How to Control Controlled School Choice: Comment

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Abstract

Echenique and Yenmez [*AER* 2015, 105(8): 2679-94] study choice rules for a school that express preferences for a diverse student body. Their Theorem 2 characterizes choice rules that are "generated by reserves for the priority". We show that the "only if part" is not correct. We exhibit a choice rule that is generated by reserves for the priority but violates one of their axioms. A similar issue arises in Theorem $D.2.$, where a priority is allowed to be endogenous. We reformulate the axioms and repair the results.

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In recent years, diversity concerns in assigning students to schools when each student belongs to one of multiple types (based on factors such as gender, socioeconomic status, or ethnicity) have been receiving increasing attention from economists. Echenique and Yenmez (2015) study choice rules for a school that express preferences for a diverse student body. In one of their results, they take a priority as given, and characterize choice rules

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that are “generated by reserves for the priority”: such a choice rule reserves a number of seats for each type, and first, for each type, chooses students from that type, respecting priority, until the reserved seats are filled or no student of that type is left; and then chooses among the remaining students, respecting priority, until all seats are filled or no student is left. Echenique and Yenmez (2015), in their Theorem 2, state that a choice rule is generated by reserves for the priority if and only if it satisfies three axioms. One of their axioms explicitly refers to the priority and formalizes the idea that the priority should take over once there are “enough” students of the type in question.

We show that an error in formalizing what enough should mean, or in other words when a type should be considered “saturated”, results in a mistake in the “only if part” of their Theorem 2. In particular, we exhibit a choice rule that is generated by reserves for the priority but violates their axiom. We repair the result by redefining when a type is saturated and by introducing a new axiom based on this definition. The proof of Theorem 2 in Echenique and Yenmez (2015) cannot be repaired by simply invoking the new axiom. Therefore, we provide a self-contained proof. A similar issue arises in their Theorem $D.2.$, where priorities are allowed to be endogenous. We show that the “only if part” of Theorem $D.2.$ is not correct, which is also due to the error in formalizing saturation. Our new definition of saturation also allows us to repair that result.

1 The Model and Notation

We first present the primitives of Echenique and Yenmez (2015) that are relevant for our discussion. Let $\mathcal{S}$ be a nonempty finite set of students. A **choice rule** C for a school maps each nonempty set $S \subseteq \mathcal{S}$ to a nonempty subset $C(S) \subseteq S$. Let q denote the **capacity** of the school. It is such that for each $S \subseteq \mathcal{S}$, $|C(S)| \leq q$.

The set of students is partitioned into d different types. Let $T \equiv \{t_1, \ldots, t_d\}$ be the set of **types** and $\tau : \mathcal{S} \to T$ be the **type function**. For each $S \subseteq \mathcal{S}$ and $t \in T$, let $S_t \equiv \{s \in S : \tau(s) = t\}$. We assume that for each $t \in T$, $q < |\mathcal{S}_t|$.

A **priority** $\succ$ is a complete, transitive, and antisymmetric binary relation on $\mathcal{S}$. A choice rule is **generated by reserves for priority** $\succ$ if there is a vector $(r_t)_{t \in T} \in \mathbb{Z}_+^{|\mathcal{S}|}$ such that $||r|| \leq q$ and for each $S \subseteq \mathcal{S}$, $C(S)$ is determined by the following two-stage procedure. In the first stage, for each type t, type-t students with the highest priority are chosen until r_t type-t students are chosen or no type-t student is left. In the second stage, for the remaining seats, students with the highest priority are chosen until all seats are filled or no student is left.

2 Results

Theorem 2 in Echenique and Yenmez (2015) claims that a choice rule is generated by reserves for priority $\succ$ if and only if it satisfies the following three axioms.

Gross Substitutes: If a student is chosen from a set S, then the student is also chosen from any subset of S that contains her.

Acceptance: A student is rejected only when all seats are filled.

A type t is **saturated** in a set of students if there is a set of students, with the same number of type-t students, in which not all type-t students are chosen.

Saturated $\succ$-compatibility: If a student s whose type is saturated is chosen over another student s', then s must have higher priority than s'.

The following example shows that the "only if part" of Theorem 2 in Echenique and Yenmez (2015) is not correct. In particular, the example illustrates a choice rule that is generated by reserves for a priority $\succ$ but violates *saturated $\succ$-compatibility*.[1]

Example 1. *Let* $\mathcal{S} \equiv \{1,2,3,4\}$, $T \equiv \{t_1, t_2\}$, $\tau(1) = \tau(2) = t_1$, $\tau(3) = \tau(4) = t_2$, $q = 1$, $r_{t_1} = 0$, $r_{t_2} = 1$, *and* $\succ$ *be defined as* $1 \succ 2 \succ 3 \succ 4$. *Let* C *be the choice rule that is generated by* (r_{t_1}, r_{t_2}) *for* $\succ$. *Note that* $C(\mathcal{S}) = \{3\}$. *Since* $4 \notin \mathcal{S}$, t_2 *is saturated in* S. *However,* $1 \notin C(\mathcal{S})$ *and* $1 \succ 3$, *implying that* C *violates* saturated $\succ$-compatibility.

We show that replacing *saturated $\succ$-compatibility* with the following two axioms repairs the result. The following axiom was introduced by Echenique and Yenmez (2015).

Within-type $\succ$-compatibility: If a student s is chosen over another student s' of the same type, then s must have higher priority than s'.

A type t is **saturated*** in a set of students S if there is a set of students S' such that the number of type-t students in S' is no more than the number of chosen type-t students in S and not all type-t students are chosen in S'. Formally, a type t is saturated* in S if there is S' such that $|S'_t| \leq |C(S) \cap S_t|$ and $S'_t \setminus C(S') \neq \emptyset$.

Saturated* $\succ$-compatibility: If a student s whose type is saturated* is chosen over another student s' of a different type, then s has higher priority than s'.[2]

Note that *within-type $\succ$-compatibility* is not implied by *saturated* $\succ$-compatibility*, although it is implied by *saturated $\succ$-compatibility*. Next, we show that replacing *saturated $\succ$-compatibility* with *within-type $\succ$-compatibility* and *saturated* $\succ$-compatibility* repairs Theorem 2 in Echenique and Yenmez (2015). Their proof cannot be repaired by simply invoking *saturated $\succ$-compatibility* and *within-type $\succ$-compatibility* instead of *saturated $\succ$-compatibility*. Therefore, we provide a self-contained new proof.

[1] In the proof of Theorem 2, Echenique and Yenmez (2015) do not attempt to prove that a choice rule that is generated by reserves for a priority $\succ$ satisfies *saturated $\succ$-compatibility*. Instead, they just claim that such an implication is immediate.

[2] Formally, if $s \in C(S)$, $s' \in S \setminus C(S)$, and $\tau(s)$ is saturated* in S, then $s \succ s'$.

Theorem 2*. *A choice rule is generated by reserves for priority $\succ$ if and only if it satisfies gross substitutes, acceptance, within-type $\succ$-compatibility, and saturated* $\succ$-compatibility.*

PROOF: Let C be a choice rule satisfying the axioms. We construct the reserve profile $(r_t)_{t\in T}$ as follows. For each $t \in T$, let $X_t \equiv \{S \subseteq \mathcal{S} : \exists s \in C(S), \exists s' \in S \setminus C(S)$ such that $\tau(s) = t, \tau(s') \neq t,$ and $s' \succ s\}$ and let $r_t \equiv \max\{|C(S) \cap S_t| : S \in X_t\}$. We claim that C is generated by $(r_t)_{t\in T}$ for priority $\succ$. Let $S \subseteq \mathcal{S}$.

We first show that, for each type t, at least $\min\{|S_t|, r_t\}$ type-t students are chosen in S. Suppose not. Then, $|C(S) \cap S_t| < |S_t|$ and $|C(S) \cap S_t| < r_t$. By definition of r_t, there is S' with $|C(S') \cap S'_t| = r_t$ such that $s \in C(S')$, $s' \in S' \setminus C(S')$, $\tau(s) = t$, $\tau(s') \neq t$, and $s' \succ s$. We consider two cases.

First, suppose that $|S_t| \leq r_t$. Now, type t is saturated* in S' because S has no more type-t students than the number of chosen type-t students in S' and not all type-t students are chosen in S. Then, by *saturated* $\succ$-compatibility*, $s \succ s'$, a contradiction.

Second, suppose that $|S_t| > r_t$. Let $S'' \subseteq S$ be the set of students obtained from S by removing $(|S_t| - r_t)$ non-chosen type-t students from S. By *gross substitutes*, $C(S'') \subseteq C(S)$. By *acceptance*, $|C(S'')| \leq |C(S)|$. Thus, $C(S'') = C(S)$.[3] Now, type t is saturated* in S' because S'' has no more type-t students than the number of chosen type-t students in S' and not all type-t students are chosen in S''. Then, by *saturated* $\succ$-compatibility*, $s \succ s'$, a contradiction.

Since at least $\min\{|S_t|, r_t\}$ type-t students are chosen in S, by *within-type $\succ$-compatibility*, for each type t, type-t students with the highest priority are chosen until the reserves for type t are filled or no type-t student is left. We claim that, for the remaining seats, students with the highest priority are chosen until all seats are filled or no student is left. By contradiction, suppose that among the remaining students there are s and s' such

[3] This also follows by Lemma 3 in Echenique and Yenmez (2015).

that $s \in C(S)$, $s' \in S \setminus C(S)$, and $s' \succ s$. Let $\tau(s) \equiv t$. By *within-type $\succ$-compatibility*, $\tau(s') \neq t$. Then, $S \in X_t$ and $|C(S) \cap S_t| \geq r_t + 1$, which is in contradiction to the definition of r_t.

Finally, let C be generated by reserves for priority $\succ$. *Acceptance* and *within-type $\succ$-compatibility* follow by definition. *Gross substitutes* is shown in Echenique and Yenmez (2015). To see *saturated* $\succ$-compatibility*, let $S \subseteq \mathcal{S}$, $s \in C(S)$, and $s' \in S \setminus C(S)$ be such that $\tau(s) = t$ and $\tau(s') \neq t$. Suppose that type-t is saturated* in S. Then, the number of chosen type-t students in S is more than r_t. Therefore, there is at least one type-t student who is considered together with s' in the second stage of the choice procedure and he is chosen over s', which implies that each chosen type-t student in S, in particular s, has priority over s'. □

In their online appendix, Echenique and Yenmez (2015) also consider the case where priorities are not exogenously given. Their Theorem $D.2.$ asserts that a choice rule is generated by reserves for some priority if and only if it satisfies *gross substitutes*, *acceptance*, and the following axiom called *saturated strong axiom of revealed preference*.

Saturated strong axiom of revealed preference (S-SARP): There are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k

1. $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
2. $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_{k+1})$ is saturated in S_{k+1} (using addition mod K).

The following example shows that the "only if part" of Theorem $D.2.$ is not correct, in particular, the example illustrates a choice rule that is generated by reserves for some priority but violates *S-SARP*.

Example 2. *Let* $\mathcal{S} \equiv \{1, 2, 3, 4, 5, 6\}$, $T \equiv \{t_1, t_2\}$, $\tau(1) = \tau(2) = \tau(3) = t_1$, $\tau(4) = \tau(5) = \tau(6) = t_2$, $q = 2$, $r_{t_1} = 0$, $r_{t_2} = 1$, *and* $\succ$ *be defined as* $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$. *Let* C *be the choice rule that is generated by* (r_{t_1}, r_{t_2}) *for* $\succ$. *Let* $s_1 = 2$, $s_2 = 5$, $S_1 = \{2, 3, 4, 5\}$, *and* $S_2 = \{1, 2, 5, 6\}$. *Note that* $C(S_1) = \{2, 4\}$ *and* $C(S_2) = \{1, 5\}$. *Also note that* t_1 *is saturated in* S_1 *and* t_2 *is saturated in* S_2. *And finally, note that the sequences* $\{s_1, s_2\}$ *and* $\{S_1, S_2\}$ *constitute a violation of S-SARP.*

We show that redefining *S-SARP* with the saturation* notion repairs the result.

Saturated* strong axiom of revealed preference (S*-SARP): There are no sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, such that, for all k

1. $s_{k+1} \in C(S_{k+1})$ and $s_k \in S_{k+1} \setminus C(S_{k+1})$;
2. $\tau(s_{k+1}) = \tau(s_k)$ or $\tau(s_{k+1})$ is saturated* in S_{k+1} (using addition mod K).

Theorem D.2*. *A choice rule is generated by reserves for some priority if and only if it satisfies gross substitutes, acceptance, and S*-SARP.*

PROOF: Let C be a choice rule satisfying the axioms. Let the binary relation $\succ^*$ over $\mathcal{S}$ be defined as follows. For each $s, s' \in \mathcal{S}$ such that $\tau(s) = \tau(s')$, $s \succ^* s'$ if and only if s is chosen over s' in a set S. For each $s, s' \in \mathcal{S}$ such that $\tau(s) \neq \tau(s')$, $s \succ^* s'$ if and only if s is chosen over s' in a set S where $\tau(s)$ is saturated*. By *S*-SARP*, $\succ^*$ has a linear extension $\succ$ to $\mathcal{S}$.

For any choice set, if a student s is chosen over another student s' of the same type, then $s \succ^* s'$ by definition of $\succ^*$. Since $\succ$ is an extension of $\succ^*$, also $s \succ s'$. Thus, C satisfies *within-type* $\succ$*-compatibility*. For any choice set, if a student s whose type is saturated* is chosen over another student s' of a different type, then $s \succ^* s'$ by definition of $\succ^*$. Since $\succ$ is an extension of $\succ^*$, also $s \succ s'$. Thus, C also satisfies *saturated** $\succ$*-compatibility*. Hence, by Theorem 1, C is generated by reserves for priority $\succ$.

For the other direction, suppose that a choice rule is generated by reserves for priority $\succ$. To see that it satisfies *S*-SARP*, suppose otherwise, that is, suppose that that there are sequences $\{s_k\}_{k=1}^K$ and $\{S_k\}_{k=1}^K$, of students and sets of students, respectively, with the properties described in the statement of *S*-SARP*. Note that for each k, $s_k \succ s_{k-1}$, which implies that $\succ$ admits a cycle, contradiction to $\succ$ being a linear order. $\square$

3 Independence of Axioms

The following examples of choice rules will be useful to show the independence of axioms.

Example 3. *Let $\mathcal{S} = \{1, 2, 3, 4\}$, $q = 2$, $\tau(1) = \tau(2) = t_1$, and $\tau(3) = \tau(4) = t_2$. For each $S \subseteq \mathcal{S}$ such that $S \neq \emptyset$, $C(S)$ is determined by the following procedure. If $|S| \leq 2$, then $C(S) = S$. Otherwise, if $|S \cap \{3, 4\}| = 1$, then first choose the type-t_2 student in S, then choose 1 if he is available, and finally choose 2 if he is available and there is an available seat; if $|S \cap \{3, 4\}| = 2$, then first choose 1 if he is available, then choose 2 if he is available, then choose 3 if he is available and there is an available seat, and finally choose 4 if he is available and there is an available seat.*

Example 4. *This is Example 1 in Echenique and Yenmez's (2015) online appendix. Let $\mathcal{S} = \{1, 2, 3\}$, $q = 2$, and $\tau(1) = \tau(2) = \tau(3) = t$. Let $C(\{1, 2, 3\}) = C(\{1, 2\}) = C(\{1, 3\}) = C(\{1\}) = \{1\}$, $C(\{2, 3\}) = \{2, 3\}$, $C(\{2\}) = \{2\}$, and $C(\{3\}) = \{3\}$.*

Example 5. *This is Example 3 in Echenique and Yenmez's (2015) online appendix. Let $\mathcal{S} = \{1, 2, 3, 4\}$, $q = 2$, and $\tau(1) = \tau(2) = \tau(3) = \tau(4) = t$. Let $C(\{1, 2, 3, 4\}) = C(\{1, 2, 3\}) = C(\{1, 2, 4\}) = \{1, 2\}$, $C(\{1, 3, 4\}) = \{1, 3\}$, $C(\{2, 3, 4\}) = \{2, 4\}$, and for any other S, $C(S) = S$.*

3.1 Independence of Axioms in Theorem 2*

Violating only *gross substitutes*: Consider the choice rule in Example 3. Let $\succ$ be such that $1 \succ 2 \succ 3 \succ 4$. Clearly, C satisfies *acceptance* and *within-type $\succ$-compatibility*. Note that a violation of *saturated* $\succ$-compatibility* is possible only if there exists a choice set S such that 3 or 4 is chosen over 1 or 2 and t_2 is saturated* in S. Now, since 3 or 4 is chosen over 1 or 2 only if $|S \cap \{3,4\}| = 1$, and since there is no choice set which includes exactly one type-t_2 student and at which the type-t_2 student is not chosen, t_2 cannot be saturated* in S. Thus, C satisfies *saturated* $\succ$-compatibility*. Finally, C violates *gross substitutes* since $2 \in C(\mathcal{S})$ but $2 \notin C(\mathcal{S} \setminus \{4\})$.

Violating only *acceptance*: Consider the choice rule in Example 4. Let $\succ$ be such that $1 \succ 2 \succ 3$. Clearly, C satisfies *gross substitutes*, *within-type $\succ$-compatibility*, and *saturated* $\succ$-compatibility*, but violates *acceptance*.

Violating only *within-type $\succ$-compatibility*: Consider the choice rule in Example 5. Let $\succ$ be such that $1 \succ 2 \succ 3 \succ 4$. Clearly, C satisfies *acceptance* and *gross substitutes*. Since all students are of the same type, C trivially satisfies *saturated* $\succ$-compatibility*. Finally, C violates *within-type $\succ$-compatibility* since 4 is chosen over 3 in $\{2,3,4\}$ although $3 \succ 4$.

Violating only *saturated* $\succ$-compatibility*: Consider the choice rule in Example 5 but suppose that each student has a different type. Let $\succ$ be such that $1 \succ 2 \succ 3 \succ 4$. Clearly, C satisfies *acceptance*, *gross substitutes*, and *within-type $\succ$-compatibility*. Finally, C violates *saturated* $\succ$-compatibility* since 4 is chosen over 3 in $\{2,3,4\}$ although $3 \succ 4$ and the type of 4 is saturated in $\{2,3,4\}$, since there is one student of 4's type in $\mathcal{S}$ and $4 \notin C(\mathcal{S})$.

3.2 Independence of Axioms in Theorem D.2*

Violating only *gross substitutes*: Consider the choice rule in Example 3. As we have already argued before, C satisfies *acceptance* and violates *gross substitutes*. We will show that C satisfies *S*-SARP*. Let R be the revealed preference relation where $s\ R\ s'$ if there is $S \in \mathcal{S}$ such that $s \in C(S)$, $s' \in S \setminus C(S)$, and either s and s' are of the same type or the type of s is saturated* in S. Clearly, $1\ R\ 2$ and $3\ R\ 4$, and there is no other s and s' of the same type with $s\ R\ s'$. In order R to have a cycle, there must be $s \in \{3, 4\}$ and $s' \in \{1, 2\}$ such that $s\ R\ s'$. However, $s \in \{3, 4\}$ is chosen over $s' \in \{1, 2\}$ in a set S only when s is the only type-t_2 student in S, in which case t_2 cannot be saturated* in S since there is no choice set which includes exactly one type-t_2 student and at which the type-t_2 student is not chosen. Thus, R is acyclic. Hence, C satisfies *S*-SARP*.

Violating only *acceptance*: Consider the choice rule in Example 4. Clearly, C satisfies *gross substitutes* and *S*-SARP*, since 1 is the only student who is chosen over some other student in some choice set. However, C violates *acceptance*.

Violating only *S*-SARP*: Consider the choice rule in Example 5. Clearly, C satisfies *acceptance* and *gross substitutes*. Note that the sequences $\{3, 4\}$ and $\{\{1, 3, 4\}, \{2, 3, 4\}\}$ constitute a violation of *S*-SARP*.

References

[1] Echenique, Federico and M. Bumin Yenmez. 2015. "How to Control Controlled School Choice." *American Economic Review*, 105(8): 2679-94.

[2] Echenique, Federico and M. Bumin Yenmez. 2015. "How to Control Controlled School Choice Online Appendix." Available at the AER website.