingness*, and *set-uncompromisingness*.

Let the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $\Phi(R) = \Phi^{a,b}(R)$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq \cdots \leq p(R_n) = \bar{p}(R)$. We show that $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme peaks-onliness* and the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = \Phi(R) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$.

Case 2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)$. By *extreme peaks-onliness*, it is without loss of generality to assume that at both profiles R and $\bar{R}$, agent 1 has the smallest peak and all other agents have the largest peak, i.e., $R = (R_1, R_n, \ldots, R_n)$ such that $p(R_1) \leq p(R_n)$ and $\bar{R} = (\bar{R}_1, R_n, \ldots, R_n)$ such that $p(\bar{R}_1) \leq p(R_n)$. Hence, $R_{-1} = \bar{R}_{-1}$ and $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(\bar{R}) = \bar{p}(R)$. By *Pareto-efficiency* and Proposition 1 (i), $p(R_1) = \underline{p}(R) \leq \underline{\Phi}(R) \leq \bar{\Phi}(R) \leq \bar{p}(R)$ and $p(\bar{R}_1) = \underline{p}(\bar{R}) \leq \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R}) \leq \bar{p}(\bar{R})$.

Case 2.1. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) \leq \underline{\Phi}(R)$. Then, $p(R_1) = \underline{p}(R) < \underline{\Phi}(R)$. By *set-uncompromisingness*, $\Phi(\bar{R}) = \Phi(R) = \Phi^{a,b}(R)$ and by the definition of $\Phi^{a,b}$, point $a \geq \underline{p}(\bar{R})$. If point $a \leq \bar{p}(R) = \bar{p}(\bar{R})$, then $\Phi^{a,b}(R) = [a, b] \cap \text{Conv}(R) = [a, b] \cap \text{Conv}(\bar{R}) = \Phi^{a,b}(\bar{R})$. If point $a > \bar{p}(R) = \bar{p}(\bar{R})$, then, $\Phi^{a,b}(R) = \{\bar{p}(R)\} = \Phi^{a,b}(\bar{R})$. Therefore, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 2.2. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) > \underline{\Phi}(R)$ and $p(\bar{R}_1) = \underline{p}(R) \leq \bar{\Phi}(R)$. Then, $\underline{\Phi}(R) \neq \bar{\Phi}(R)$ and $p(R_1) < \bar{\Phi}(R)$. By *uncompromisingness*, maxima $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R)$. Recall that by *Pareto-efficiency* and Proposition 1 (i), minimum $\underline{\Phi}(\bar{R}) \geq \underline{p}(\bar{R}) = p(\bar{R}_1)$. Next, assuming that minimum $\underline{\Phi}(\bar{R}) > \underline{p}(\bar{R}) = p(\bar{R}_1) > \underline{\Phi}(R)$ results in a contradiction as follows: since $p(\bar{R}_1) < \underline{\Phi}(\bar{R})$ and $p(R_1) < \underline{\Phi}(\bar{R})$, by *uncompromisingness*, minimum $\underline{\Phi}(R) = \underline{\Phi}(\bar{R}) \neq \underline{\Phi}(R)$, a contradiction. Hence, minimum $\underline{\Phi}(\bar{R}) = \underline{p}(\bar{R})$ and thus, $\Phi(\bar{R}) = [\underline{p}(\bar{R}), \bar{\Phi}(R)]$. Since $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and $\Phi(R) = [a, b] \cap \text{Conv}(R)$, $\Phi(\bar{R}) = \Phi(R) \cap \text{Conv}(\bar{R}) = [a, b] \cap \text{Conv}(\bar{R})$. Therefore, by the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = [a, b] \cap \text{Conv}(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 2.3. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) > \bar{\Phi}(R) \geq \underline{\Phi}(R)$. By the definition of $\Phi^{a,b}$, points $a, b < \underline{p}(\bar{R})$. Next, assuming that maximum $\bar{\Phi}(\bar{R}) > \underline{p}(\bar{R}) = p(\bar{R}_1) > \bar{\Phi}(R)$ results in a contradiction as follows: since $p(\bar{R}_1) < \bar{\Phi}(\bar{R})$ and $p(R_1) < \bar{\Phi}(\bar{R})$, by *uncompromisingness*, maximum $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) \neq \bar{\Phi}(R)$, a contradiction. Hence, maximum $\bar{\Phi}(\bar{R}) = \underline{p}(\bar{R})$ and thus $\Phi(\bar{R}) = \{\underline{p}(\bar{R})\}$. Since point $b < \underline{p}(\bar{R})$, by the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = \{\underline{p}(\bar{R})\} = \Phi^{a,b}(\bar{R})$.

Case 3. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) = \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$. By a symmetric proof to Case 2, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 4. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$. Let profile $R^1 \in \mathcal{R}^N$ such that $\underline{p}(R^1) = \underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(R^1) = \bar{p}(R)$. By Case 2, $\Phi(R^1) = \Phi^{a,b}(R^1)$. Next, since $\underline{p}(\bar{R}) = \underline{p}(R^1)$ and $\bar{p}(\bar{R}) < \bar{p}(R^1)$, by Case 3, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$. □

Proof of Theorem 1. *If part.* By Propositions 1 and 4, each fc-target set correspondence satisfies *Pareto-efficiency* and *one-sided replacement-domination*.

Only if part. Let fixed coalition $N \in \mathcal{P}$ such that $|N| \geq 3$ and fc-choice correspondence $\Phi \in \Omega^N$ satisfy *Pareto-efficiency* and *one-sided replacement-domination*. By Lemma 5 (Appendix B), Lemma 9, and Corollary 4, Φ satisfies *extreme peaks-onliness*, *uncompromisingness*, and *set-uncompromisingness*.

For each pair of points $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, define a profile $R^{\alpha,\beta} \in \mathcal{R}^N$ such that $\underline{p}(R^{\alpha,\beta}) = \alpha$ and $\bar{p}(R^{\alpha,\beta}) = \beta$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\alpha = p(R_1^{\alpha,\beta}) \leq \ldots \leq p(R_n^{\alpha,\beta}) = \beta$. By *Pareto-efficiency* and Proposition 1 (i), $\alpha \leq \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) \leq \beta$.

We prove that there exists an fc-target set correspondence $\Phi^{a,b} \in \Omega^N$ such that for each profile $R \in \mathcal{R}^N$, $\Phi(R) = \Phi^{a,b}(R)$.

There are four cases. Loosely speaking, in all but the last case the proof proceeds as follows. Given a profile $R^{\alpha,\beta} \in \mathcal{R}^N$ and for all profiles $R \in \mathcal{R}^N$ we select a profile such that the convex hull of its peaks is a superset of both $\text{Conv}(R^{\alpha,\beta})$ and $\text{Conv}(R)$ and then, we apply Lemma 10 to show that $\Phi(R) = \Phi^{a,b}(R)$.

Case 1. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) < \beta$. Define points $a := \underline{\Phi}(R^{\alpha,\beta})$ and $b := \bar{\Phi}(R^{\alpha,\beta})$. Since $\Phi(R^{\alpha,\beta}) = [a, b] = [a, b] \cap \text{Conv}(R^{\alpha,\beta})$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R_1^1 such that his peak

$$p(R_1^1) = \begin{cases} p(R_1^{\alpha,\beta}) & \text{if } p(R_1^{\alpha,\beta}) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $p(R_1^{\alpha,\beta}) < \underline{\Phi}(R^{\alpha,\beta})$ and $p(R_1^1) < \underline{\Phi}(R^{\alpha,\beta})$, by *set-uncompromisingness*, $\Phi(R^1) = \Phi(R^{\alpha,\beta}) = [a, b]$. Then, change agent n's preferences to R_n^2 such that his peak

$$p(R_n^2) = \begin{cases} p(R_n^1) & \text{if } p(R_n^1) \geq \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R_{-n}^1, R_n^2)$. Since $p(R_n^1) > \bar{\Phi}(R^1)$ and $p(R_n^2) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = [a, b]$. Since $\Phi(R^2) = [a, b] = [a, b] \cap \text{Conv}(R^2)$, by the definition of $\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since, $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 10, $\Phi(R) = \Phi^{a,b}(R)$.

Case 2. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) < \beta$, and for each $\bar{\alpha} \leq \alpha$ and its associated $R^{\bar{\alpha},\beta} \in \mathcal{R}^N$, $\bar{\alpha} = \underline{\Phi}(R^{\bar{\alpha},\beta}) \leq \bar{\Phi}(R^{\bar{\alpha},\beta}) < \beta$.

Case 2.1. There exist $\alpha, \beta \in \mathbb{R}$ as specified in Case 2 and in addition, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) < \bar{\Phi}(R^{\alpha,\beta}) < \beta$. Define points $a := -\infty$ and $b := \bar{\Phi}(R^{\alpha,\beta})$. Since $\Phi(R^{\alpha,\beta}) = [\underline{p}(R^{\alpha,\beta}), b] = [a, b] \cap \text{Conv}(R^{\alpha,\beta})$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R_1^1 such that his peak

$$p(R_1^1) = \begin{cases} p(R_1^{\alpha,\beta}) & \text{if } p(R_1^{\alpha,\beta}) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $\underline{p}(R^1) \leq \alpha$ and $\bar{p}(R^1) = \beta$, as specified in Case 2 and by *extreme peaks-onliness*, $\underline{p}(R^1) = \underline{\Phi}(R^1)$. Since $p(R_1^{\alpha,\beta}) < \bar{\Phi}(R^{\alpha,\beta})$ and $p(R_1^1) < \bar{\Phi}(R^{\alpha,\beta})$, by *uncompromisingness*, maxima $\bar{\Phi}(R^1) = \bar{\Phi}(R^{\alpha,\beta}) = b$. Hence, $\Phi(R^1) = [\underline{p}(R^1), b]$. Then, change agent n's preferences to R_n^2 such that his peak

$$p(R_n^2) = \begin{cases} p(R_n^1) & \text{if } p(R_n^1) \geq \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R^1_{-n}, R^2_n)$. Since $p(R^1_n) > \bar{\Phi}(R^1)$ and $p(R^2_n) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = [\underline{p}(R^2), b]$. Since $\Phi(R^2) = [\underline{p}(R^2), b] = [a, b] \cap \text{Conv}(R^2)$, by the definition of $\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 10, $\Phi(R) = \Phi^{a,b}(R)$.

Case 2.2. There exist $\alpha, \beta \in \mathbb{R}$, as specified in Case 2 and in addition, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) = \bar{\Phi}(R^{\alpha,\beta}) < \beta$, and for each $\bar{\alpha} \leq \alpha$ and its associated $R^{\bar{\alpha},\beta} \in \mathcal{R}^N$, $\bar{\alpha} = \underline{\Phi}(R^{\bar{\alpha},\beta}) = \bar{\Phi}(R^{\bar{\alpha},\beta}) < \beta$. Define points $a, b := -\infty$. Since $b < \underline{p}(R^{\alpha,\beta})$ and $\Phi(R^{\alpha,\beta}) = \{\underline{p}(R^{\alpha,\beta})\}$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R^1_1 such that his peak

$$p(R^1_1) = \begin{cases} p(R^{\alpha,\beta}_1) & \text{if } p(R^{\alpha,\beta}_1) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R^{\alpha,\beta}_{-1}, R^1_1)$. Since $\underline{p}(R^1) \leq \alpha$ and $\bar{p}(R^1) = \beta$, as specified in this case and by *extreme peaks-onliness*, $\Phi(R^1) = \{\underline{p}(R^1)\}$. Then, change agent n's preferences to R^2_n such that his peak

$$p(R^2_n) = \begin{cases} p(R^1_n) & \text{if } p(R^1_n) \geq \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R^1_{-n}, R^2_n)$. Since $p(R^1_n) > \bar{\Phi}(R^1)$ and $p(R^2_n) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = \{\underline{p}(R^2)\}$. Since $b < \underline{p}(R^2)$, by the definition of $\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 10, $\Phi(R) = \Phi^{a,b}(R)$.

Case 3. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) = \beta$, and for each $\bar{\beta} \geq \beta$ and its associated $R^{\alpha,\bar{\beta}} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\bar{\beta}}) \leq \bar{\Phi}(R^{\alpha,\bar{\beta}}) = \bar{\beta}$. The proof of this case is symmetric to Case 2.

Case 4. For each $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) = \beta$. Define points $a := -\infty$ and $b := \infty$. Since for each $\alpha, \beta \in \mathbb{R}$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) = \beta$, by *extreme peaks-onliness*, for each $R \in \mathcal{R}^N$, $\Phi(R) = \text{Conv}(R)$. Therefore, since $a < \underline{p}(R)$ and $b > \bar{p}(R)$, by the definition of $\Phi^{a,b}$, $\Phi(R) = \Phi^{a,b}(R)$. □

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