Non-Revelation Mechanisms for Many-to-Many Matching: Equilibria versus Stability*

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Abstract

We study many-to-many matching markets in which agents from a set A are matched to agents from a disjoint set B through a two-stage non-revelation mechanism. In the first stage, A-agents, who are endowed with a quota that describes the maximal number of agents they can be matched to, simultaneously make proposals to the B-agents. In the second stage, B-agents sequentially, and respecting the quota, choose and match to available A-proposers.

We study the subgame perfect Nash equilibria of the induced game. We prove that stable matchings are equilibrium outcomes if all A-agents' preferences are substitutable. We also show that the implementation of the set of stable matchings is closely related to the quotas of the A-agents. In particular, implementation holds when A-agents' preferences are substitutable and their quotas are non-binding.

Keywords: implementation; matching, mechanisms, stability, substitutability

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1 Introduction

We study many-to-many matching markets in which agents from a set A are matched to agents from a disjoint set B through a two-stage non-revelation mechanism. In the first stage, A-agents, who are endowed with a quota that describes the maximal number

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of agents they can be matched to, simultaneously make proposals to the B-agents. In the second stage, B-agents sequentially, and respecting the quota, choose and match to available A-proposers. Mechanisms where the agents on one side of the market apply simultaneously and then the agents on the other side choose sequentially are very common, e.g., in college admission and school choice (Roth and Sotomayor, 1990; Vulkan et al., 2013).

We study the subgame perfect Nash equilibria of the induced game. We prove that stable matchings are equilibrium outcomes if all A-agents' preferences are substitutable (Theorem 1); even if only one A-agent does not have substitutable preferences it can happen that some stable matching is not an equilibrium outcome (Example 1). We also show that the implementation of the set of stable matchings is closely related to the quotas of the A-agents (Theorem 2). In particular, implementation holds when A-agents' preferences are substitutable and their quotas are non-binding (Corollary 1). Interestingly, Corollary 1 essentially generalizes the simultaneous move game implementation results obtained by Alcalde and Romero-Medina (2000, Theorem 4.1) and Sotomayor (2004, Theorem 1) (see Remark 2). In a recent and independent paper, Romero-Medina and Triossi (2016) analyze subgame perfect Nash equilibria of a closely related simultaneous move game for many-to-many matching with contracts markets; due to the absence of quota on the proposing side, one of their results implies our Theorem 2 (see Remark 2).

In the context of many-to-one matching between students and colleges, Romero-Medina and Triossi (2014) introduce two sequential non-revelation mechanisms. They show that if colleges' preferences are substitutable, then the mechanisms implement the set of stable matchings in subgame perfect Nash equilibrium. More specifically, Romero-Medina and Triossi (2014) propose a mechanism, called the CSM (students apply Colleges Sequentially choose Mechanism), which coincides with our mechanism by taking the set A to be students, the set B to be colleges, and setting the quota for each agent (student) in A to be equal to one. Assuming furthermore that preferences of the agents in set B (colleges) are substitutable, Romero-Medina and Triossi (2014, Proposition 1) show that CSM implements the set of stable matchings. We provide examples that show that Proposition 1 of Romero-Medina and Triossi (2014) is tight in our more general many-to-many framework in the sense that under a slight relaxation of the assumptions, implementation needs no longer be possible (Examples 2 and 3).

Romero-Medina and Triossi (2014) also consider a mechanism, called the SSM (colleges apply Students Sequentially choose Mechanism), where colleges first simultaneously propose to students and then students sequentially pick a college. The SSM coincides with our mechanism by taking the set A to be colleges, the set B to be students, and not limiting the quota for each agent (college) in A. Romero-Medina and Triossi (2014, Proposition 2) show that SSM implements the set of stable matchings. Our Corollary 1 generalizes Romero-Medina and Triossi (2014, Proposition 2).

Finally, in Section 4, we discuss the validity of our results when using the stronger stability notion of setwise stability instead of (pairwise) stability: while Theorem 1 remains valid, Theorem 2 does not hold anymore.

2 Preliminaries

2.1 Many-to-many matching

There are two disjoint and finite sets of agents A and B. Let $I = A \cup B$ denote the *set of agents*. Generic elements of A, B, and I are denoted by a, b, and i, respectively. The *set of (possible) partners* of agent i is $T_i = B$ if $i \in A$, and $T_i = A$ if $i \in B$. The *preferences* of agent i are given by a linear order P_i over all subsets of set T_i, 2^{T_i}.[1] Let $\mathcal{P}_i$ denote the collection of all possible preferences for agent i. Since we fix the set of agents, a *(many-to-many matching) market* is given by a preference profile, i.e., a tuple $P = (P_i)_{i \in I}$. For each agent $i \in I$, let R_i denote the 'at least as desirable as' relation associated with P_i, i.e., for each pair $j, k \in T_i$, $j \, R_i \, k$ if and only if $j = k$ or $j \, P_i \, k$. For each agent i with preferences P_i, let $\mathsf{Ch}(\cdot, P_i)$ be the induced *choice function* on 2^{T_i}. In other words, for each set $T \subseteq T_i$, $\mathsf{Ch}(T, P_i)$ is agent i's most preferred subset of T according to P_i. A set of agents $T \subseteq T_i$ is *acceptable* to agent i at P if $T \, R_i \, \emptyset$.

A *matching* is a mapping from the set of agents I into $2^A \cup 2^B$ such that for each agent $a \in A$ and each agent $b \in B$, $\mu(a) \in 2^B$, $\mu(b) \in 2^A$, and $[a \in \mu(b) \Leftrightarrow b \in \mu(a)]$. For any agent $i \in I$, set $\mu(i)$ is called agent i's *match* (at μ). Next, we introduce (pairwise) stability.[2] Since the matching markets we consider are based on voluntary participation, we require a matching to be individually rational. Formally, a matching μ is *individually rational* if for all agents $i \in I$, $\mu(i)$ is individually rational, i.e., $\mathsf{Ch}(\mu(i), P_i) = \mu(i)$. Matching μ is *blocked by a pair* (of agents) $(a, b) \in A \times B$, $a \notin \mu(b)$,[3] if for all agents $i, j \in \{a, b\}$ with $i \neq j$, $j \in \mathsf{Ch}(\mu(i) \cup \{j\}, P_i)$. A matching μ is *(pairwise) stable* if it is individually rational and not blocked by any pair $(a, b) \in A \times B$. Let $\Sigma(P)$ denote the *set of stable matchings*. Note that the set of stable matchings $\Sigma(P)$ can be empty (see, e.g., Roth and Sotomayor, 1990, Example 2.7). A well-known sufficient condition for the non-emptiness of $\Sigma(P)$ is substitutability of all agents' preferences. The preferences P_i of an agent $i \in I$ are *substitutable*[4] if for all sets $T' \subseteq T_i$ and for all agents $j, j' \in T'$ with $j \neq j'$, $[\, j \in \mathsf{Ch}(T', P_i) \implies j \in \mathsf{Ch}(T' \backslash \{j'\}, P_i) \,]$. For a subset of agents $I' \subseteq I$, we say that $P_{I'} = (P_i)_{i \in I'}$ is substitutable if for all $i \in I'$, P_i is substitutable.

2.2 A class of non-revelation mechanisms

We assume that for each agent $a \in A$, there is an exogenous *quota*, given by a positive integer q_a, so that any match for agent a cannot have cardinality larger than q_a (for instance due to legal or physical constraints). We suppose that q_a is not smaller than the

[1] In other words, P_i is transitive, antisymmetric (strict), and total.

[2] In Section 4, we explain how our results would be affected if we used a stronger stability notion that is also often considered for many-to-many matching markets, setwise stability, instead of pairwise stability.

[3] When formulating blocking like this we need to make sure a and b are not already matched (otherwise a matched pair could block).

[4] Substitutability is an adaptation of the gross substitutability property (Kelso and Crawford, 1982) by Roth (1984) and Roth and Sotomayor (1990) to matching problems without monetary transfers.

largest acceptable match for agent a.[5] Let $q = (q_a)_{a \in A}$ denote the *quota vector*.

Let the set of agents $B = \{b_1, \ldots, b_k\}$. Let $\beta = (b_1, \ldots, b_k)$ be an order of the B-agents.

The [$\boldsymbol{A}$ simultaneously apply – $\boldsymbol{B}$ sequentially choose] mechanism $\boldsymbol{\varphi^{\beta,q}}$:

For each $a \in A$, let $r_a := q_a$.

STEP 0 (applications): A-agents simultaneously apply to sets of B-agents.

For each $a \in A$, *agent a's strategy* is the set $s_a \in 2^B$ of B-agents agent a applies to.[6] Let $s_A = (s_a)_{a \in A}$.

STEPS $l = 1, \ldots, k$ (choices): The set of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents are the A-agents that applied to b_l in Step 0 and that are still available, i.e., the set of agents $a \in A$ with $b_l \in s_a$ and $r_a > 0$. Agent b_l chooses a subset of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents. If an agent $a \in A$ is chosen by b_l, then they are (permanently) matched and we set $r_a := r_a - 1$.

For each agent $b_l \in B$, *agent b_l's strategy* is a choice function s_b that for each $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$ describes agent b_l's choice from the $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents.[7]

For any *strategy profile* $s = (s_i)_{i \in I}$, the outcome of non-revelation mechanism $\varphi^{\beta,q}$ is a well-defined matching and the mechanism induces an extensive form game. Let $\mathcal{E}^{\beta,q}(P)$ (or $\mathcal{E}(P)$ if no confusion is possible) denote the set of subgame perfect Nash equilibria (in pure strategies), SPE, at P, i.e., $\mathcal{E}^{\beta,q}(P)$ is the set of subgame perfect Nash equilibria strategy profiles. Similarly, let $\mathcal{O}^{\beta,q}(P)$ (or $\mathcal{O}(P)$ if no confusion is possible) denote the set of SPE outcomes at P, i.e., $\mathcal{O}^{\beta,q}(P)$ is the set of matchings that result from the set of SPE.

For any strategy profile s and any agent $i \in I$, let $s_{-i} = (s_j)_{j \in I \setminus \{i\}}$.

An example of a mechanism $\varphi^{\beta,q}$ is the application of students to public schools: a student cannot consume more that one school admission, but he is allowed to apply to more than one public school. Public schools process applications in sequence and once a student accepts an admission he is no longer available for later admissions.

3 Results

Our first result shows that when A-agents have substitutable preferences, the [A simultaneously apply – B sequentially choose] mechanism $\varphi^{\beta,q}$ implements in SPE a *super*set of the set of stable matchings.

[5] It could very well be that an agent finds matches that exceed a legally prescribed quota acceptable. We assume that, for all practical purposes, such an agent derives and uses "legal preferences" and a "legal choice function."

[6] The strategy-space for an A-agent is the set of all subsets of B, 2^B.

[7] The strategy-space for a B-agent is the set of all choice functions $\mathsf{Ch} : 2^A \to 2^A$ such that for each $A' \subseteq A$, $\mathsf{Ch}(A') \subseteq A'$. Note that the strategic choice function Ch of a B-agent does not need to be consistent with any strict preference ranking over the set of A-agents.

Theorem 1. (All stable matchings can be obtained as SPE outcomes) *For any (β, q) and any preference profile P where P_A is substitutable,*

$$\Sigma(P) \subseteq \mathcal{O}^{\beta,q}(P).$$

Examples 2 and 3 show that under the assumptions of Theorem 1, $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$ is possible.

Proof. Without loss of generality, let $\beta = (b_1, \ldots, b_k)$. Let P be a preference profile. Let matching μ be stable, i.e., $\mu \in \Sigma(P)$. Consider the following strategy profile s. Each agent $a \in A$ (only) applies to set $s_a = \mu(a)$. For each $b_l \in B$, let s_{b_l} be the strategy such that for each of agent b_l's decision nodes, he accepts the set of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents that he prefers most according to his true preferences P_{b_l}. Since for all $a \in A$, $|s_a| = |\mu(a)| \leq q_a$,[8] when we apply the mechanism $\varphi^{\beta,q}$ to strategy profile s, at each step $l = 1, \ldots, k$, the set of $(s_A, s_{b_1}, \ldots, s_{b_{l-1}})$-available agents is precisely $\mu(b_l)$. Since μ is stable at P, we have for all $l = 1, \ldots, k$, $\mathsf{Ch}(\mu(b_l), P_{b_l}) = \mu(b_l)$. Hence, $\varphi^{\beta,q}(s) = \mu$. In view of the optimality of the decisions of the B-agents, it only remains to show that no agent $a \in A$ has a profitable unilateral deviation, i.e., he cannot get matched to a more preferred set of B-agents.

Suppose to the contrary that for some agent $a' \in A$ such a deviation, say $s'_{a'}$, does exist. We show that then there exists a blocking pair for matching μ. Let strategy profile $s' = (s'_{a'}, s_{-a'})$ and matching $\mu' = \varphi^{\beta,q}(s')$. Since strategy $s'_{a'}$ is a beneficial deviation, $\mathsf{Ch}(\mu(a) \cup \mu'(a'), P_{a'})\ R_{a'}\ \mu'(a')\ P_{a'}\ \mu(a')$. Since matching μ is individually rational, $\mathsf{Ch}(\mu(a') \cup \mu'(a'), P_{a'}) \not\subseteq \mu(a')$. Let $b_l \in \mathsf{Ch}(\mu(a') \cup \mu'(a'), P_{a'}) \setminus \mu(a')$. Note that $b_l \notin \mu(a')$ and $b_l \in \mu'(a')$.

Since $b_l \in \mu'(a')$ it follows that when we apply the mechanism $\varphi^{\beta,q}$ to strategy profile s', at step l agent a' is one of the $(s'_A, s'_{b_1}, \ldots, s'_{b_{l-1}})$-available agents. In fact, the (only) other $(s'_A, s'_{b_1}, \ldots, s'_{b_{l-1}})$-available agents are $\mu(b_l)$. To see this, note first that for all $a \in A \backslash \{a'\}$, $s'_a = s_a = \mu(a)$ and $|\mu(a)| \leq q_a$. Therefore, agent $a \in A \backslash \{a'\}$ is $(s'_A, s'_{b_1}, \ldots, s'_{b_{l-1}})$-available if and only if $b_l \in \mu(a)$, i.e., $a \in \mu(b_l)$.

Then, in view of the optimality of agent b_l's decision at strategy profile s', it follows that $\mathsf{Ch}(\mu(b_l) \cup \{a'\}, P_{b_l}) = \mu'(b_l)$. Hence,

$$a' \in \mathsf{Ch}(\mu(b_l) \cup \{a'\}, P_{b_l}). \tag{1}$$

For agent a', by substitutability of preferences $P_{a'}$, $b_l \in \mathsf{Ch}(\mu(a') \cup \mu'(a'), P_{a'})$ implies

$$b_l \in \mathsf{Ch}(\mu(a') \cup \{b_l\}, P_{a'}). \tag{2}$$

Hence, (1) and (2) imply that (a', b_l) is a blocking pair for μ; a contradiction. □

The following example shows that substitutability of P_A cannot be omitted in Theorem 1. In fact, even if only one A-agent does not have substitutable preferences, then it can happen that some stable matching is not an equilibrium outcome.

[8] This follows from the fact that by stability of μ, $\mu(a)$ is an acceptable match for agent a, and our assumption that q_a is not smaller than the largest acceptable match for agent a.

Example 1. (P_A not substitutable and $\Sigma(P) \not\subseteq \mathcal{O}^{\beta,q}(P)$)
Consider the market with $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and preference profile P given by Table 1: in this and the following examples, we list only individually rational matches and better matches are ranked higher. Note that all preferences except for those of agent a_1 are substitutable.

a_1	a_2	b_1	b_2
$\{b_1, b_2\}$	$\boxed{\{b_2\}}$	$\{a_1\}$	$\{a_1\}$
$\boxed{\emptyset}$	$\emptyset$	$\boxed{\emptyset}$	$\boxed{\{a_2\}}$
			$\emptyset$

Table 1: Preference profile P in Example 1

Let quota vector $q = (q_{a_1}, q_{a_2}) = (2, 1)$ and let $\beta = (b_1, b_2)$ be the order of the B-agents. One easily verifies that the (boxed) matching

$$\mu : \begin{array}{cc} a_1 & a_2 \\ | & | \\ \emptyset & b_2 \end{array}$$

is stable, i.e., $\mu \in \Sigma(P)$. However, matching $\mu \notin \mathcal{O}^{\beta,q}(P)$. To see this, suppose $\mu \in \mathcal{O}^{\beta,q}(P)$. Let strategy profile $s \in \mathcal{E}^{\beta,q}(P)$ such that $\varphi^{\beta,q}(s) = \mu$. Let strategy $s'_{a_1} = \{b_1, b_2\}$ and strategy profile $s' = (s'_{a_1}, s_{-a_1})$. Then, at matching $\mu' = \varphi^{\beta,q}(s')$ agent a_1's match is $\mu'(a_1) = \{b_1, b_2\}$ which he strictly prefers to $\mu(a_1) = \emptyset$. Hence, $\mu \notin \mathcal{O}^{\beta,q}(P)$. $\diamond$

Romero-Medina and Triossi (2014) study a many-to-one matching model where a set of students S has to be matched to a set of colleges C. They assume that each student $s \in S$ finds it unacceptable to being matched to a set of two or more colleges (so, in particular each student s has substitutable preferences). Romero-Medina and Triossi (2014) propose a mechanism, called the CSM (students apply Colleges Sequentially choose Mechanism), which coincides with our mechanism $\varphi^{\beta,q}$ by taking set $A = S$, set $B = C$, and setting for each agent $a \in A$, quota $q_a = 1$.[9] Assuming furthermore that preferences P_B are substitutable, Romero-Medina and Triossi (2014, Proposition 1) show that in this particular case the mechanism implements the set of stable matchings, i.e., it is possible to obtain the other inclusion in Theorem 1: $\Sigma(P) \supseteq \mathcal{O}^{\beta,q}(P)$. Hence, Romero-Medina and Triossi (2014, Proposition 1) implies the following result in our more general many-to-many framework.

[9] Note that the CSM (Romero-Medina and Triossi, 2014, page 626) is formulated slightly differently in steps $l = 1, \ldots, k$ in that the set of available agents that a college can choose from when it is its turn is equal to the set of students that applied to that college and that were not matched with another college before. However, it is easy to see that an A-agent a with quota $q_a = 1$ (i.e., a student) is not matched with another college before if and only if $r_a = 1$ (recall that at the beginning of step 1, $r_a = 1$ and that r_a is only reduced when agent a is matched). Hence, if $A = S$, $B = C$, and for each agent $a \in A$, quota $q_a = 1$, then $\varphi^{\beta,q} = CSM$.

Proposition 1. (Romero-Medina and Triossi, 2014, Proposition 1) *For any* (β, q) *and any preference profile* P *where* P_B *is substitutable and for all* $a \in A$, $q_a = 1$,

$$\mathcal{O}^{\beta,q}(P) = \Sigma(P).$$

The next two examples show that Proposition 1 of Romero-Medina and Triossi (2014) is tight in the sense that under a slight relaxation of the assumptions, implementation needs no longer be possible.

The first example related to Proposition 1 of Romero-Medina and Triossi (2014) shows that an unstable SPE outcome may exist if some B-agent has preferences that are not substitutable (even when all other preferences are substitutable and all quotas equal 1).

Example 2. (For some $\boldsymbol{b \in B}$, $\boldsymbol{P_b}$ is not substitutable, for all $\boldsymbol{a \in A}$, $\boldsymbol{q_a = 1}$, and $\boldsymbol{\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)}$) Consider the market with $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and preference profile P given by Table 2. Note that all preferences except for those of agent b_2 are substitutable. Hence, by Theorem 1, $\Sigma(P) \subseteq \mathcal{O}^{\beta,q}(P)$.

a_1	a_2	b_1	b_2
$\{b_2\}$	$\{b_1\}$	$\boxed{\{a_1\}}$	$\{a_1, a_2\}$
$\boxed{\{b_1\}}$	$\boxed{\{b_2\}}$	$\{a_2\}$	$\boxed{\{a_2\}}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 2: Preference profile P in Example 2

Let quota vector $q = (q_{a_1}, q_{a_2}) = (1, 1)$ and let $\beta = (b_1, b_2)$ be the order of the B-agents. We show that $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$.

Let s be the strategy profile where $s_{a_1} = \{b_1\}$, $s_{a_2} = \{b_1, b_2\}$, and both B-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching

$$\mu : \begin{array}{cc} a_1 & a_2 \\ | & | \\ b_1 & b_2 \end{array}$$

is the resulting matching, i.e., $\mu = \varphi^{\beta,q}(s)$. We claim that strategy profile s is an SPE, i.e., $s \in \mathcal{E}^{\beta,q}(P)$. To see this, suppose there is a profitable deviation s'_{a_1} for agent a_1. Then, $s'_{a_1} = \{b_1, b_2\}$ or $s'_{a_1} = \{b_2\}$. However, in the first case, agent a_1 would again be matched to b_1. In the second case, agent a_1 would remain unmatched. Suppose now that there is a profitable deviation s'_{a_2} for agent a_2. Then, $s'_{a_2} = \{b_1\}$ which however would leave agent a_2 unmatched. Thus, $s \in \mathcal{E}^{\beta,q}(P)$ and $\mu \in \mathcal{O}^{\beta,q}(P)$. But since (a_1, b_2) is a blocking pair for μ, μ is not stable; i.e., $\mu \notin \Sigma(P)$. Hence, $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$. $\diamond$

The second example related to Proposition 1 of Romero-Medina and Triossi (2014) shows that an unstable SPE outcome may exist if some A-agent has a quota that is larger than 1 (even when all preferences are substitutable and all other quotas equal 1).

Example 3. (P substitutable, for some $a \in A$, $q_a > 1$, and $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$) Consider the market with $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and preference profile P given by Table 3. Note that all preferences are substitutable. Hence, by Theorem 1, $\Sigma(P) \subseteq \mathcal{O}^{\beta,q}(P)$.

a_1	a_2	b_1	b_2
$\{b_1, b_2\}$	$\boxed{\{b_1\}}$	$\{a_1\}$	$\{a_2\}$
$\boxed{\{b_2\}}$	$\{b_2\}$	$\boxed{\{a_2\}}$	$\boxed{\{a_1\}}$
$\{b_1\}$	$\emptyset$	$\emptyset$	$\emptyset$
$\emptyset$			

Table 3: Preference profile P in Example 3

Let quota vector $q = (q_{a_1}, q_{a_2}) = (2, 1)$ and let $\beta = (b_1, b_2)$ be the order of the B-agents. We show that $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$.

Let s be the strategy profile where $s_{a_1} = \{b_2\}$, $s_{a_2} = \{b_1, b_2\}$, and both B-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching

$$\mu : \begin{array}{cc} a_1 & a_2 \\ | & | \\ b_2 & b_1 \end{array}$$

is the resulting matching, i.e., $\mu = \varphi^{\beta,q}(s)$. We claim that strategy profile s is an SPE, i.e., $s \in \mathcal{E}^{\beta,q}(P)$. To see this, note that agent a_2 gets his most preferred match and that B-agents choose optimally. Hence, a_1 is the only possible candidate for a profitable deviation. Suppose there is a profitable deviation s'_{a_1} for agent a_1. Then, $s'_{a_1} = \{b_1\}$ or $s'_{a_1} = \{b_1, b_2\}$. However, in both cases one easily verifies that at strategy profile $s' = (s'_{a_1}, s_{-a_1})$ agent a_1 is matched to $\{b_1\}$. Hence, s'_{a_1} is not a profitable deviation for agent a_1. Thus, $s \in \mathcal{E}^{\beta,q}(P)$ and $\mu \in \mathcal{O}^{\beta,q}(P)$. But since (a_1, b_1) is a blocking pair for matching μ, μ is not stable; i.e., $\mu \notin \Sigma(P)$. Hence, $\Sigma(P) \subsetneq \mathcal{O}^{\beta,q}(P)$. $\diamond$

Example 3 shows that if some quota is larger than 1, then not all equilibrium outcomes need to be stable. We next show that if however *all* quotas are *sufficiently large*, then all equilibrium outcomes are guaranteed to be stable matchings (without any assumptions on the preferences!).

We say that quotas are *non-binding* if for all agents $a \in A$, $q_a \geq |B|$. When quotas are non-binding, at any strategy profile s, no A-agent ever becomes unavailable, i.e., if an agent a decides to apply to set s_a, then any agent $b \in s_a$ can choose agent a in any of its decision nodes.

Our second result shows that when quotas are non-binding, the [A simultaneously apply – B sequentially choose] mechanism $\varphi^{\beta,q}$ implements in SPE a *sub*set of the set of stable matchings.

Theorem 2. (Non-binding quotas guarantee stability in equilibrium)
For any (β, q) and any preference profile P where quotas are non-binding,

$$\mathcal{O}^{\beta,q}(P) \subseteq \Sigma(P).$$

Proof. Let P be a preference profile. Let matching μ be an SPE outcome, i.e., $\mu \in \mathcal{O}^{\beta,q}(P)$. Suppose matching μ is not stable, i.e., $\mu \notin \Sigma(P)$. Let strategy profile $s \in \mathcal{E}^{\beta,q}(P)$ such that $\mu = \varphi^{\beta,q}(s)$.

We first show that since μ is an equilibrium outcome, it is individually rational. In view of the optimality of the decisions of the B-agents in equilibrium, a B-agent always chooses an individually rational set of A-agents. Suppose that there exists an agent $a \in A$ for whom $\mu(a)$ is not individually rational. Hence, there exist a non-empty set of "undesirable agents" $\{\bar{b}_1, \ldots, \bar{b}_k\} \subseteq B$ such that $\mu(a) = \mathsf{Ch}(\mu(a), P_a) \cup \{\bar{b}_1, \ldots, \bar{b}_k\}$, i.e., agent a would be better off not being matched to the agents in $\{\bar{b}_1, \ldots, \bar{b}_k\}$. Let strategy $s'_a = \mathsf{Ch}(\mu(a), P_a)$ and strategy profile $s' = (s'_a, s_{-a})$. Note that each agent in s'_a receives the same set of applications at strategy profile s and at strategy profile s'. Since quotas are non-binding, at strategy profile s' each agent in set s'_a chooses the same set of agents as at strategy profile s and hence the chosen set will again include agent a. Hence, at strategy profile s' agent a will be matched to $\mathsf{Ch}(\mu(a), P_a)$ and s'_a is a profitable deviation for agent a; a contradiction.

Thus, since matching μ is not stable, there is a blocking pair $(a, b) \in A \times B$ with $a \notin \mu(b)$, $b \in \mathsf{Ch}(\mu(a) \cup \{b\}, P_a)$, and $a \in \mathsf{Ch}(\mu(b) \cup \{a\}, P_b)$.

Let strategy $s''_a = \mathsf{Ch}(\mu(a) \cup \{b\}, P_a)$ and strategy profile $s'' = (s''_a, s_{-a})$. We show that strategy s''_a is a profitable deviation for agent a. Since $b \in \mathsf{Ch}(\mu(a) \cup \{b\}, P_a)$, $\mathsf{Ch}(\mu(a) \cup \{b\}, P_a)\, P_a\, \mu(a)$ and it suffices to show that at strategy profile s'' each agent in s''_a chooses a.

Note that $s''_a \subseteq \mu(a) \cup \{b\}$. Hence, each agent in $s''_a \backslash \{b\}$ receives the same set of applications at strategy profile s and at strategy profile s''. Since quotas are non-binding, at strategy profile s'' each agent in set $s''_a \backslash \{b\}$ chooses the same set of agents as at strategy profile s and hence the chosen set will again include agent a.

Next, we prove that $b \notin s_a$. Suppose to the contrary that $b \in s_a$. Then, agent $a \in \{\bar{a} \in A : b \in s_{\bar{a}}\}$. Since $\mu(b) = \mathsf{Ch}(\{\bar{a} \in A : b \in s_{\bar{a}}\}, P_b)$ it follows that $\mu(b) = \mathsf{Ch}(\mu(b) \cup \{a\}, P_b)$. Thus, $a \in \mathsf{Ch}(\mu(b) \cup \{a\}, P_b)$ implies $a \in \mu(b)$; a contradiction. So, $b \notin s_a$.

Since $b \in s''_a \backslash s_a$ and strategy profile s'' only contains a unilateral deviation from strategy profile s, at strategy profile s'' agent b receives the same set of applications as at strategy profile s *and* in addition the application of a. In other words, $\{\bar{a} \in A : b \in s''_{\bar{a}}\} = \{\bar{a} \in A : b \in s_{\bar{a}}\} \cup \{a\}$. Suppose agent b does not choose agent a at strategy profile s''. Then, $a \notin \mathsf{Ch}(\{\bar{a} \in A : b \in s''_{\bar{a}}\}, P_b) = \mathsf{Ch}(\{\bar{a} \in A : b \in s_{\bar{a}}\} \cup \{a\}, P_b) = \mathsf{Ch}(\mu(b) \cup \{a\}, P_b)$, where the last equality follows from $a \notin \mu(b) = \mathsf{Ch}(\{\bar{a} \in A : b \in s_{\bar{a}}\})$. Since we obtain a contradiction to $a \in \mathsf{Ch}(\mu(b) \cup \{a\}, P_b)$, it follows that agent b chooses agent a at strategy profile s''. This shows that strategy s''_a is a profitable deviation for agent a; a contradiction. □

Remark 1. Note that in Theorem 2, instead of requiring that quotas are non-binding, we could restrict A-agents' strategies to not exceed their quotas: the result and the proof would then remain the same. However note that limiting the number of applications an A-agent can submit might be difficult to enforce in practice, e.g., a student is usually allowed to apply to more than one college even though in the end (s)he will only be able to attend one college at a time. ⋄

An immediate consequence of Theorem 2 is that for any (β, q) whenever preference profile P is such that quotas are non-binding and no stable matching exists, there is no SPE (in pure strategies).

The following result is a corollary to Theorems 1 and 2.

Corollary 1. (Implementation)
For any (β, q) and any preference profile P where P_A is substitutable and quotas are non-binding,

$$\mathcal{O}^{\beta,q}(P) = \Sigma(P).$$

The assumption that P_A is substitutable cannot be omitted from Corollary 1: if we let quota vector $q = (2, 2)$ (instead of $(2, 1)$) in Example 1, then for the resulting market P we have $\mathcal{O}^{\beta,q}(P) \neq \Sigma(P)$.

Remark 2. (Simultaneous move games)
Note that when quotas are non-binding, the second phase of mechanism $\varphi^{\beta,q}$ essentially transforms into a simultaneous-move game among B-agents. Games in which first a set of A-agents move simultaneously and then a set of B-agents move simultaneously are also studied in Alcalde and Romero-Medina (2000) for many-to-one matching markets with substitutable preferences and Sotomayor (2004) for many-to-many matching markets with responsive preferences. As a consequence, Corollary 1 essentially generalizes the simultaneous move game implementation results of the set of stable matchings obtained by Alcalde and Romero-Medina (2000, Theorem 4.1) and Sotomayor (2004, Theorem 1).

In a recent and independent paper, Romero-Medina and Triossi (2016) analyze subgame perfect Nash equilibria of a closely related simultaneous move game for many-to-many matching with contracts markets. They show that all SPE outcomes are stable (Romero-Medina and Triossi, 2016, Proposition 1), that SPE exist when A-agents have substitutable and B-agents have unilaterally substitutable preferences (Romero-Medina and Triossi, 2016, Proposition 2), and that the set of SPE outcomes forms a non-empty lattice if all agents have substitutable preferences. Due to the absence of quota for A-agents, all results for SPE of the simultaneous move game also hold when B-agents move sequentially (Romero-Medina and Triossi, 2016, Theorem 2). Hence, in particular, Romero-Medina and Triossi (2016, Proposition 1) implies our Theorem 2. ⋄

Corollary 1 subsumes results obtained by Romero-Medina and Triossi (2014, Proposition 2) for many-to-one matching markets with substitutable preferences and Sotomayor (2003, Theorems 1 and 2) for one-to-one matching markets. More specifically, Romero-Medina and Triossi (2014) consider a mechanism, called the SSM (colleges apply Students

Sequentially choose Mechanism), where colleges first simultaneously propose to students and then students sequentially pick a college. The SSM coincides with our mechanism $\varphi^{\beta,q}$ by taking set $A = C$, set $B = S$, and setting for each $a \in A$, $q_a = |B|$.

Corollary 2. (Romero-Medina and Triossi, 2014, Proposition 2)
For any (β, q) and any preference profile P where P_A is substitutable, for all agents $b \in B$ and for all $T \subseteq A$, $[|T| \geq 2 \Rightarrow \emptyset \; P_b \; T]$, and for all agents $a \in A$, $q_a = |B|$,

$$\mathcal{O}^{\beta,q}(P) = \Sigma(P).$$

4 Concluding remark: setwise stability

For many-to-many matching markets, the following stronger stability notion is also often considered. Let P be a preference profile. Then, matching μ is *blocked by a set* (of agents) $I' = A' \cup B' \subseteq A \cup B$, $I' \neq \emptyset$, if there exists a matching μ' such that (a) for all $i \in I$, $\mu'(i) \setminus \mu(i) \subseteq I'$ —new matches are among the members of the blocking coalition only— and (b) for all $i \in I'$, $\mu'(i) \; P_i \; \mu(i)$ and $\mu'(i) = \mathsf{Ch}(\mu'(i), P_i)$ —all members of the blocking coalition receive a better and individually rational match. Note that agents outside the blocking coalition are not matched to new agents, but possibly some of their matches are canceled by members of the blocking coalition. A matching μ is *setwise stable* if it is individually rational and not blocked by any set of agents $I' = A' \cup B'$. Let $\Omega(P)$ denote the *set of setwise stable matchings*.

First, note that a setwise stable matching is always (pairwise) stable, i.e., for all preference profiles P, $\Omega(P) \subseteq \Sigma(P)$. Hence, Theorem 1 would also hold if we used setwise stability instead of (pairwise) stability.

Second, we show that a result similar to Theorem 2 cannot be obtained if we used setwise stability instead of (pairwise) stability.

Example 4. (Setwise stability not obtained in equilibrium)
Consider the market introduced by Blair (1988, Example 2.6) where $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2, b_3\}$, and preference profile P is given by Table 4. Note that all preferences are substitutable.

a_1	a_2	a_3	b_1	b_2	b_3
$\{b_1, b_2\}$	$\{b_2, b_3\}$	$\{b_1, b_3\}$	$\{a_1, a_2\}$	$\{a_2, a_3\}$	$\{a_1, a_3\}$
$\boldsymbol{\{b_2, b_3\}}$	$\boldsymbol{\{b_1, b_3\}}$	$\boldsymbol{\{b_1, b_2\}}$	$\boldsymbol{\{a_2, a_3\}}$	$\boldsymbol{\{a_1, a_3\}}$	$\boldsymbol{\{a_1, a_2\}}$
$\boxed{\{b_1\}}$	$\boxed{\{b_2\}}$	$\boxed{\{b_3\}}$	$\boxed{\{a_1\}}$	$\boxed{\{a_2\}}$	$\boxed{\{a_3\}}$
$\{b_2\}$	$\{b_1\}$	$\{b_1\}$	$\{a_2\}$	$\{a_1\}$	$\{a_1\}$
$\{b_3\}$	$\{b_3\}$	$\{b_2\}$	$\{a_3\}$	$\{a_3\}$	$\{a_2\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Table 4: Preference profile P in Example 4

Let quota vector $q = (q_{a_1}, q_{a_2}, q_{a_3}) = (2, 2, 2)$ and let $\beta = (b_1, b_2, b_3)$ be the order of the B-agents. We show that $\Omega(P) \subsetneq \mathcal{O}^{\beta,q}(P)$. First, Blair (1988) shows that even though a unique stable (boxed) matching

$$\mu : \begin{array}{ccc} a_1 & a_2 & a_3 \\ | & | & | \\ b_1 & b_2 & b_3 \end{array}$$

exists, it can be setwise blocked by $I' = A \cup B$ through the boldfaced matching

$$\mu' : \begin{array}{ccc} a_1 & a_2 & a_3 \\ | & | & | \\ \{b_2, b_3\} & \{b_1, b_3\} & \{b_1, b_2\}. \end{array}$$

Thus, $\Sigma(P) = \{\mu\}$ and $\Omega(P) = \emptyset$.

Next, we show that matching μ is an SPE outcome, i.e., $\mu \in \mathcal{O}^{\beta,q}(P)$. Let s be the strategy profile where $s_{a_1} = \{b_1\}$, $s_{a_2} = \{b_2\}$, $s_{a_3} = \{b_3\}$, and all B-agents choose optimally according to their preferences in all their decision nodes. One easily verifies that the (boxed) matching μ is the resulting matching, i.e., $\mu = \varphi^{\beta,q}(s)$.

We claim that strategy profile s is an SPE, i.e., $s \in \mathcal{E}^{\beta,q}(P)$. To see this, suppose there is a profitable deviation s'_{a_1} for agent a_1. Then, $s'_{a_1} = \{b_1, b_2\}$, $s'_{a_1} = \{b_2, b_3\}$, or $s'_{a_1} = \{b_1, b_2, b_3\}$. However, in the first case, agent a_1 would again be matched with $\{b_1\}$, in the second case, agent a_1 would be matched with $\{b_3\}$, and in the third case, agent a_1 would be matched with $\{b_1, b_3\}$. Hence, s'_{a_1} is not a profitable deviation for agent a_1. Similarly, we can show that neither agent a_2 nor agent a_3 has a profitable deviation. Thus, $s \in \mathcal{E}^{\beta,q}(P)$ and $\mu \in \mathcal{O}^{\beta,q}(P)$. Hence, $\Omega(P) \subsetneq \mathcal{O}^{\beta,q}(P)$. ◇

Note that Example 4 remains valid with non-binding quotas, e.g., $q = (3, 3, 3)$. Thus, Example 4 shows that in many-to-many matching markets with substitutable preferences and non-binding quotas, an implementation result for *setwise* stable matchings similar to Corollary 1 need not hold. Interestingly, for the variation of our mechanisms where A-agents simultaneously apply and B-agents simultaneously choose, Echenique and Oviedo (2006, Corollary 7.2) show that the set of setwise stable matchings can be implemented if A-agents have substitutable preferences and B-agents have so-called strongly substitutable preferences. The preferences P_i of an agent $i \in I$ are *strongly substitutable* if for all $j \in T_i$ and for all sets $T', T \subseteq T_i$ with $T' \, P_i \, T$, $[\, j \in \mathsf{Ch}(T' \cup \{j\}, P_i)$ implies $j \in \mathsf{Ch}(T \cup \{j\}, P_i)\,]$. In Example 4, all agents' preferences are substitutable but not strongly so.[10] Because in our setting the effect of B-agents moving simultaneously can be obtained via non-binding quota, an implication of Echenique and Oviedo (2006, Corollary 7.2) is the following corollary.

Corollary 3. (Implementation)
For any (β, q) and any preference profile P where P_A is substitutable, P_B is strongly substitutable, and quotas are non-binding,

$$\mathcal{O}^{\beta,q}(P) = \Omega(P).$$

[10]For instance, P_{b_1} violates strong substitutability since $T' = \{a_2, a_3\} \, P_{b_1} \, \{a_1\} = T$ and $a_3 \in \mathsf{Ch}(T' \cup \{a_3\}, P_{b_1})$, but $a_3 \notin \mathsf{Ch}(T \cup \{a_3\}, P_{b_1})$.

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