Solidarity for public goods under single-peaked preferences: Characterizing target set correspondences*

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Abstract

We consider the problem of choosing a set of locations of a public good on the real line $\mathbb{R}$. Similarly to Klaus and Storcken (2002), we ordinally extend the agents' preferences over compact subsets of $\mathbb{R}$, and extend the results of Ching and Thomson (1996), Vohra (1999), and Klaus (2001) to choice correspondences. Specifically, we show that *efficiency* and either *population-monotonicity* or *one-sided replacement-dominance* characterize the class of target set correspondences on the domains of single-peaked preferences and symmetric single-peaked preferences.

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1 Introduction

We study the social choice problem where a non-empty and compact set (of points) is chosen on the real line $\mathbb{R}$. We consider this (chosen) set to represent a public good such that each point in the set represents an option for the public good together with its location. We assume that agents have single-peaked preferences, that is, an agent's welfare is strictly increasing up to a certain point, his "peak", and is strictly decreasing beyond this point. Given a non-empty and compact set (of points) that represents the public good's options and their locations, an agent is unable to compute his chance of obtaining the public good at a particular location, e.g., in the case of parking spaces along a street, an agent might know that he will (eventually) find a parking spot but he does not know where this will be. We therefore assume that agents, when comparing sets, only consider their best (most favorite) point(s) and their worst (least favorite) point(s) in each set. Finally, we assume that the set has adequate capacity to accommodate all agents, that is, all agents have access to the public good but possibly at different locations.

More specifically, we look into the situation where the social planner wishes to make a choice by providing the public good in a way that is *efficient*, according to the agents' preferences, and that satisfies some notion of solidarity between agents towards changes in circumstances. Loosely speaking, solidarity requires that all agents not responsible for the change should be affected in the same direction. The changes in circumstances we study in this paper are changes in the agents' population, by considering the property of *population-monotonicity*, and changes in some agents' preferences, by considering the property of *replacement-dominance*. *Population-monotonicity*, introduced in the context of bargaining (Thomson, 1983a,b), applies to a model with a variable population of agents and requires that if additional agents join a population, then the agents who were initially present should all be made at least as well off, as they were initially, or they should all be made at most as well off. *Replacement-dominance*, introduced in the context of quasi-linear binary public decision (Moulin, 1987), applies to a model with a fixed population of agents and requires that if the preferences of an agent change, then the other agents whose preferences remained unchanged should all be made at least as well off, as they were initially, or they should all be made at most as well off.

Further to the parking zone example, already briefly mentioned and further explained in Section 2, another example of the described situation could be the following. A social planner drafts an "if-needed" list of candidate locations to build a public hospital according to the agents' preferences. She does so in an effort to narrow down future construction scenarios while at the same time respecting (in an *efficient* sense) the agents' preferences and adhering to some notion of solidarity, as described above. Then, if at some future time the need to build a hospital materializes, each location in this list is scrutinized and one of them is chosen for the hospital to be built at, with this final verdict assumed unpredictable at the time when the list is drafted.

Many more social choice problems can be phrased as problems of providing a public good by choosing the location of it on the real line $\mathbb{R}$ or an interval of it, or more generally, on a tree network,[1] when agents have single-peaked preferences. In these types of problems, it is very natural for changes in the population (e.g., through a change in the birth or migration rate) or changes in the agents' preferences (e.g., through the influence of public media or social networks) to arise. Hence, the properties of *population-monotonicity* and *replacement-dominance* have been studied, together or individually, in a variety of contexts. For the special case where the tree network is a closed interval, the problem coincides with the problem of providing a public good by choosing its level when agents have single-peaked preferences (Moulin, 1980). Apart from the provision of public parking or the provision of a hospital by choosing an "if-needed" list of locations, further examples of providing a public good in one or more locations include the provision of (one or more) schools, parks, or libraries on a tree network that represents an infrastructure, e.g., the network of roads in a neighborhood.

For choice functions that assign a public good on an interval, or on a tree network, the solidarity properties *population-monotonicity* and *replacement-dominance*, have been considered. Specifically, for the location problem on an interval (on a tree network), it was shown that *efficiency* and *population-monotonicity* characterize the class of "target point functions" on the domain of single-peaked preferences (Ching and Thomson, 1996; Thomson, 1993).[2] and for constant sets of agents *efficiency* and *replacement-dominance* characterize

[1]A tree network is a connected graph that contains no cycles.

[2]Each target point function is determined by its target point: if the target point is *efficient*, it is chosen; if it is not *efficient*, the closest *efficient* point is chosen. Such functions are sometimes called status quo rules

the class of "target point functions" on the domains of single-peaked preferences and symmetric single-peaked preferences (Vohra, 1999). Moreover, it turns out that *efficiency* and *population-monotonicity* imply *replacement-dominance* and also, that the former characterization also holds on the domain of symmetric single-peaked preferences and on tree networks (Klaus, 2001). In addition, both aforementioned characterizations hold under much looser assumptions on the set of locations (alternatives) and the domain of preferences (Gordon, 2007a).[3] Finally, if the set of admissible preferences is constrained on attribute-based preference domains,[4] *efficiency* and either one of the two solidarity properties are only compatible on discrete trees, where equivalent characterizations are obtained (Gordon, 2015).

For the location problem on an interval, if the property of *replacement-dominance* is weakened to ϵ-*replacement-dominance*[5] the characterization of target point functions still holds for the domain of single-peaked preferences (Harless, 2015b). However, for the location problem on a circle when a constant set of agents exists, no choice function satisfies *efficiency* and either *replacement-dominance* or *population-monotonicity* on the domain of symmetric single-peaked preferences (Gordon, 2007b).

Regarding choice correspondences, the case of providing a public good at exactly two locations, when one or both of the aforementioned solidarity properties are being considered, has been studied under different settings. On the domain of single-peaked preferences and if the agents compare pairs of locations using the max-extension,[6] the following holds. For an interval in $\mathbb{R}$ and a constant set of agents, the class of choice functions satisfying *efficiency* and *replacement-dominance* are the "left-peaks choice function" and the "right-peaks choice function"[7] (Miyagawa, 2001). However, if this model is extended to trees, then no choice function satisfies *efficiency* and *replacement-dominance* on the symmetric single-peaked domain (Umezawa, 2012).

or status quo solutions.

[3]The critical assumptions are: (i) the set of alternatives is fixed, (ii) the agents' preferences are defined over all alternatives, and (iii) the domain of preferences is common to all agents.

[4]Given a finite set of alternatives A, the non-empty and finite family of subsets $\mathcal{H} \subseteq 2^A$ is an attribute space if [for each attribute $H \in \mathcal{H}$, $H \neq \emptyset$ and the complement $H^C \in \mathcal{H}$] and [for each pair $x, y \in A$ with $x \neq y$, there exists $H \in \mathcal{H}$ such that $x \in H$ and $y \notin H$].

[5]Agents' solidarity is only required if the change in an agent's preferences are below a certain threshold.

[6]Under the max-extension, an agent prefers set X to set Y if and only if he prefers his best point(s) in set X to his best point(s) in set Y.

[7]The left (right) peaks choice function chooses the two unique left-most (right-most) peaks.

For the problem of providing a public good at exactly two locations on an interval, on the domain of single-peaked preferences and if agents compare pairs of locations using the leximin-extension,[8] the following two results have been obtained that consider *population-monotonicity* or *replacement-dominance*. First, for a constant set of agents the class of choice functions satisfying *efficiency*, *anonymity*, and *population-monotonicity* is the class of "single-plateaued preference choice functions"[9] (Ehlers, 2003); and second, in the same setting, the class of choice functions satisfying *efficiency* and *replacement-dominance* is the class of "single-peaked preference choice functions"[10] (Ehlers, 2002).

In the setting of preference aggregation problems, where agents strictly rank a finite set of alternatives and a (not necessarily strict) social ranking over the alternatives must be chosen, the aforementioned solidarity properties have also been studied. It is shown that on the domain of strict rankings, *efficiency* and *population-monotonicity* characterize the class of "strict status-quo functions"[11] (Bossert and Sprumont, 2014). Moreover, in this result, *population-monotonicity* can be substituted with *adjacent replacement-dominance*.[12] Furthermore, if the domain is enlarged to also include weak rankings, *efficiency* and either *population-monotonicity* or *adjacent replacement-dominance* characterize the class of "status-quo functions"[13] (Harless, 2016).

[8]Under the leximin-extension, in the case of sets containing exactly two points, an agent prefers set X to set Y if and only if he either [prefers his best point(s) in set X to his best point(s) in set Y] or [he is indifferent between his best point in set X and his best point in set Y and prefers his second best point in set X to his second best point in set Y].

[9]Each single-plateaued preference choice function is determined by fixed single-plateaued preferences R and plateau $[\underline{r}, \bar{r}]$: if all the agents' peaks lie outside of $[\underline{r}, \bar{r}]$, then loosely speaking, the best of the agents' peaks and its indifferent point are chosen (according to R); otherwise, the two locations in the convex hull of the agents' peaks lying closest to $\underline{r}$ and $\bar{r}$ respectively are chosen.

[10]Each single-peaked preference choice function is essentially a single-plateaued preference choice function determined by a fixed single-plateaued preference relation R with the plateau being a point, i.e., $\underline{r} = \bar{r}$.

[11]Each strict status-quo function is determined by a strict ranking R over the alternatives and reaches a unique efficient strict ranking as follows: beginning from R it reverses the order of an adjacently ranked pair of alternatives if all agents prefer the reverse to the initial ranking of the pair.

[12]*Adjacent replacement-dominance* is weaker than *replacement-dominance*: solidarity is only required when an agent reverses a single pair of adjacently ordered alternatives.

[13]Each status-quo function is determined by a ranking $\bar{R}$ over the alternatives and reaches a unique efficient ranking as follows: beginning from $\bar{R}$ it reverses the order of an adjacently ranked pair of single alternatives if all agents prefer the reverse to the initial ranking of the pair. Moreover, it "creates" order in an indifference class (of alternatives) if all agents prefer the alternative moved up in the order to the one (or

Finally, in the binary social choice model (i.e., when there are exactly two alternatives to choose from) and if agents can be indifferent between the two alternatives, a choice function satisfies *replacement-dominance* or *population-monotonicity* if and only if it is a "generalized mixed-consensus rule"[14] (Harless, 2015a).

All the above mentioned work analyzes solidarity properties where at each preference profile, either at most two alternatives are chosen or a ranking over the alternatives is chosen. In this paper we study a class of problems where more than two alternatives might be chosen, which are viewed as locations to provide a public good. This has been considered in a median voter context where the standard choice function setup is extended to choice correspondences since for an even number of agents or voters, a set of median voter locations exists, hence choosing the median implies choosing a set of median points (Klaus and Storcken, 2002). To capture the full spirit of this median voter result, Klaus and Storcken (2002) considered choice correspondences. Our motivation for extending choice from one or two locations to a set of locations is that we study situations in which the public good is usually provided through "larger" sets of options, e.g., the assignment of neighborhood parking spots along a street.

On the domain of single-peaked preferences as well as the smaller domain of symmetric single-peaked preferences, we show that the class of choice correspondences satisfying *efficiency* and either *one-sided replacement-dominance*[15] or *population-monotonicity*, is the class of target set correspondences (Theorems 1 and 2). Each target set correspondence is determined by a target set $[a, b]$: if this set is *efficient*, it is chosen; if it is not *efficient*, then its largest *efficient* subset is chosen, if such a subset exists; otherwise, the closest *efficient* point to the target set is chosen. We also show that *efficiency* and *replacement-dominance* characterize the sub-class of target set correspondences where $a = b$, i.e., we obtain the class of target point functions (Corollary 3). Hence, we obtain corresponding results with the literature

more) alternatives moved down. Reversals in the order between a single alternative and an indifference class or between two indifference classes occur in a similar way.

[14] Each generalized mixed-consensus rule chooses for each profile either alternative a or alternative b. The only further requirement concerns cases where at least one agent prefers a over b and at least one agent prefers b over a; specifically, either a is selected in all such cases or b is selected in all such cases.

[15] *One-sided replacement-dominance* is weaker than *replacement-dominance*: solidarity is not required when the preferences of the agent with the unique smallest peak are changed such that he becomes the agent with the unique largest peak, and vice-versa.

(Ching and Thomson, 1996; Thomson, 1993; Vohra, 1999).

Our results are parallel to the case where the public good is provided via a lottery over locations on an interval, and probabilistic target choice functions are characterized on the basis of *efficiency* and either *one-sided replacement-dominance* or *population-monotonicity* (Ehlers and Klaus, 2001).

The paper proceeds as follows. Section 2 explains the model and states some preliminary results. Section 3 contains the definition of target set correspondences. Section 4 contains the solidarity properties and further preliminary results. Section 5 presents characterizations of target set correspondences.

2 The model

Denote the set of natural numbers by $\mathbb{N}$. There is a *grand population* of "potential" agents, indexed by $\mathbb{P} \subseteq \mathbb{N}$, where $\mathbb{P}$ contains at least 3 agents. We denote the class of non-empty and finite subsets of $\mathbb{P}$ by $\mathcal{P}$. A set of agents $N \in \mathcal{P}$ is called a *population*.

Each agent $i \in \mathbb{P}$ is equipped with *preferences* R_i, defined on the real line $\mathbb{R}$, that are *complete*, *transitive*, and *reflexive*. As usual, $x\, R_i\, y$ is interpreted as "x is at least as desirable as y", $x\, P_i\, y$ as "x is preferred to y", and $x\, I_i\, y$ as "x is indifferent to y". Moreover, for preferences R_i there exists a number $p(R_i) \in \mathbb{R}$, called the *peak (level) of agent* i, with the following property: for each pair x, $y \in \mathbb{R}$ such that either $y < x \leq p(R_i)$, or $y > x \geq p(R_i)$, we have $x\, P_i\, y$. We call such preferences *single-peaked*. We denote the *domain of all single-peaked preferences* on $\mathbb{R}$ by $\mathcal{R}$. Preferences R_i are *symmetric* if for each pair $x, y \in \mathbb{R}$, $|x - p(R_i)| = |y - p(R_i)|$ implies $x\, I_i\, y$. We denote the *domain of all symmetric single-peaked preferences* on $\mathbb{R}$ by $\mathcal{S}$.

For each population $N \in \mathcal{P}$, we denote the set of *(preference) profiles* $R = (R_i)_{i \in N}$ where for each $i \in N$, $R_i \in \mathcal{R}$, by $\mathcal{R}^N$. Similarly, we denote the set of profiles $R = (R_i)_{i \in N}$, where for each $i \in N$, $R_i \in \mathcal{S}$ by $\mathcal{S}^N$. For each pair of populations $N, M \in \mathcal{P}$, with $N \subseteq M$, we denote the restriction $(R_i)_{i \in N} \in \mathcal{R}^N$ of profile $R \in \mathcal{R}^M$ to population N by R_N. Given profile $R \in \mathcal{R}^N$, for each pair $i, j \in N$ we also use the notation R_{-i} instead of $R_{N \setminus \{i\}}$ and $R_{-i,j}$ instead of $R_{N \setminus \{i,j\}}$.

In the sequel, all notation and definitions refer to single-peaked preferences but also apply to symmetric single-peaked preferences.

Given $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$, we denote the (set of) *peaks* in R as $p(R) = \{p(R_i)\}_{i \in N}$. Let the *smallest peak* in R be $\underline{p}(R) \equiv \min \{p(R_i)\}_{i \in N}$ and the *largest peak* in R be $\bar{p}(R) \equiv \max \{p(R_i)\}_{i \in N}$. Let the *convex hull* of the peaks in R be $\mathrm{Conv}(R) \equiv [\underline{p}(R), \bar{p}(R)]$.

Denote the class of non-empty and compact subsets of $\mathbb{R}$ by $\mathcal{C}$.[16] Given a set $X \in \mathcal{C}$, let the *minimum (point)* of X be $\underline{X} \equiv \min X$ and the *maximum (point)* of X be $\bar{X} \equiv \max X$. Given a set $X \in \mathcal{C}$ and preferences $R_i \in \mathcal{R}$, let the set of most preferred point(s) or *best point(s)* of agent i in set X be $b_X(R_i) \equiv \{x \in X : \text{for each } y \in X,\ x\, R_i\, y\}$. Similarly, let the set of least preferred point(s) or *worst point(s)* of agent i in set X be $w_X(R_i) \equiv \{x \in X : \text{for each } y \in X,\ y\, R_i\, x\}$. Note that by single-peakedness the sets $b_X(R_i)$ and $w_X(R_i)$ contain one or two elements. When $b_X(R_i)$ (respectively $w_X(R_i)$) contains two elements, agent i is indifferent between them. Hence, with some abuse of notation, we treat sets $b_X(R_i)$ and $w_X(R_i)$ as if they are points and for each $x \in X$, we write $b_X(R_i)\, R_i\, x\, R_i\, w_X(R_i)$.

Before describing the "best-worst" extension of preferences over sets that we use, we first introduce the properties of *weak-dominance* and *weak-independence* that characterize it (Barberà et al., 1984) denoting preferences defined over $\mathcal{C}$ by $R^{\mathcal{C}}$. In the following examples, we illustrate why these properties are reasonable.

Weak-dominance. Let $x, y \in \mathbb{R}$. If $x\, P^{\mathcal{C}}\, y$, then $\{x\}\, P^{\mathcal{C}}\, \{x, y\}\, P^{\mathcal{C}}\, \{y\}$.

Weak-independence. Let $X, Y, Z \in \mathcal{C}$ such that $[X \cap Z = \emptyset$ and $Y \cap Z = \emptyset]$. If $X\, P^{\mathcal{C}}\, Y$, then $[X \cup Z]\, R^{\mathcal{C}}\, [Y \cup Z]$.

The following two examples illustrate these properties. Both pertain to a linear city whose residents own one car each and have single-peaked preferences over where to park.

Example 1 (**Weak-dominance).** All public parking is located in two (parking) garages at $x, y \in \mathbb{R}$, with $x \neq y$, that we simply refer to as zone x and y. Neither garage's capacity can accommodate all residents but the joint capacity is sufficient. Initially, a one-zone scheme is in place and all residents are assigned to either zone x or zone y: residents assigned to zone x (zone y) are only allowed to park at garage x (y), which has the capacity to accommodate them. Later, a two-zone scheme is adopted: each resident can use either one of the two

[16] As discussed in Remark 5, the requirement for sets in $\mathcal{C}$ to be compact is without loss of generality.

garages. Consider a resident i of zone x who prefers x to y. Under the one-zone scheme he always parks at x, while under the two-zone scheme he sometimes parks at y (whenever x is full). We expect resident i to be worse off under the two-zone scheme, that is, if $x \, P_i \, y$, then $\{x\} \, P_i \, \{x, y\} \, P_i \, \{y\}$ and *weak-dominance* holds. □

Example 2 (Weak-independence). Two (single-zone parking) schemes, X and Y, are being considered for adoption. Before a final decision is made, and following a development project on some previously unused land, (parking) space Z now becomes available. Now assume that instead of schemes X and Y, two new schemes are being considered for adoption, $X \cup Z$ and $Y \cup Z$. Suppose resident i initially prefers X to Y. Since Z was unavailable under X and Y and is now available under both $X \cup Z$ and $Y \cup Z$, we expect i to find $X \cup Z$ at least as desirable as $Y \cup Z$. That is, if $X \cap Z = \emptyset$, $Y \cap Z = \emptyset$, and $X \, P_i \, Y$, then $[X \cup Z] \, R_i \, [Y \cup Z]$ and *weak-independence* holds. □

Under the best-worst extension of preferences over sets, when comparing two sets, an agent only considers his best and his worst point(s) in each of them. Given two sets $X, Y \in \mathcal{C}$, an agent prefers X to Y if he prefers his best point(s) in X to his best point(s) in Y and his worst point(s) in X to his worst point(s) in Y. The following definition also covers three more cases arising if an agent is indifferent between his best or worst point(s) in two sets.

With some abuse of notation, we use the same symbols to denote preferences over points and preferences over sets.

Best-worst extension of preferences to sets. For each agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and each pair of sets $X, Y \in \mathcal{C}$, we have

$$X \, R_i \, Y \text{ if and only if } \begin{cases} b_X(R_i) \, R_i \, b_Y(R_i) \\ \qquad\quad \text{and} \\ w_X(R_i) \, R_i \, w_Y(R_i) \end{cases}$$

and

$$X \, P_i \, Y \text{ if and only if } X \, R_i \, Y \text{ and } \begin{cases} b_X(R_i) \, P_i \, b_Y(R_i) \\ \qquad\quad \text{or} \\ w_X(R_i) \, P_i \, w_Y(R_i). \end{cases}$$

This extension of preferences is *transitive*, i.e., for each triple $X, Y, Z \in \mathcal{C}$, if $X \; R_i \; Y$ and $Y \, R_i \, Z$, then $X \, R_i \, Z$. However, it is not *complete*: there exist sets $X, Y \in \mathcal{C}$ such that neither $X \; R_i \; Y$ nor $Y \; R_i \; X$. To be more precise, we now make the following definition.

Comparability. Sets $X, Y \in \mathcal{C}$ are *comparable* by agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ if and only if $[b_X(R_i) \; P_i \; b_Y(R_i)$ implies $w_X(R_i) \; R_i \; w_Y(R_i)]$ and $[w_X(R_i) \; P_i \; w_Y(R_i)$ implies $b_X(R_i) \; R_i \; b_Y(R_i)]$.

Regarding the best-worst extension of preferences over sets, we now define *Pareto-efficiency*, *Pareto-dominance*, and *Pareto-equivalence*, henceforth, *efficiency*, *dominance*, and *equivalence* respectively.

Efficiency (of sets). Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$. Set $X \in \mathcal{C}$ is *efficient* if and only if there is no set $Y \in \mathcal{C}$ such that for each $i \in N$, $Y \; R_i \; X$, and for at least one $j \in N$, $Y \; P_j \; X$. We denote the class containing all *efficient* sets for $R \in \mathcal{R}^N$ by $\mathrm{PE}(R)$.

Dominance and equivalence. Let $N \in \mathcal{P}$ and $R \in \mathcal{R}^N$. Let pair $X, Y \in \mathcal{C}$ such that for each $i \in N$, $Y \; R_i \; X$. If for at least one $j \in N$, $Y \; P_j \; X$, then Y *dominates* X, otherwise Y and X are *equivalent*.

We now proceed to characterize *efficient* sets.

Proposition 1 (**Efficient sets**). *For each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$, a set $X \in \mathcal{C}$ is efficient if and only if the following two conditions hold.*

(i) *X is a subset of the convex hull of the agents' peaks. That is,*

$$X \subseteq \mathrm{Conv}(R).$$

(ii) *All of the agents' peaks that lie in the convex hull of X are included in X. That is,*

$$\mathrm{Conv}\,(X) \cap p(R) \subseteq X.$$

We prove Proposition 1 in Appendix A and illustrate it in Figure 1.

When considering convex sets, the characterization in Proposition 1 simplifies.

Remark 1 (**Efficient convex sets**). For each $N \in \mathcal{P}$, each $R \in \mathcal{R}^N$, and each convex set $X = \mathrm{Conv}(X) \in \mathcal{C}$, $X \in \mathrm{PE}(R)$ if and only if $X \subseteq \mathrm{Conv}(R)$. $\square$

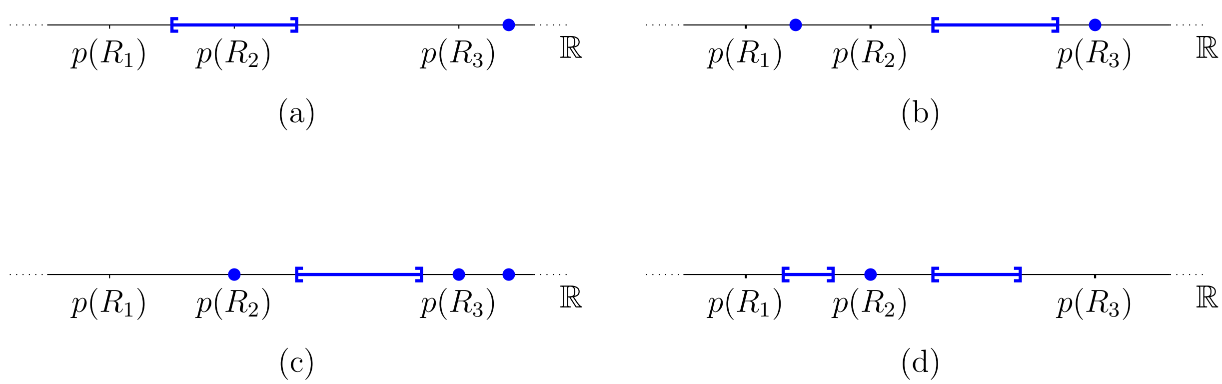

Figure 1: Let $N = \{1, 2, 3\}$ with $R \in \mathcal{R}^N$ and $p(R) = \{p(R_1), p(R_2), p(R_3)\}$. Sets under consideration are shown in bold. The set in (a) satisfies neither (i) nor (ii). The set in (b) satisfies (i) but not (ii). The set in (c) does not satisfy (i) but it satisfies (ii). The set in (d) satisfies both (i) and (ii), hence it is *efficient*.

Further consequences of Proposition 1 are Corollaries 1 and 2. Essentially, Corollary 1 states that given a population M with profile R, if $X \in \mathcal{C}$ is *efficient*, then it is also *efficient* for each population $N \subsetneq M$ such that the convex hull of population N's peaks at profile R_N, and that of population M's peaks at profile R, are the same.

Corollary 1. *Let $M \in \mathcal{P}$, $R \in \mathcal{R}^M$, and $X \in \mathrm{PE}(R)$. Then, for each $N \in \mathcal{P}$ such that $N \subsetneq M$ and $\mathrm{Conv}(R_N) = \mathrm{Conv}(R)$, $X \in \mathrm{PE}(R_N)$.*

Proof. Let $N, M \in \mathcal{P}$ be such that $N \subsetneq M$, $R \in \mathcal{R}^M$, and $X \in \mathrm{PE}(R)$. By Proposition 1 (i), $X \subseteq \mathrm{Conv}(R)$. Since, $\mathrm{Conv}(R) = \mathrm{Conv}(R_N)$, $X \subseteq \mathrm{Conv}(R_N)$. By Proposition 1 (ii), $X \cap p(R) \subseteq X$. Since, $p(R_N) \subsetneq p(R)$, $X \cap p(R_N) \subseteq X$. By Proposition 1, $X \in \mathrm{PE}(R_N)$. $\square$

Corollary 2 provides some consequences for *efficient* and *equivalent* sets.

Corollary 2. *Let $N \in \mathcal{P}$, $R \in \mathcal{R}^N$, and $X \in \mathrm{PE}(R)$. Then, $\mathrm{Conv}(X)$ is equivalent to X. Moreover, if Y is equivalent to X, then $\mathrm{Conv}(Y) = \mathrm{Conv}(X)$.*

We prove Corollary 2 in Appendix A.

To simplify notation, in the sequel we always represent any *efficient* set by its convex hull.

3 Choice correspondences

A *choice correspondence* Φ assigns to each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$ a set $\Phi(R) \in \mathcal{C}$, i.e., $\Phi\colon \bigcup_{N\in\mathcal{P}} \mathcal{R}^N \to \mathcal{C}$. We denote the family of choice correspondences Φ by $\mathcal{F}$.

In the sequel, when the properties of *replacement-dominance* and *one-sided replacement-dominance* (defined in Section 4) are considered, the population of agents does not change. For this reason, we introduce *fixed-population choice correspondences*, henceforth *fp-choice correspondences*.

Given $N \in \mathcal{P}$, an *fp-choice correspondence* Φ for N assigns to each $R \in \mathcal{R}^N$ a set $\Phi(R) \in \mathcal{C}$, i.e., $\Phi\colon \mathcal{R}^N \to \mathcal{C}$. Let $\mathcal{F}^N$ denote the family of fp-choice correspondences for N. A choice correspondence is a collection of fp-choice correspondences indexed by $N \in \mathcal{P}$.

Remark 2 (Choice functions). Given population $N \in \mathcal{P}$, if an fp-choice correspondence for N assigns to each $R \in \mathcal{R}^N$ a set consisting of a single point, it is essentially an fp-choice function. Similarly, if a choice correspondence assigns to each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$ a set consisting of a single point, it is essentially a choice function. $\square$

We now proceed to our *efficiency* notion for fp-choice correspondences and choice correspondences.

Efficiency (of choice correspondences).

(a) Let $N \in \mathcal{P}$ and $\Phi \in \mathcal{F}^N$ be an fp-choice correspondence. For each $R \in \mathcal{R}^N$, $\Phi(R) \in \mathrm{PE}(R)$.

(b) Let choice correspondence $\Phi \in \mathcal{F}$. For each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$, $\Phi(R) \in \mathrm{PE}(R)$.

The following classes of "target (choice) correspondences" and "fp-target (choice) correspondences" play an important role in the sequel.

Any *fp-target point correspondence* is determined by its fixed population and its target point. Similarly, any *target point correspondence* is determined by its target point. In both cases: if the target point is *efficient*, then it is chosen. If the target point is not *efficient*, then the (unique) closest *efficient* point to it is chosen.

Target point correspondences. Let $a \in \mathbb{R} \cup \{-\infty, \infty\}$. We define:

(a) for population $N \in \mathcal{P}$, the *fp-target point correspondence* with target a, $\varphi^a \in \mathcal{F}^N$, such that for each $R \in \mathcal{R}^N$,

(b) the *target point correspondence* with target a, $\varphi^a \in \mathcal{F}$, such that for each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$,

$$\varphi^a(R) = \begin{cases} \{\underline{p}(R)\} & \text{if } a < \underline{p}(R) \\ \{\bar{p}(R)\} & \text{if } a > \bar{p}(R) \\ \{a\} & \text{otherwise.} \end{cases}$$

A (fp-)target point correspondence φ^a is essentially a *(fp-)target point function.*[17]

Any *fp-target set correspondence* is determined by its population and its non-empty, closed, and convex target set. Similarly, any *target set correspondence* is determined by its non-empty, closed, and convex target set. In both cases: if the target set is *efficient*, it is chosen. If the target set is not *efficient*, the (unique) maximal *efficient* subset of the target set is chosen, if one exists; otherwise, the (unique) closest *efficient* point to the target set is chosen.

Target set correspondences. Let $[a, b] \subseteq \mathbb{R} \cup \{-\infty, \infty\}$. We define:

(a) for population $N \in \mathcal{P}$, the *fp-target set correspondence* with target $[a, b]$, $\Phi^{a,b} \in \mathcal{F}^N$, such that for each $R \in \mathcal{R}^N$,

(b) the *target set correspondence* with target set $[a, b]$, $\Phi^{a,b} \in \mathcal{F}$, such that for each $N \in \mathcal{P}$ and each $R \in \mathcal{R}^N$,

$$\Phi^{a,b}(R) = \begin{cases} \{\underline{p}(R)\} & \text{if } b < \underline{p}(R) \\ \{\bar{p}(R)\} & \text{if } a > \bar{p}(R) \\ [a, b] \cap \text{Conv}(R) & \text{otherwise.} \end{cases}$$

Each target set correspondence is a set of fp-target set correspondences, one for each $N \in \mathcal{P}$, where the target set is constant and independent of the population. Also, each (fp-)target set correspondence with a target set $[a, b] \subseteq \mathbb{R} \cup \{-\infty, \infty\}$ such that $a = b$, is a (fp-)target point correspondence.

By Proposition 1, it follows that each (fp-)target set correspondence satisfies *efficiency*.

[17] The difference is that a (fp-)target point correspondence φ^a only assigns singleton sets while the corresponding (fp-)target point function assigns the points in these sets.

We illustrate the concept of an fp-target set correspondence in Figure 2. Since each target set correspondence is a collection of fp-target set correspondences indexed by $N \in \mathcal{P}$, a similar example for target set correspondences can be easily obtained if in Figure 2 we allow for the population to change.

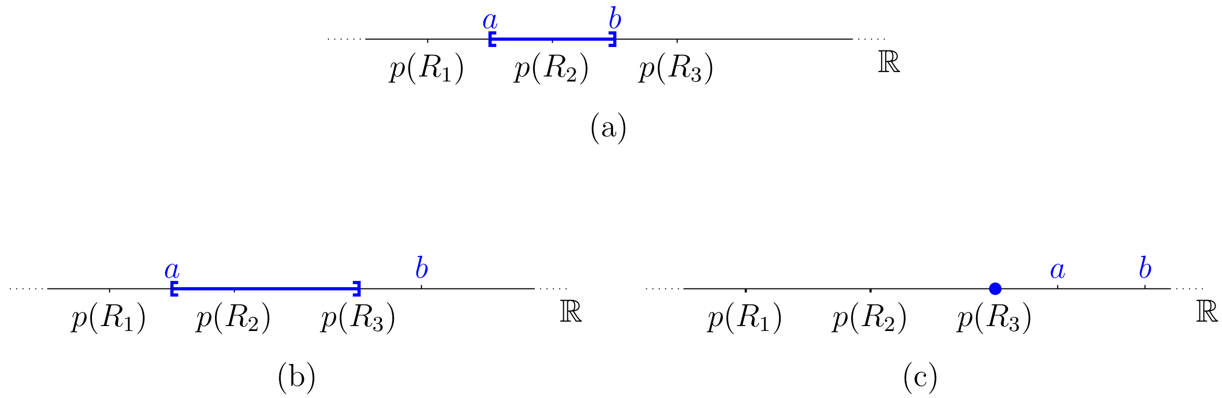

Figure 2: Let $N = \{1, 2, 3\}$ with $R \in \mathcal{R}^N$ and $p(R) = \{p(R_1), p(R_2), p(R_3)\}$. Let $\Phi^{a,b} \in \mathcal{F}^N$. The chosen sets in each case are shown in bold. The target set in (a) is *efficient* and is chosen. The target set in (b) is not *efficient* but the maximal *efficient* subset exists and it is chosen. The target set in (c) is not *efficient* and no maximal *efficient* subset exists; hence the closest *efficient* point is chosen.

Remark 3 (**Properties of fp-choice correspondences extend to choice correspondences**)**.** In Section 4, we introduce properties of fp-choice correspondences. Since a choice correspondence is a collection of fp-choice correspondences, these properties easily extend to choice correspondences. □

4 Properties of choice correspondences

In the sequel, all properties and results refer to single-peaked preferences but also apply to symmetric single-peaked preferences.

We consider two solidarity properties of choice correspondences. The first solidarity property, expresses the solidarity among agents against changes in the population (Thomson, 1983a,b): if agents are added to the population, the agents initially present should all be made at least as well off or they should all be made at most as well off by this change.

Population-monotonicity. Let $\Phi \in \mathcal{F}$ be a choice correspondence. For each pair $N, M \in \mathcal{P}$ such that $N \subseteq M$ and each $R \in \mathcal{R}^M$ the following holds:

$$\text{for each } i \in N,\ \Phi(R_N)\, R_i\, \Phi(R) \text{ or for each } i \in N,\ \Phi(R)\, R_i\, \Phi(R_N).$$

Population-monotonicity implies that the chosen sets, before and after the change in population, are *comparable* by all agents present before and after this change.

The next lemma states that if a choice correspondence satisfies *efficiency* and *population-monotonicity*, then if agents are added to the population, all agents who were initially present are at most as well off.

Lemma 1 (Efficiency and population-monotonicity). *Let choice correspondence $\Phi \in \mathcal{F}$ satisfy efficiency and population-monotonicity. Then, for each pair $N, M \in \mathcal{P}$ such that $N \subseteq M$, each $R \in \mathcal{R}^M$, and each $i \in N$, $\Phi(R_N)\, R_i\, \Phi(R)$. In particular, if* $\mathrm{Conv}(R_N) = \mathrm{Conv}(R)$, *then $\Phi(R_N) = \Phi(R)$.*

Proof. Let choice correspondence $\Phi \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*. Let $N, M \in \mathcal{P}$ be such that $N \subseteq M$. Let $R \in \mathcal{R}^M$.

By *efficiency*, $\Phi(R) \in \mathrm{PE}(R)$ and $\Phi(R_N) \in \mathrm{PE}(R_N)$. By *population-monotonicity*, for each $i \in N$, $\Phi(R)\, R_i\, \Phi(R_N)$ or for each $i \in N$, $\Phi(R_N)\, R_i\, \Phi(R)$. If for each $i \in N$, $\Phi(R)\, R_i\, \Phi(R_N)$ and since $\Phi(R_N) \in \mathrm{PE}(R_N)$, then for each $i \in N$, $\Phi(R_N)\, I_i\, \Phi(R)$. Therefore, for each $i \in N$, $\Phi(R_N)\, R_i\, \Phi(R)$.

In particular, if $\mathrm{Conv}(R_N) = \mathrm{Conv}(R)$, then by $\Phi(R) \in \mathrm{PE}(R)$ and Corollary 1, $\Phi(R) \in \mathrm{PE}(R_N)$. Since for each $i \in N$, $\Phi(R_N)\, R_i \Phi(R)$, and moreover [$\Phi(R) \in \mathrm{PE}(R_N)$ and $\Phi(R_N) \in \mathrm{PE}(R_N)$], then for each $i \in N$, $\Phi(R_N)\, I_i \Phi(R)$. By Corollary 2, $\mathrm{Conv}(\Phi(R_N)) = \mathrm{Conv}(\Phi(R))$, and since we always represent any *efficient* set by its convex hull, $\Phi(R_N) = \Phi(R)$. □

Proposition 2 ($\mathbf{\Phi^{a,b}}$ is population-monotonic). *Each target set correspondence satisfies population-monotonicity.*

Proof. Let $\Phi^{a,b} \in \mathcal{F}$ be a target set correspondence. Let $N \in \mathcal{P}$ be such that $|N| \geq 2$ and $R \in \mathcal{R}^N$. We prove *population-monotonicity* of $\Phi^{a,b}$ by showing that if $j \in N$ leaves all remaining agents end up at least as well off, i.e., for each $i \in N \setminus \{j\}$, $\Phi^{a,b}(R_{-j})\, R_i\, \Phi^{a,b}(R)$.

Case 1. $\mathrm{Conv}(R_{-j}) = \mathrm{Conv}(R)$. Then, the chosen set remains the same, $\Phi^{a,b}(R_{-j}) = \Phi^{a,b}(R)$.

Case 2. $\mathrm{Conv}(R_{-j}) \neq \mathrm{Conv}(R)$. Then, j has either the unique smallest peak at R or the unique largest peak at R. By symmetry of arguments, assume that j has the unique smallest peak at R, $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(R_{-j})$. There are 3 possibilities.

(i) $a, b < \underline{p}(R_{-j})$. Then, $\Phi^{a,b}(R_{-j}) = \underline{p}(R_{-j})$. Furthermore, if $b \leq p(R_j)$, then $\Phi^{a,b}(R) = p(R_j)$; if $a \leq p(R_j)$ and $b > p(R_j)$, then $\Phi^{a,b}(R) = [p(R_j), b]$; and if $a > p(R_j)$, then $\Phi^{a,b}(R) = [a, b]$. Hence, for each $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(R_{-j})}(R_i) = w_{\Phi^{a,b}(R_{-j})}(R_i) = \underline{p}(R_{-j})$, $b_{\Phi^{a,b}(R)}(R_i) \in \{p(R_j), b\}$, and $w_{\Phi^{a,b}(R)}(R_i) \in \{p(R_j), a\}$. Thus, for each $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(R)}(R_i) < b_{\Phi^{a,b}(R_{-j})}(R_i) \leq p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(R_{-j})}(R_i) \leq p(R_i)$. By single-peakedness, for each $i \in N \setminus \{j\}$, the best and worst points are improved. Hence, $\Phi^{a,b}(R_{-j}) \; P_i \; \Phi^{a,b}(R)$.

(ii) $a < \underline{p}(R_{-j})$ and $b \geq \underline{p}(R_{-j})$. Then, $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(R_{-j}) = \underline{p}(R_{-j})$ and $\bar{\Phi}^{a,b}(R) = \bar{\Phi}^{a,b}(R_{-j})$. Thus, for each $i \in N \setminus \{j\}$, $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(R_{-j}) \leq p(R_i)$. If $\bar{\Phi}^{a,b}(R_{-j}) < p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(R_{-j})}(R_i) < p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(R_{-j})}(R_i) < p(R_i)$. Hence, by single-peakedness, i's best point is at least as desirable and his worst point is improved. If $\bar{\Phi}^{a,b}(R_{-j}) \geq p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(R_{-j})}(R_i) = p(R_i)$ and $w_{\Phi^{a,b}(R_{-j})}(R_i) \in \Phi^{a,b}(R_{-j}) \subseteq \Phi^{a,b}(R)$. Thus, i's best and worst points are at least as desirable. Hence, for each $i \in N \setminus \{j\}$, $\Phi^{a,b}(R_{-j}) \; R_i \; \Phi^{a,b}(R)$.

(iii) $a, b \geq \underline{p}(R_{-j})$. Then, the chosen set remains the same, $\Phi^{a,b}(R_{-j}) = \Phi^{a,b}(R)$. □

The second solidarity property we consider expresses the solidarity among agents against changes in preferences (Moulin, 1987): if the preferences of an agent change, then the other agents should all be made at least as well off or they should all be made at most as well off. We formulate this requirement for fp-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Replacement-dominance. Let $N \in \mathcal{P}$ and $\Phi \in \mathcal{F}^N$ be an fp-choice correspondence. For each $j \in N$, and each pair $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ the following holds:

$$\text{for each } i \in N \setminus \{j\},\ \Phi(R) \; R_i \; \Phi(\bar{R}) \text{ or for each } i \in N \setminus \{j\},\ \Phi(\bar{R}) \; R_i \; \Phi(R).$$

Replacement-dominance implies that the chosen sets, before and after the change in preferences of some agent, are *comparable* by all other agents.

Note that for a population of one or two agents, *replacement-dominance* imposes no restriction on fp-choice correspondences. Hence, for each fixed population with one or two agents, each fp-target set correspondence satisfies *replacement-dominance*. However, if the fixed population contains at least three agents, then the target set must equal a point.

Proposition 3 (**$\Phi^{[a,b]}$ is replacement-dominant $\Leftrightarrow$ $a = b$**). *If a population consists of at least 3 agents, then an associated fp-target set correspondence satisfies replacement-dominance if and only if it is an fp-target point correspondence.*

***Proof*.** Let $N \in \mathcal{P}$ be such that $|N| \geq 3$ and $\Phi^{a,b} \in \mathcal{F}^N$ be an fp-target set correspondence.

First, if $a = b$, we prove *replacement-dominance* of φ^a ($\Phi^{a,b}$, $a = b$) by showing that for each pair $R, \bar{R} \in \mathcal{R}^N$ such that $\bar{R} \in \mathcal{R}^N$ and $R_{-j} = \bar{R}_{-j}$, [for each $i \in N \setminus \{j\}$, $\varphi^a(R)\, R_i\, \varphi^a(\bar{R})$] or [for each $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, R_i\, \varphi^a(R)$].

Case 1. $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

Case 2.1. $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$. Then, j has either the unique smallest peak at R or the unique largest peak at R. By symmetry of arguments, assume that $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) = \bar{p}(\bar{R})$. There are 2 possibilities.

(i) $a < \underline{p}(\bar{R})$. Then, $\varphi^a(\bar{R}) = \{\underline{p}(\bar{R})\}$. Furthermore, if $a \leq \underline{p}(R)$, then $\varphi^a(R) = \{\underline{p}(R)\}$ and if $a > \underline{p}(R)$, then $\varphi^a(R) = \{a\}$. Hence, for each $i \in N \setminus \{j\}$, $\varphi^a(R) < \varphi^a(\bar{R}) \leq p(\bar{R}_i)$. Hence, by single-peakedness, for each $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, P_i\, \varphi^a(R)$.

(ii) $a \geq \underline{p}(\bar{R})$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

Case 2.2. $\mathrm{Conv}(\bar{R}) \supsetneq \mathrm{Conv}(R)$. Then, by Case 2.1 (with the roles of R and $\bar{R}$ reversed), for each $i \in N \setminus \{j\}$, $\varphi^a(R)\, R_i\, \varphi^a(\bar{R})$.

Case 3. $\mathrm{Conv}(\bar{R}) \not\subseteq \mathrm{Conv}(R)$ and $\mathrm{Conv}(\bar{R}) \not\supseteq \mathrm{Conv}(R)$. Then, j has either [the unique smallest peak at R and the unique largest peak at $\bar{R}$] or [the unique largest peak at R and the unique smallest peak at $\bar{R}$]. By symmetry of arguments, assume that $p(R_j) = \underline{p}(R)$ and $p(\bar{R}_j) = \bar{p}(\bar{R})$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) < \bar{p}(\bar{R})$. There are 3 possibilities.

(i) $a < \underline{p}(\bar{R})$. Then, as shown in Case 2.1, for each $i \in N \setminus \{j\}$, $\varphi^a(\bar{R})\, P_i\, \varphi^a(R)$.

(ii) $\underline{p}(\bar{R}) \leq a \leq \bar{p}(R)$. Then, the set (point) chosen remains the same, $\varphi^a(\bar{R}) = \varphi^a(R)$.

(iii) $a > \bar{p}(R)$. Then, $\varphi^a(R) = \{\bar{p}(R)\}$. Furthermore, if $a \geq \bar{p}(\bar{R})$, then $\varphi^a(\bar{R}) = \{\bar{p}(\bar{R})\}$ and if $a < \bar{p}(\bar{R})$, then $\varphi^a(\bar{R}) = \{a\}$. Hence, for each $i \in N \setminus \{j\}$, $p(\bar{R}_i) \leq \varphi^a(R) < \varphi^a(\bar{R})$. Hence, by single-peakedness, for each $i \in N \setminus \{j\}$, $\varphi^a(R)\ P_i\ \varphi^a(\bar{R})$.

Second, we prove that if $a < b$, then $\Phi^{a,b}$ does not satisfy *replacement-dominance*. Without loss of generality, assume that $1, 2, 3 \in N$.

If $a = -\infty$, let $\bar{a} \in \mathbb{R}$ be such that $\bar{a} < b$, otherwise, let $\bar{a} = a$. If $b = \infty$, then let $\bar{b} \in \mathbb{R}$ be such that $\bar{b} > \bar{a}$, otherwise, let $\bar{b} = b$. Hence, $[\bar{a}, \bar{b}] \subseteq [a, b]$. We divide the interval $[\bar{a}, \bar{b}]$ into three equal parts and use the four points $a_1 = \bar{a}$, $a_2 = \big(\bar{a} + \frac{1}{3}(\bar{b} - \bar{a})\big)$, $a_3 = \big(\bar{a} + \frac{2}{3}(\bar{b} - \bar{a})\big)$, and $a_4 = \bar{b}$ to construct (symmetric) profiles $R, \bar{R} \in \mathcal{S}^N$ such that $p(R_1) = a_1$, $p(R_2) = p(\bar{R}_2) = a_2$, $p(R_3) = p(\bar{R}_3) = a_3$, $p(\bar{R}_1) = a_4$, and for each $i \in N \setminus \{1, 2, 3\}$, $p(R_i) = p(\bar{R}_i) = a_2$. Note that $R_{-1} = \bar{R}_{-1}$.

By the definition of $\Phi^{a,b}$, we have $\Phi^{a,b}(R) = [a_1, a_3]$ and $\Phi^{a,b}(\bar{R}) = [a_2, a_4]$. Under both R and $\bar{R}$, the best points of agents 2 and 3 remain the same, $b_{\Phi^{a,b}(R)}(R_2) = b_{\Phi^{a,b}(\bar{R})}(R_2) = p(R_2)$ and $b_{\Phi^{a,b}(R)}(R_3) = b_{\Phi^{a,b}(\bar{R})}(R_3) = p(R_3)$. However, the worst points of agent 2 and 3 change as follows. For agent 2, $w_{\Phi^{a,b}(R)}(R_2) = \{a_1, a_3\}$ and $w_{\Phi^{a,b}(\bar{R})}(R_1) = \{a_4\}$. Since $p(R_2) = a_2 < a_3 < a_4$, single-peakedness implies $\Phi^{a,b}(R)\ P_2\ \Phi^{a,b}(\bar{R})$. For agent 3, $w_{\Phi^{a,b}(R)}(R_3) = \{a_1\}$ and $w_{\Phi^{a,b}(\bar{R})}(R_3) = \{a_2, a_4\}$. Since $a_1 < a_2 < a_3 = p(R_3)$, single-peakedness implies $\Phi^{a,b}(\bar{R})\ P_3\ \Phi^{a,b}(R)$. This contradicts *replacement-dominance*. □

We next introduce a property weaker than *replacement-dominance* in the sense that it does not require solidarity when the preferences of the agent with the unique smallest peak are changed such that he becomes the agent with the unique largest peak, or vice-versa. We formulate this requirement for fp-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

One-sided replacement-dominance. Let $N \in \mathcal{P}$ and $\Phi \in \mathcal{F}^N$ be an fp-choice correspondence. For each $j \in N$ and each pair $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\mathrm{Conv}(R) \subseteq \mathrm{Conv}(\bar{R})$ or $\mathrm{Conv}(R) \supseteq \mathrm{Conv}(\bar{R})$ the following holds:

$$\text{for each } i \in N \setminus \{j\},\ \Phi(R)\ R_i\ \Phi(\bar{R}) \text{ or for each } i \in N \setminus \{j\},\ \Phi(\bar{R})\ R_i\ \Phi(R).$$

One-sided replacement-dominance implies that the chosen sets, before and after the change in preferences of some agent, are *comparable* by all other agents. Moreover, *replacement-dominance* implies *one-sided replacement-dominance*.

The next lemma states that given a population of at least three agents and an associated fp-choice correspondence satisfying *efficiency* and *one-sided replacement-dominance*, if the preferences of an agent change in such a way that the new set of peaks is a subset of the initial one, all other agents end up at least as well off.

Lemma 2 (**Efficiency and one-sided replacement-dominance**). *Let* $N \in \mathcal{P}$ *be such that* $|N| \geq 3$ *and fp-choice correspondence* $\Phi \in \mathcal{F}^N$ *satisfy efficiency and one-sided replacement-dominance. Then, for each* $j \in N$*, each pair* $R, \bar{R} \in \mathcal{R}^N$ *such that* $[R_{-j} = \bar{R}_{-j}$ *and* $\mathrm{Conv}(\bar{R}) \subseteq \mathrm{Conv}(R)]$*, and each* $i \in N \setminus \{j\}$*,* $\Phi(\bar{R})\ R_i\ \Phi(R)$*. In particular, if* $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$*, then* $\Phi(\bar{R}) = \Phi(R)$*.*

We prove Lemma 2 in Appendix B.

Recall that for a population $N \in \mathcal{P}$ with one or two agents *(one-sided) replacement-dominance* imposes no restriction on an associated fp-choice correspondence. The following example illustrates why Lemma 2 does not hold for a population of two agents and an associated fp-choice correspondence.

Example 3. Let $N \in \mathcal{P}$ be such that $N = \{1, 2\}$ and $\Phi \in \mathcal{F}^N$ be an fp-choice correspondence such that

$$\Phi(R) = \begin{cases} p(R_2) & \text{if } p(R_2) = 1 \\ p(R_1) & \text{otherwise.} \end{cases}$$

Hence, Φ satisfies *efficiency*, and since $|N| = 2$, it trivially satisfies *(one-sided) replacement-dominance*. Let $R, \bar{R} \in \mathcal{R}^N$ be such that $p(R_1) = p(\bar{R}_1) = 0$, $p(R_2) = 2$, and $p(\bar{R}_2) = 1$. Hence, $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$. It follows, that $\Phi(R) = 0$ and $\Phi(\bar{R}) = 1$. Hence, agent 1's peak $p(R_1) = \Phi(R) < \Phi(\bar{R})$. By single-peakedness, $\Phi(R)\ P_1\ \Phi(\bar{R})$. □

Proposition 4 (**$\Phi^{a,b}$ is one-sided replacement-dominant**). *Each fp-target set correspondence satisfies one-sided replacement-dominance.*

Proof. Let $N \in \mathcal{P}$ and $\Phi^{a,b} \in \mathcal{F}^N$ be an fp-target set correspondence. Since for $|N| \leq 2$, (*one-sided*) *replacement-dominance* imposes no restriction on fp-choice correspondence $\Phi^{a,b}$, fix $|N| \geq 3$.

We prove that $\Phi^{a,b}$ satisfies *one-sided replacement-dominance*, i.e., we show that for each $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and $\mathrm{Conv}(R) \subseteq \mathrm{Conv}(\bar{R})$ or $\mathrm{Conv}(\bar{R}) \subseteq \mathrm{Conv}(R)$, the following holds. For each $i \in N \setminus \{j\}$, $\Phi^{a,b}(R)\ R_i\ \Phi^{a,b}(\bar{R})$ or for each $i \in N \setminus \{j\}$, $\Phi^{a,b}(\bar{R})\ R_i$ $\Phi^{a,b}(R)$.

Case 1. $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$. Then, the chosen set remains the same, $\Phi^{a,b}(\bar{R}) = \Phi^{a,b}(R)$.

Case 2.1. $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$. Then, j has either the unique smallest peak at R or the unique largest peak at R. By symmetry of arguments, assume that j has the unique smallest peak at R, $p(R_j) = \underline{p}(R)$. Then, $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(R) = \bar{p}(\bar{R})$. There are 3 possibilities.

(i) $a, b < \underline{p}(\bar{R})$. Then $\Phi^{a,b}(\bar{R}) = \underline{p}(\bar{R})$. Furthermore, if $a, b \leq \underline{p}(R)$, then $\Phi^{a,b}(R) = \underline{p}(R)$; if $a \leq \underline{p}(R)$ and $b > \underline{p}(R)$, then $\Phi^{a,b}(R) = [\underline{p}(R), b]$; and if $a, b > \underline{p}(R)$, then $\Phi^{a,b}(R) = [a, b]$. Hence, for each $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(\bar{R})}(R_i) = w_{\Phi^{a,b}(\bar{R})}(R_i) = \{\underline{p}(\bar{R})\}$, $b_{\Phi^{a,b}(R)}(R_i) \in \{\underline{p}(R), b\}$, and $w_{\Phi^{a,b}(R)}(R_i) \in \{\underline{p}(R), a\}$. Thus, for each $i \in N \setminus \{j\}$, $b_{\Phi^{a,b}(R)}(R_i) < b_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$. By single-peakedness, for each $i \in N \setminus \{j\}$, best and worst points improve. Hence, $\Phi^{a,b}(\bar{R})\ P_i\ \Phi^{a,b}(R)$.

(ii) $a < \underline{p}(\bar{R})$ and $b \geq \underline{p}(\bar{R})$. Then, for the minima $\underline{\Phi}^{a,b}(R)$ and $\underline{\Phi}^{a,b}(\bar{R})$ we have $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(\bar{R}) = \underline{p}(\bar{R})$ and for the maxima $\bar{\Phi}(R)$ and $\bar{\Phi}(\bar{R})$ we have $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$. Thus, for each $i \in N \setminus \{j\}$, minimum $\underline{\Phi}^{a,b}(R) < \underline{\Phi}^{a,b}(\bar{R}) \leq p(R_i)$. If maximum $\bar{\Phi}^{a,b}(\bar{R}) < p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(\bar{R})}(R_i) < p(R_i)$ and $w_{\Phi^{a,b}(R)}(R_i) < w_{\Phi^{a,b}(\bar{R})}(R_i) \leq p(R_i)$. Hence, by single-peakedness, i's best point is at least as desirable and his worst point improves. If maximum $\bar{\Phi}^{a,b}(\bar{R}) \geq p(R_i)$, then $b_{\Phi^{a,b}(R)}(R_i) = b_{\Phi^{a,b}(\bar{R})}(R_i) = p(R_i)$ and $w_{\Phi^{a,b}(\bar{R})}(R_i) \in \Phi^{a,b}(\bar{R}) \subseteq \Phi^{a,b}(R)$. Thus, i's best and worst points are at least as desirable. It follows, that for each $i \in N \setminus \{j\}$, $\Phi^{a,b}(\bar{R})\ R_i\ \Phi^{a,b}(R)$.

(iii) $a, b \geq \underline{p}(\bar{R})$. Then, the set chosen remains the same, $\Phi^{a,b}(\bar{R}) = \Phi^{a,b}(R)$.

Case 2.2. $\mathrm{Conv}(\bar{R}) \supsetneq \mathrm{Conv}(R)$. Then, by Case 2.1 (with the roles of R and $\bar{R}$ reversed), for each $i \in N \setminus \{j\}$, $\Phi^{a,b}(R)\ R_i\ \Phi^{a,b}(\bar{R})$. □

The next proposition states an important relation between the two solidarity properties we study.

Proposition 5 (Efficiency and population-monotonicity ⇒ one-sided replacement-dominance). *Each choice correspondence satisfying efficiency and population-monotonicity also satisfies one-sided replacement-dominance.*

We prove Proposition 5 in Appendix C.

5 Characterizing target set correspondences

In the sequel, all results presented refer to single-peaked preferences but also apply to symmetric single-peaked preferences.

Our first theorem states that the properties of *efficiency* and *one-sided replacement-dominance* characterize fp-target set correspondences.

Theorem 1 (Φ **is efficient and one-sided replacement-dominant** $\Leftrightarrow$ $\boldsymbol{\Phi = \Phi^{a,b}}$**).** *If a fixed population consists of at least 3 agents, then an associated fp-choice correspondence satisfies efficiency and one-sided replacement-dominance if and only if it is an fp-target set correspondence.*

We prove Theorem 1 in Appendix D.

Corollary 3 that follows, strengthens a result for choice functions by Thomson (1993).

Corollary 3 (Φ **is efficient and replacement-dominant** $\Leftrightarrow$ $\boldsymbol{\Phi = \varphi^a}$**).** *If a fixed population consists of at least 3 agents, then an associated fp-choice correspondence satisfies efficiency and replacement-dominance if and only if it is an fp-target point correspondence.*

Proof. ***If part.*** By Propositions 1 and 3, all fp-target point correspondences satisfy *efficiency* and *replacement-dominance*.

Only if part. Let $N \in \mathcal{P}$ be such that $|N| \geq 3$ and let the fp-choice correspondence $\varPhi \in \mathcal{F}^N$ satisfy *efficiency* and *replacement-dominance*. Then, $\varPhi$ satisfies *one-sided replacement-dominance* and by Theorem 1 it is an fp-target set correspondence $\varPhi^{a,b} \in \mathcal{F}^N$. By Proposition 3, $\varPhi^{a,b}$ satisfies *replacement-dominance* if and only if it is an fp-target point correspondence $\varphi^a \in \mathcal{F}^N$. $\square$

We have formulated Theorem 1 and Corollary 3 for fp-choice correspondences where the fixed population contains at least 3 agents. If instead we consider choice correspondences, then *efficiency* and *one-sided replacement-dominance* (*replacement-dominance*) imply that for each population with at least 3 agents, a different target set or target point can be chosen, while for each population with at most 2 agents, the choice correspondence can equal any *efficient* fp-choice correspondence.

Our second theorem states that the properties of *efficiency* and *population-monotonicity* characterize target set correspondences.

Theorem 2 ($\mathbf{\Phi}$ is efficient and population-monotonic $\Leftrightarrow$ $\mathbf{\Phi = \Phi^{a,b}}$). *A choice correspondence satisfies efficiency and population-monotonicity if and only if it is a target set correspondence.*

Proof. If part. By Propositions 1 and 2, all target set correspondences satisfy *efficiency* and *population-monotonicity*.

Only if part. Let choice correspondence $\Phi \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*. By Proposition 5, Φ satisfies *one-sided replacement-dominance*. Let $M \in \mathcal{P}$ be such that $|M| \geq 3$. By Theorem 1, for each $R \in \mathcal{R}^M$, $\Phi = \Phi^{a_M,b_M} \in \mathcal{F}^M$. Define points $a := a_M$ and $b := b_M$.

We show that for each $N \in \mathcal{P}$ and each $\bar{R} \in \mathcal{R}^N$, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$. We do so by showing that for each $N \in \mathcal{P}$, each $\bar{R} \in \mathcal{R}^N$, and each $R \in \mathcal{R}^M$, if $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R)$, then $\Phi(\bar{R}) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$ (the latter equality follows by the definition of $\Phi^{a,b}$).

Let $R \in \mathcal{R}^M$ and $\bar{R} \in \mathcal{R}^N$. Recall that $\Phi(R) = \Phi^{a,b}(R)$. Begin from $R \in \mathcal{R}^M$ and construct $R^1 \in \mathcal{R}^{M\cup N}$ by adding the population $N \setminus M$ with profile $\bar{R}_{N\setminus M}$, i.e., $R^1 = (R, \bar{R}_{N\setminus M})$. Since $\mathrm{Conv}(R^1) = \mathrm{Conv}(R)$, by *population-monotonicity* and Lemma 1, $\Phi(R^1) = \Phi(R)$. Next, change the preferences of each $i \in N$ to $\bar{R}_i$ and denote the new profile $R^2 = (R^1_{M\setminus N}, \bar{R}) \in \mathcal{R}^{M\cup N}$. Since $\mathrm{Conv}(R^2) = \mathrm{Conv}(R^1)$, by *population-monotonicity* and Lemma 1, $\Phi(R^2) = \Phi(R^1)$. Finally, remove the population $M \setminus N$ and notice that the new profile $R^2_N = \bar{R} \in \mathcal{R}^N$. Since $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R^2)$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R^2)$. Hence, $\Phi(\bar{R}) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$. □

All the properties we consider are independent.

Remark 4 (Independence of properties). Note that the properties in all our characterization results are independent. A constant choice correspondence that always chooses a fixed set satisfies *(one-sided) replacement-dominance* and *population-monotonicity* but violates *efficiency*. A choice correspondence that always chooses the peak of the agent with the lowest index satisfies *efficiency*, but it violates *one-sided replacement-dominance* and *population-monotonicity*. □

Finally, we comment on the validity of our results for some natural model variations.

Remark 5 (Chosen sets are not necessarily compact). Although we only study compact subsets of $\mathbb{R}$, the compactness requirement is without loss of generality for the following reasons. First, the agents' peaks being real numbers and Proposition 1 (i) imply that unbounded sets are not *efficient*. Hence, by Theorems 1 and 2, the two classes of correspondences we characterize satisfy *efficiency* and therefore only select bounded sets. Second, concerning open (and bounded) sets, after assuming that each agent is indifferent between a set and its *closure*,[18] all our results hold and the target sets of target set correspondences and fp-target set correspondences can be open. Notice that in this case, the second requirement for the *efficiency* of a set, that is, Proposition 1 (ii), must change slightly to $\mathrm{Conv}(\mathrm{closure}(X)) \cap p(R) \subseteq \mathrm{closure}(X)$; moreover, to accommodate for the possible openness of sets, throughout the text and for each set X, references to $\mathrm{Conv}(X)$ must be substituted with $\mathrm{Conv}(\mathrm{closure}(X))$. □

Remark 6 (Monotonic preferences). Allowing for agents to have monotonic preferences, i.e., have minus infinity or plus infinity as peaks, poses the following problem. If all agents have minus infinity or all agents have plus infinity as their peak, then by Proposition 1, no *efficient* set exists in $\mathcal{C}$. Moreover, if unbounded sets of $\mathbb{R}$ are considered, then in this case the only *efficient* sets are $\{-\infty\}$ (when all agents have minus infinity as their peak) and $\{+\infty\}$ (when all agents have plus infinity as their peak). However, a policy interpretation for these two sets, as well as other unbounded sets, is not clear and we therefore do not add monotonic preferences to our model. □

Remark 7 (Closed interval alternative set). All our results hold if the preferences of the agents are defined on some closed interval $[a, b] \subsetneq \mathbb{R}$. In this case and since *efficiency* is required, by Proposition 1 (i), the class of sets considered equals the class of non-empty subsets of $[a, b]$ and closedness is not required (see Remark 5). Moreover, agents can have monotonic preferences, i.e., have a or b as peaks, since the policy interpretation of "locating the public good at a" or "locating the public good at b" is straightforward, in contrast to our original model (see Remark 6). Finally, it should be mentioned that this restriction on the set of alternatives facilitates our main proof (Theorem 1) as follows. Since a profile with a as the minimum peak and b as the maximum peak can be chosen (in contrast to our original

[18]Given $X \subsetneq \mathbb{R}$, the *closure* of X, $\mathrm{closure}(X)$, is defined as the union of X with all its limit / boundary points.

model, where a profile with $-\infty$ as the minimum peak and $+\infty$ as the maximum peak is not available), the proof essentially follows from Lemma 11. $\square$

Appendices

Throughout the Appendices we use the domain of single-peaked preferences $\mathcal{R}$, with the exception of Lemma 9 (Appendix D), where we use the domain of symmetric single-peaked preferences $\mathcal{S}$. All results proven for $\mathcal{R}$ also hold on $\mathcal{S}$; however, for Lemma 9, the proof for $\mathcal{S}$ requires a different approach (and additional "proof steps") that also holds on $\mathcal{R}$.

A Proofs of Proposition 1 and Corollary 2

The following terms describe a set obtained by a truncation of a given set $X \in \mathcal{C}$ on one side at a specific point x, which is added to the new set to ensure that this new set is closed.

Left truncaddition (of a set at a point). Let point $x \in \mathbb{R}$ and set $X \in \mathcal{C}$. Then, set $Y \in \mathcal{C}$ is a *left truncaddition* of X at x if $Y = [X \cap (x, \infty)] \cup \{x\}$.

Right truncaddition (of a set at a point). Let point $x \in \mathbb{R}$ and set $X \in \mathcal{C}$. Then, set $Y \in \mathcal{C}$ is a *right truncaddition* of X at x if $Y = [X \cap (-\infty, x)] \cup \{x\}$.

Before proceeding with the proof of Proposition 1 we present two lemmas. First, we describe some cases where a truncaddition of a set at a point makes an agent weakly better off.

Lemma 3 (Truncadditions). *Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $X \in \mathcal{C}$.*

(i) *Let minimum $\underline{X} < p(R_i)$, point $\underline{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i)$, and set $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$ be a left truncaddition of set X at point $\underline{x}$. Then, $Y \, R_i \, X$. Moreover, if the unique worst point $w_X(R_i) = \underline{X}$, then $Y \, P_i \, X$.*

(ii) *Let maximum $\bar{X} > p(R_i)$, point $\bar{x} \in \mathbb{R}$ be such that $\bar{X} > \bar{x} \geq p(R_i)$, and set $Y = [X \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$ be a right truncaddition of set X at point $\bar{x}$. Then, $Y \, R_i \, X$. Moreover, if the unique worst point $w_X(R_i) = \bar{X}$,then $Y \, P_i \, X$.*

(iii) *Let minimum* $\underline{X} < p(R_i)$*, maximum* $\bar{X} > p(R_i)$*, and points* $\underline{x}, \bar{x} \in \mathbb{R}$ *be such that* $\underline{X} < \underline{x} \leq p(R_i) \leq \bar{x} < \bar{X}$*, set* $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$ *be a left truncaddition of set* X *at point* $\underline{x}$*, and set* $Z = [Y \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$ *be a right truncaddition of set* Y *at point* $\bar{x}$*. Then,* $Z \, P_i \, X$*.*

Proof. Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $X \in \mathcal{C}$.

(i) Let minimum $\underline{X} < p(R_i)$, point $\underline{x} \in \mathbb{R}$ such that $\underline{X} < \underline{x} \leq p(R_i)$, truncaddition $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$, and Z be the set of truncated points, $Z = X \setminus Y$. By single-peakedness, for each $z \in Z$, agent i prefers $\underline{x}$ to z, $\underline{x} \, P_i \, z$. Hence, his best and worst points in Y are at least as desirable as his (respective) best and worst points in X. It follows, that $Y \, R_i \, X$. If additionally his worst point $w_X(R_i) = \underline{X} \notin Y$ is unique, then $\bar{X} \, P_i \, w_X(R_i)$ and $\underline{x} \, P_i \, w_X(R_i)$. Since by single-peakedness, $w_Y(R_i) \subseteq \{\underline{x}, \bar{X}\}$, it follows that $Y \, P_i \, X$.

(ii) Symmetric proof to (i).

(iii) Let minimum $\underline{X} < p(R_i)$, maximum $\bar{X} > p(R_i)$, points $\underline{x}, \bar{x} \in \mathbb{R}$ be such that $\underline{X} < \underline{x} \leq p(R_i) \leq \bar{x} < \bar{X}$, truncaddition $Y = [X \cap (\underline{x}, \infty)] \cup \{\underline{x}\}$, and truncaddition $Z = [Y \cap (-\infty, \bar{x})] \cup \{\bar{x}\}$. By part (i), $Y \, R_i \, X$. By part (ii), $Z \, R_i \, Y$. Hence, by transitivity, $Z \, R_i \, X$. Moreover, by single-peakedness, his worst point(s) $w_X(R_i) \subseteq \{\underline{X}, \bar{X}\}$ and $w_Z(R_i) \subseteq \{\underline{x}, \bar{x}\}$. Since by single-peakedness $\underline{x} \, P_i \, w_X(R_i)$ and $\bar{x} \, P_i \, w_X(R_i)$, his worst point(s) improves. It follows that $Z \, P_i \, X$. $\square$

Second, adding a closed interval to a set, without changing its convex hull, makes an agent indifferent, unless his best point improves, in which case he is better off. Furthermore, removing an open interval from a set, without changing its convex hull, makes an agent indifferent, unless his best point worsens, in which case he is worse off.

Lemma 4. *Let agent* $i \in \mathbb{P}$ *with preferences* $R_i \in \mathcal{R}$ *and set* $X \in \mathcal{C}$*.*

(i) *Let closed interval* $[x, y] \subseteq \mathrm{Conv}(X)$ *and set* $Y = X \cup [x, y]$*. Then,* $Y \, I_i \, X$ *unless agent* i*'s best point(s) improves, i.e.,* $b_Y(R_i) \, P_i \, b_X(R_i)$*, in which case,* $Y \, P_i \, X$*.*

(ii) *Let open interval* $(x, y) \subsetneq \mathrm{Conv}(X)$ *and set* $Y = X \setminus (x, y)$*. Then,* $X \, I_i \, Y$ *unless agent* i*'s best point(s) worsens, i.e.,* $b_X(R_i) \, P_i \, b_Y(R_i)$*, in which case,* $X \, P_i \, Y$*.*

Proof. Let agent $i \in \mathbb{P}$ with preferences $R_i \in \mathcal{R}$ and set $Y \in \mathcal{C}$.

(i) Let $[x, y] \subseteq \text{Conv}(X)$ and $Y = X \cup [x, y]$. By single-peakedness, agent i's worst point(s) does not change, $w_X(R_i) = w_Y(R_i) \subseteq \{\underline{X}, \bar{X}\}$. If for his best point(s) we have $b_X(R_i)\ I_i\ b_Y(R_i)$, then $b_X(R_i) \subseteq b_Y(R_i)$ and $Y\ I_i\ X$. Otherwise, $b_X(R_i) \not\subseteq b_Y(R_i)$, his best point(s) improves, $b_Y(R_i)\ P_i\ b_X(R_i)$, and $Y\ P_i\ X$.

(ii) Let $(x, y) \subsetneq \text{Conv}(X)$ and $Y = X \setminus (x, y)$. By single-peakedness, agent i's worst point(s) does not change, $w_X(R_i) = w_Y(R_i) \subseteq \{\underline{X}, \bar{X}\}$. If for his best point(s) we have $b_X(R_i)\ I_i\ b_Y(R_i)$, then $b_X(R_i) \supseteq b_Y(R_i)$ and $Y\ I_i\ X$. Otherwise, $b_X(R_i) \not\supseteq b_Y(R_i)$, his best point(s) worsens, $b_X(R_i)\ P_i\ b_Y(R_i)$, and $X\ P_i\ Y$. $\square$

Proof of Proposition 1. Let population $N \in \mathcal{P}$, profile $R \in \mathcal{R}^N$, and set $X \in \mathcal{C}$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq \ldots \leq p(R_n) = \bar{p}(R)$.

The proof follows in three steps.

Step 1. We show that if set $X \in \text{PE}(R)$ then condition (i) holds, that is, $X \subseteq \text{Conv}(R)$.

Let set $X \in \text{PE}(R)$. Assume by contradiction that $X \not\subseteq \text{Conv}(R)$. Then, minimum $\underline{X} < p(R_1)$ or maximum $\bar{X} > p(R_n)$. By symmetry of arguments, assume that $\underline{X} < p(R_1)$.

Case 1. Let maximum $\bar{X} > p(R_n)$. Then, for each $i \in N$, minimum $\underline{X} < p(R_1) \leq p(R_i) \leq p(R_n) < \bar{X}$. Let $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$ be a left truncaddition of X at $p(R_1)$, and $Z = [Y \cap (-\infty, p(R_n))] \cup \{p(R_n)\}$ be a right truncaddition of Y at $p(R_n)$. Therefore, by Lemma 3 (iii), for each $i \in N$, $Z\ P_i\ X$. Hence, $X \notin \text{PE}(R)$; a contradiction.

Case 2. Let maximum $\bar{X} \leq p(R_n)$. Then, for each $i \in N$, minimum $\underline{X} < p(R_1) \leq p(R_i)$. Let $Y = [X \cap (p(R_1), \infty)] \cup \{p(R_1)\}$ be a left truncaddition of X at $p(R_1)$. By Lemma 3 (i), for each $i \in N$, $Y\ R_i\ X$. Furthermore, agent n's worst point $w_X(R_n) = \underline{X}$ is unique. Therefore, by Lemma 3 (i), $Y\ P_n\ X$. Hence, $X \notin \text{PE}(R)$; a contradiction.

Step 2. We show that if set $X \in \text{PE}(R)$ then condition (ii) holds, that is, $(\text{Conv}(X) \cap p(R)) \subseteq X$.

Let set $X \in \text{PE}(R)$. By Step 1, $X \subseteq \text{Conv}(R)$. Assume by contradiction that $(\text{Conv}(X) \cap p(R)) \not\subseteq X$. Then, there exists agent $j \in N$ such that $p(R_j) \in \text{Conv}(X)$ and $p(R_j) \notin X$.

Let set $Y = X \cup \{p(R_j)\}$. By Lemma 4 (i), for each $i \in N$, $Y\ R_i\ X$. Furthermore, agent j's best point $b_Y(R_j) = p(R_j)\ P_j\ b_X(R_j)$. Therefore, by Lemma 4 (i), $Y\ P_j\ X$. Hence, $X \notin \text{PE}(R)$; a contradiction.

Step 3. We show that if conditions (i) and (ii) hold for set $X \in \mathcal{C}$, then $X \in \mathrm{PE}(R)$.

Let set $X \in \mathcal{C}$ be such that $X \subseteq \mathrm{Conv}(R)$ and $(\mathrm{Conv}(X) \cap p(R)) \subseteq X$. Assume by contradiction that $X \notin \mathrm{PE}(R)$. Hence, there exists a set $Y \subseteq \mathbb{R}$ that *dominates* set X, i.e., for each agent $i \in N$, $Y\, R_i\, X$, and for at least one agent $j \in N$, $Y\, P_j\, X$.

Case 1. Let agent j's peak $p(R_j) \in \mathrm{Conv}(X)$. By condition (ii), $p(R_j) \in X$. Agent j's best point $b_X(R_j) = p(R_j) \in X$ cannot be improved. By single-peakedness, agent j's worst point(s) $w_X(R_j) \subseteq \{\underline{X}, \bar{X}\}$; if his worst point(s) $w_Y(R_j)\, P_j\, w_X(R_j)$, by single-peakedness, minimum $\underline{X} < \underline{Y}$ or maximum $\bar{X} > \bar{Y}$. By symmetry of arguments, assume minimum $\underline{X} < \underline{Y}$. Consider agent 1; by condition (i), his peak $p(R_1) \leq \underline{X} < \underline{Y}$. By single-peakedness, his best point $b_X(R_1)\, P_1\, b_Y(R_1)$. It follows that for agent 1 set Y is not at least as desirable as set X. Hence, set Y does not *dominate* set X; a contradiction.

Case 2. Let agent j's peak $p(R_j) \notin \mathrm{Conv}(X)$. Then, either $p(R_j) < \underline{X}$ or $p(R_j) > \bar{X}$. By symmetry of arguments, assume that $p(R_j) > \bar{X}$. By single-peakedness, agent j's best point $b_X(R_j) = \bar{X}$ and agent j's worst point $w_X(R_j) = \underline{X}$. If his best point(s) $b_Y(R_j)\, P_j\, b_X(R_j)$, by single-peakedness, maximum $\bar{X} < \bar{Y}$. If his worst point(s) $w_Y(R_j)\, P_j\, w_X(R_j)$, by single-peakedness, minimum $\underline{X} < \underline{Y}$. Consider now agent 1. By condition (i), his peak $p(R_1) \leq \underline{X} \leq \bar{X}$. By single-peakedness, his best and worst point(s) are $b_X(R_1) = \underline{X}$ and $w_X(R_1) = \bar{X}$. If minimum $\underline{X} < \underline{Y}$, by single-peakedness, $b_X(R_1)\, P_1\, b_Y(R_1)$. If maximum $\bar{X} < \bar{Y}$, by single-peakedness, $w_X(R_1)\, P_1\, w_Y(R_1)$. It follows that for agent 1 set Y is not at least as desirable as set X. Hence, set Y does not *dominate* set X; a contradiction. □

Proof of Corollary 2. Let population $N \in \mathcal{P}$, profile $R \in \mathcal{R}^N$, and set $X \in \mathrm{PE}(R)$.

First, we show that $\mathrm{Conv}(X)$ and X are *equivalent* sets. By single-peakedness, for each agent $i \in N$ such that $p(R_i) \in \mathrm{Conv}(X)$, the best point $b_{\mathrm{Conv}(X)}(R_i) = p(R_i)$ and by Proposition 1 (ii), $(\mathrm{Conv}(X) \cap p(R)) \subseteq X$. Hence, the best point $b_{\mathrm{Conv}(X)}(R_i) = b_X(R_i)$. By single-peakedness, for each agent $i \in N$ such that $p(R_i) \notin \mathrm{Conv}(X)$, the best point $b_{\mathrm{Conv}(X)}(R_i) \in \{\underline{X}, \bar{X}\}$. Since $\{\underline{X}, \bar{X}\} \subseteq X$, the best point $b_{\mathrm{Conv}(X)}(R_i) = b_X(R_i)$. Moreover, since $\mathrm{Conv}(X)$ is a closed interval and (trivially) $\mathrm{Conv}(X) = X \cup \mathrm{Conv}(X)$, by Lemma 4 (i), for each agent $i \in N$, $\mathrm{Conv}(X)\, I_i\, X$.

Second, we show that if X and Y are *equivalent* sets, then $\mathrm{Conv}(X) = \mathrm{Conv}(Y)$. Let $Y \in \mathcal{C}$ be an equivalent set to $X \in \mathrm{PE}(R)$. Let agent $1 \in N$ have the smallest peak at profile R, $p(R_1) = \underline{p}(R)$. By Proposition 1 (i), $X, Y \subseteq \text{Conv}(R)$, hence, $p(R_1) \leq \underline{X} \leq \bar{X}$ and $p(R_1) \leq \underline{Y} \leq \bar{Y}$. By single-peakedness, for agent 1, [best points are $b_X(R_1) = \underline{X}$ and $b_Y(R_1) = \underline{Y}$] and [worst points are $w_X(R_1) = \bar{X}$ and $w_Y(R_1) = \bar{Y}$]. Since $X \; I_1 \; Y$, $b_X(R_1) = b_Y(R_1)$ and $w_X(R_1) = w_Y(R_1)$. Therefore, $\text{Conv}(X) = \text{Conv}(Y)$. □

B Proof of Lemma 2

Before proceeding with the proof of Lemma 2, we first prove an implication of *efficiency* and *(one-sided) replacement-dominance*.

An fp-choice correspondence satisfies *extreme-peaks-onliness* if the chosen set only depends on the convex hull of the peaks of the profile. We formulate *extreme-peaks-onliness* for fp-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Extreme-peaks-onliness. Let fixed population $N \in \mathcal{P}$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$. For each pair of profiles $R, \bar{R} \in \mathcal{R}^N$, if $\text{Conv}(R) = \text{Conv}(\bar{R})$, then $\Phi(R) = \Phi(\bar{R})$.

Notice that *extreme-peaks-onliness* not only implies the properties of *anonymity*[19] and *peaks-onliness*,[20] but since it only depends on the extreme agents' peaks, it is a much stronger property.

Lemma 5 (**Efficiency and one-sided replacement-dominance ⇒ extreme-peaks-onliness**). *If a fixed population consists of at least 3 agents, then each associated fp-choice correspondence satisfying efficiency and one-sided replacement-dominance also satisfies extreme-peaks-onliness.*

Proof. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. Let the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $\text{Conv}(R) = \text{Conv}(\bar{R})$. Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

[19] *Anonymity*: the identities of the agents do not affect the chosen set.

[20] *Peaks-Onliness*: only the peaks of the agents affect the chosen set.

We prove that $\Phi(R) = \Phi(\bar{R})$ in three steps.

Step 1. We show that if the preferences of one agent change and the convex hull of the peaks does not change, the chosen set does not change.

Case 1.1. The preferences of a middle agent at profile R change such that the convex hull of the peaks does not change. Let agent $k \in N$ be a middle agent at profile R and let profile $\bar{R} \in \mathcal{R}^N$ be such that $\bar{R}_{-k} = R_{-k}$, and $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Notice that agent k is also a middle agent at profile $\bar{R}$.[21]

By *efficiency*, $\Phi(\bar{R}) \in PE(\bar{R})$ and $\Phi(R) \in PE(R)$. Since agent k is a middle agent at both profiles R and $\bar{R}$, $\text{Conv}(\bar{R}) = \text{Conv}(R) = \text{Conv}(R_{-k})$, and by Corollary 1, $\Phi(\bar{R}), \Phi(R) \in \text{PE}(R_{-k})$. Since $\bar{R}_{-k} = R_{-k}$, by *one-sided replacement-dominance*, for each agent $i \in N \setminus \{k\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ or for each agent $i \in N \setminus \{k\}$, $\Phi(R)\ R_i\ \Phi(\bar{R})$. By *efficiency* of both sets $\Phi(R)$ and $\Phi(\bar{R})$ at profile R_{-k}, for each agent $i \in N \setminus \{k\}$, $\Phi(R)\ I_i\ \Phi(\bar{R})$. By Corollary 2, $\text{Conv}(\Phi(\bar{R})) = \text{Conv}(\Phi(R))$ and since we always represent any *efficient* set by its convex hull, $\Phi(\bar{R}) = \Phi(R)$.

Case 1.2. Either the preferences of the agent with the unique smallest peak at profiles R and $\bar{R}$ change (agent 1), or the preferences of the agent with the unique largest peak at profiles R and $\bar{R}$ change (agent n), such that the convex hull of the peaks does not change. By symmetry of arguments, assume that profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$ and $\text{Conv}(\bar{R}) = \text{Conv}(R)$. Hence, $p(\bar{R}_1) = p(R_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing middle agent 2's preferences to $R^1_2 = R_1$, i.e., $R^1 = (R_{-2}, R^1_2)$ where $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change middle agent 1's preferences to $R^2_1 = \bar{R}_1$ such that the new profile is $R^2 = (R^1_{-1}, R^2_1)$ where $\text{Conv}(R^2) = \text{Conv}(R^1)$. By Case 1.1, $\Phi(R^2) = \Phi(R^1)$. Finally, change middle agent 2's preferences back to R_2 and notice that the new profile $(R^2_{-2}, R_2) = \bar{R}$ where $\text{Conv}(\bar{R}) = \text{Conv}(R^2)$. By Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Step 2. We show that if two agents swap preferences, then the chosen set does not change.

Case 2.1. At least one of the swapping agents is a middle agent at profile R. Assume profile $\bar{R}$ is obtained from profile R by agents $j, k \in N$ swapping preferences, i.e., $\bar{R}_{-j,k} = R_{-j,k}$, $\bar{R}_j = R_k$, and $\bar{R}_k = R_j$. Let agent $k \in N$ be a middle agent at profile R. Begin

[21]Note that if agent 1 (agent n) does not have the unique smallest (largest) peak, then he is a middle agent.

from profile R and construct profile R^1 by changing agent k's preferences to $R^1_k = R_j$, i.e., $R^1 = (R_{-k}, R^1_k)$ where $\text{Conv}(R^1) = \text{Conv}(R)$. By Case 1.1, $\Phi(R^1) = \Phi(R)$. Finally, change agent j's preferences to $R^2_j = R_k$ and notice that the new profile $(R^1_{-j}, R^2_j) = \bar{R}$ where $\text{Conv}(\bar{R}) = \text{Conv}(R^1)$. By Case 1.1, $\Phi(\bar{R}) = \Phi(R^1)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 2.2. None of the swapping agents is a middle agent at profile R. Hence, $p(R_1) < p(R_2) \leq \ldots < p(R_n)$. Note that in this case, $\bar{R} \in \mathcal{R}^N$ is such that $\bar{R}_{-1,n} = R_{-1,n}$, $\bar{R}_1 = R_n$, and $\bar{R}_n = R_1$. Begin from profile R and construct profile R^1 by swapping middle agent 2's preferences with agent 1's preferences, denoting the new profile by R^1. By Case 2.1, $\Phi(R^1) = \Phi(R)$. Next, swap middle agent 1's preferences with agent n's preferences, denoting the new profile by R^2. By Case 2.1, $\Phi(R^2) = \Phi(R^1)$. Finally, swap middle agent n's preferences with agent 2's preferences and notice that the new profile is $\bar{R}$. By Case 2.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Step 3. We show how each profile $\bar{R}$, where $\text{Conv}(\bar{R}) = \text{Conv}(R)$, can be constructed from profile R by sequentially repeating the first two steps of the proof. Let profile $\bar{R}$ be such that $\bar{R} = (\bar{R}_{\bar{1}}, \ldots, \bar{R}_{\bar{n}})$ and, without loss of generality, assume $\underline{p}(\bar{R}) = p(\bar{R}_{\bar{1}}) \leq \ldots \leq p(\bar{R}_{\bar{n}}) = \bar{p}(\bar{R})$. Notice that set $\{\bar{1}, \ldots, \bar{n}\}$ is a permutation of set $N = \{1, \ldots, n\}$.

Begin from profile R and construct profile R^1 by sequentially replacing each agent' preferences R_i with $\bar{R}_{\bar{i}}$, i.e., for each $i \in N$, $R^1_i = \bar{R}_{\bar{i}}$. Note that the stepwise change of agents' preferences never changes the convex hull of peaks and that $\text{Conv}(R^1) = \text{Conv}(R)$. By Step 1, $\Phi(R^1) = \Phi(R)$. Finally, permute the agents' preferences such that each agent $\bar{i}$ obtains the preferences of agent i, i.e., the new profile R^2 is such that for each $i \in N$, $R^2_{\bar{i}} = R^1_i$. Hence, for each $i \in N$, $R^2_{\bar{i}} = \bar{R}_{\bar{i}}$ and $R^2 = \bar{R}$. Since all permutations can be obtained via sequential pairwise swaps, by Step 2, $\Phi(\bar{R}) = \Phi(R)$. $\square$

We use Lemma 5 in the proof of Lemma 2.

Proof of Lemma 2. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. By Lemma 5, Φ satisfies *extreme-peaks-onliness*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{-j} = \bar{R}_{-j}$.

We show that if $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, then all remaining agents end up at least as well off, i.e., for each $i \in N \setminus \{j\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$. Without loss of generality, assume that

$N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme-peaks-onliness*, $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, at profile R, either agent $j = 1$ has the unique smallest peak or agent $j = n$ has the unique largest peak. By symmetry of arguments, assume that $j = 1$ has the unique smallest peak and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$.

Case 2.1. Agent 1 is a middle agent at profile $\bar{R}$. Then, $\text{Conv}(\bar{R}) = \text{Conv}(R_{-1})$. By *efficiency*, $\Phi(\bar{R}) \in PE(\bar{R})$ and $\Phi(R) \in PE(R)$. By Corollary 1, $\Phi(\bar{R}) \in PE(R_{-1})$.

Assume that $\Phi(R) \subseteq \text{Conv}(R_{-1})$. Since $\Phi(R) \in PE(R)$, by Proposition 1 (ii), $\text{Conv}(\Phi(R)) \cap p(R) \subseteq \Phi(R)$. Hence, $\text{Conv}(\Phi(R)) \cap p(R_{-1}) \subseteq \Phi(R)$ and by Proposition 1, $\Phi(R) \in \text{PE}(R_{-1})$. Since $\bar{R}_{-1} = R_{-1}$ and $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$, by *one-sided replacement-dominance*, for each agent $i \in N \setminus \{1\}$, $\Phi(\bar{R})\, R_i\, \Phi(R)$ or for each agent $i \in N \setminus \{1\}$, $\Phi(R)\, R_i\, \Phi(\bar{R})$. By *efficiency* of both sets $\Phi(R)$ and $\Phi(\bar{R})$ at profile R_{-1}, for each agent $i \in N \setminus \{1\}$, $\Phi(R)\, I_i\, \Phi(\bar{R})$. By Corollary 2, $\text{Conv}(\Phi(\bar{R})) = \text{Conv}(\Phi(R))$, and since we always represent any *efficient* set by its convex hull, $\Phi(\bar{R}) = \Phi(R)$.

Assume that $\Phi(R) \not\subseteq \text{Conv}(R_{-1})$. Then, minimum $\underline{\Phi}(R) < \underline{p}(R_{-1}) \leq \underline{\Phi}(\bar{R}) \leq p(R_n)$. Hence, agent n's worst points are $w_{\Phi(R)}(R_n) = \{\underline{\Phi}(R)\}$ and $w_{\Phi(\bar{R})}(R_n) = \{\underline{\Phi}(\bar{R})\}$. By single-peakedness, $w_{\Phi(\bar{R})}(R_n)\, P_n\, w_{\Phi(R)}(R_n)$. By *one-sided replacement-dominance*, agent n is better off, $\Phi(\bar{R})\, P_n\, \Phi(R)$. Hence, by *one-sided replacement-dominance*, for each agent $i \in N \setminus \{1\}$, $\Phi(\bar{R})\, R_i\, \Phi(R)$.

Case 2.2. Recall that $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ and that agent 1 has the unique smallest peak at profile R. In addition, let agent 1 also have the unique smallest peak at profile $\bar{R}$. Then, $\text{Conv}(R_{-1}) \subsetneq \text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, $p(R_1) < p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing middle agent 2's preferences to $R_2^1 = \bar{R}_1$, i.e., $R^1 = (R_{-2}, R_2^1)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$, by *extreme-peaks-onliness*, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R_1^2 = \bar{R}_1$ such that the new profile is $R^2 = (R_{-1}^1, R_1^2)$. Since agent 1 has the unique smallest peak at profile R^1 and is a middle agent at profile R^2, by Case 2.1, for each agent $i \in N \setminus \{1, 2\}$, $\Phi(R^2) R_i \Phi(R^1)$. Finally, change middle agent 2's preferences back to R_2 and notice that the new profile $(R_{-2}^2, R_2) = \bar{R}$. Since

$\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R^2)$, by *extreme-peaks-onliness*, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, for each agent $i \in N \setminus \{1,2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$. In particular, $\Phi(\bar{R})\ R_n\ \Phi(R)$. Since agent n has the largest peak, *efficiency* and single-peakedness imply $\underline{\Phi}(R) \leq \underline{\Phi}(\bar{R})$ and $\bar{\Phi}(R) \leq \bar{\Phi}(\bar{R})$. Hence, either $\Phi(\bar{R}) = \Phi(R)$ or $\Phi(\bar{R})\ P_n\ \Phi(R)$. Then, since $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ and $\bar{R}_{-1} = R_{-1}$, by *one-sided replacement-dominance*, for each agent $i \in N \setminus \{1\}$ (including agent 2 now), $\Phi(\bar{R})\ R_i\ \Phi(R)$. □

C Proof of Proposition 5

Before proceeding with the proof of Proposition 5, we first prove an implication of *efficiency* and *population-monotonicity*.

Lemma 6. *Let choice correspondence $\Phi \in \mathcal{F}$ satisfy efficiency and population-monotonicity. Then, for each population $N \in \mathcal{P}$ such that $|N| \geq 3$ and each profile $R \in \mathcal{R}^N$, the following hold.*

(i) *Without loss of generality, let agents $1, 2 \in N$ where $p(R_1) = \underline{p}(R)$ and $p(R_2) = \underline{p}(R_{-1})$. If maximum $\bar{\Phi}(R) \in \mathrm{Conv}(R_{-1})$ and maximum $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$, then maxima $\bar{\Phi}(R) = \bar{\Phi}(R_{-1})$. Moreover, if $\Phi(R) \subseteq \mathrm{Conv}(R_{-1})$, then $\Phi(R) = \Phi(R_{-1})$.*

(ii) *Without loss of generality, let agents $n-1, n \in N$ where $p(R_n) = \bar{p}(R)$ and $p(R_{n-1}) = \bar{p}(R_{-n})$. If minimum $\underline{\Phi}(R) \in \mathrm{Conv}(R_{-n})$ and minimum $\underline{\Phi}(R) \in w_{\Phi(R)}(R_{n-1})$, then minima $\underline{\Phi}(R) = \underline{\Phi}(R_{-n})$. Moreover, if $\Phi(R) \subseteq \mathrm{Conv}(R_{-n})$, then $\Phi(R) = \Phi(R_{-n})$.*

Proof. Let choice correspondence $\Phi \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*. Let population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and profile $R \in \mathcal{R}^N$.

(i) Let agents $1, 2 \in N$ be such that $p(R_1) = \underline{p}(R)$ and $p(R_2) = \underline{p}(R_{-1})$. Let maximum $\bar{\Phi}(R) \in \mathrm{Conv}(R_{-1})$ and maximum $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$. Hence, $p(R_2) \leq \bar{\Phi}(R)$. By *population-monotonicity* and Lemma 1, for each agent $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. Let agent $n \in N \setminus \{1,2\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R) = \bar{p}(R_{-1})$. Since agent n has the largest peak at profiles R and R_{-1}, $\Phi(R_{-1})\ R_n\ \Phi(R)$ and *efficiency* imply $\underline{\Phi}(R) \leq \underline{\Phi}(R_{-1}) \leq p(R_n)$ and $\bar{\Phi}(R) \leq \bar{\Phi}(R_{-1}) \leq p(R_n)$. Since agent 2 has the smallest peak at profile R_{-1}, $p(R_2) \leq \bar{\Phi}(R)$, and $\bar{\Phi}(R) \in w_{\Phi(R)}(R_2)$, $\Phi(R_{-1})\ R_1\ \Phi(R)$ and *efficiency* imply $p(R_2) \leq \bar{\Phi}(R_{-1}) \leq \bar{\Phi}(R)$. Therefore, maxima $\bar{\Phi}(R) = \bar{\Phi}(R_{-1})$.

Moreover, let $\Phi(R) \subseteq \text{Conv}(R_{-1})$. Hence, $p(R_2) \leq \underline{\Phi}(R)$. Since agent 2 has the smallest peak at profile R_{-1} and $p(R_2) \leq \underline{\Phi}(R)$, $\Phi(R_{-1})\ R_1\ \Phi(R)$ and *efficiency* imply $p(R_2) \leq \underline{\Phi}(R_{-1}) \leq \underline{\Phi}(R)$. Therefore, minima $\underline{\Phi}(R) = \underline{\Phi}(R_{-1})$ and thus, $\Phi(R) = \Phi(R_{-n})$.

(ii) Symmetric proof to (i). □

Proof of Proposition 5. Let choice correspondence $\Phi \in \mathcal{F}$ satisfy *efficiency* and *population-monotonicity*. Recall that for each population $N \in \mathcal{P}$, each choice correspondence $\Phi \in \mathcal{F}$ specifies an fp-choice correspondence $\Phi \in \mathcal{F}^N$. Since for each $N \in \mathcal{P}$ such that $|N| \leq 2$, (*one-sided*) *replacement-dominance* imposes no restriction on fp-choice correspondence $\Phi \in \mathcal{F}^N$, let $N \in \mathcal{P}$ be such that $|N| \geq 3$.

We show that for each profile $R \in \mathcal{R}^N$, if the preferences of an agent $j \in N$ change, such that $R_{-j} = \bar{R}_{-j}$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, then the other agents whose preferences remained unchanged all end up at least as well off, as they were initially, i.e., for each $i \in N \setminus \{j\}$, $\Phi(R)\ R_i\ \Phi(\bar{R})$.[22] Without loss of generality, assume that $N = \{1, 2, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq p(R_2) \leq \ldots \leq p(R_n) = \bar{p}(R)$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$.

Case 1.1. Let agent j be a middle agent at both profiles R and $\bar{R}$. Then, $\text{Conv}(\bar{R}) = \text{Conv}(R) = \text{Conv}(R_{-j})$. Remove agent j from profile R to obtain profile R_{-j}. Since $\text{Conv}(R_{-j}) = \text{Conv}(R)$, by *population-monotonicity* and Lemma 1, $\Phi(R_{-j}) = \Phi(R)$. Next, add agent j with preferences $\bar{R}_j$ to obtain profile $\bar{R}$. Since $\text{Conv}(\bar{R}) = \text{Conv}(R_{-j})$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R_{-j})$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 1.2. Let agent j have the unique smallest (largest) peak at both profiles R and $\bar{R}$. Hence, either agent $j = 1$ has the unique smallest peak at both profiles R and $\bar{R}$ or agent $j = n$ has the unique largest peak at both profiles R and $\bar{R}$. By symmetry of arguments, assume that $j = 1$ and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$. Hence, $p(R_1) = p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$.

Begin from profile R and construct profile R^1 by changing agent 2's preferences to $R^1_2 = R_1$, i.e., $R^1 = (R_{-2}, R^1_2)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$ and agent 2 is a middle agent at both

[22] Notice that the roles of profiles R and $\bar{R}$ can be reversed, hence the case where $\text{Conv}(R) \subseteq \text{Conv}(\bar{R})$ is also covered.

profiles R^1 and R, by Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R_1^2 = \bar{R}_1$ such that the new profile is $R^2 = (R_{-1}^1, R_1^2)$. Since $\mathrm{Conv}(R^2) = \mathrm{Conv}(R^1)$ and agent 1 is a middle agent at both profiles R^2 and R^1, by Case 1.1, $\Phi(R^2) = \Phi(R^1)$. Finally, change agent 2's preferences back to R_2 and notice that the new profile $(R_{-2}^2, R_2) = \bar{R}$. Since $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R^2)$ and agent 2 is a middle agent at both profiles $\bar{R}$ and R^2, by Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$. Hence, either agent $j = 1$ has the unique smallest peak at profile R or agent $j = n$ has the unique largest peak at profile R. By symmetry of arguments, assume that $j = 1$ and profile $\bar{R}$ is such that $\bar{R}_{-1} = R_{-1}$.

Case 2.1. Let agent 1 be a middle agent at profile $\bar{R}$. Then, $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R_{-1})$. Begin from profile R and remove agent 1 from profile R to obtain profile R_{-1}. By *population-monotonicity* and Lemma 1, for each agent $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. Next, add agent 1 with preferences $\bar{R}_1$ to obtain profile $\bar{R}$. Since $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R_{-1})$, by *population-monotonicity* and Lemma 1, $\Phi(\bar{R}) = \Phi(R_{-1})$. Therefore, for each agent $i \in N \setminus \{1\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$.

Case 2.2. Recall that $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ and let agent 1 have the unique smallest peak at profile R. In addition, let agent 1 also have the unique smallest peak at profile $\bar{R}$. Then, $\mathrm{Conv}(R_{-1}) \subsetneq \mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$. Hence, $p(R_1) < p(\bar{R}_1) < p(R_2) \leq \ldots \leq p(R_n)$. The proof of this case proceeds in two parts.

First, we show that for each agent $i \in N \setminus \{1, 2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ and $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$. Begin from profile R and construct profile R^1 by changing agent 2's preferences to $R_2^1 = \bar{R}_1$, i.e., $R^1 = (R_{-2}, R_2^1)$. Since $\mathrm{Conv}(R^1) = \mathrm{Conv}(R)$ and agent 2 is a middle agent at both profiles R^1 and R, by Case 1.1, $\Phi(R^1) = \Phi(R)$. Next, change agent 1's preferences to $R_1^2 = \bar{R}_1$ such that the new profile is $R^2 = (R_{-1}^1, R_1^2)$. Since agent 1 is a middle agent at profile R^2, by Case 2.1, for each agent $i \in N \setminus \{1\}$, $\Phi(R^2)\ R_i^1\ \Phi(R^1)$. Hence, for each agent $i \in N \setminus \{1, 2\}$, $\Phi(R^2)\ R_i\ \Phi(R^1)$ and $\Phi(R^2)\ \bar{R}_1\ \Phi(R^1)$. Finally, change agent 2's preferences back to R_2 and notice that the new profile $(R_{-2}^2, R_2) = \bar{R}$. Since $\mathrm{Conv}(\bar{R}) = \mathrm{Conv}(R^2)$ and agent 2 is a middle agent at both profiles $\bar{R}$ and R^2, by Case 1.1, $\Phi(\bar{R}) = \Phi(R^2)$. Therefore, for each agent $i \in N \setminus \{1, 2\}$, $\Phi(\bar{R})\ R_i\ \Phi(R)$ and $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$.

Second, we prove that $\Phi(\bar{R}) R_2 \Phi(R)$. Since agent n has the largest peak at both profiles R and $\bar{R}$, $\Phi(\bar{R})\ R_n\ \Phi(R)$ and *efficiency* imply $\underline{\Phi}(R) \leq \underline{\Phi}(\bar{R}) \leq p(R_n)$ and $\bar{\Phi}(R) \leq \bar{\Phi}(\bar{R}) \leq p(R_n)$.

Hence, either $\Phi(\bar{R}) = \Phi(R)$ or $\Phi(\bar{R})\ P_n\ \Phi(R)$. If $\Phi(\bar{R}) = \Phi(R)$, then $\Phi(\bar{R})\ R_2\ \Phi(R)$. If $\Phi(\bar{R})\ P_n\ \Phi(R)$, then (a) $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) \leq p(R_n)$ or (b) $\bar{\Phi}(R) < \bar{\Phi}(\bar{R}) \leq p(R_n)$.

If $\underline{\Phi}(R) \geq p(R_2)$, then $\Phi(\bar{R}) \subseteq \mathrm{Conv}(R_{-1})$ and by Lemma 6 (i), $\Phi(\bar{R}) = \Phi(R_{-1})$. Next, consider the change from profile R to R_{-1}. By *population-monotonicity* and Lemma 1, for each agent $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. Therefore, for each agent $i \in N \setminus \{1\}$ (including agent 2 now), $\Phi(\bar{R})\ R_i\ \Phi(R)$.

The remaining case is that $\underline{\Phi}(R) < p(R_2)$. Since agent 1 has the smallest peak at profile $\bar{R}$, *efficiency* implies $p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R})$. If (a) $\underline{\Phi}(R) < \underline{\Phi}(\bar{R})$, then $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$ implies $\underline{\Phi}(R) < p(\bar{R}_1)$ and if (b) $\bar{\Phi}(R) < \bar{\Phi}(\bar{R})$, then $\Phi(\bar{R})\ \bar{R}_1\ \Phi(R)$ implies $\bar{\Phi}(R) < p(\bar{R}_1)$ and thus, $\underline{\Phi}(R) < p(\bar{R}_1)$.

Hence, there are two cases (2.2.α) $\underline{\Phi}(R) < p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) < p(R_2)$ and $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ and (2.2.β) $\underline{\Phi}(R) \leq \bar{\Phi}(R) < p(\bar{R}_1) \leq \underline{\Phi}(\bar{R}) < p(R_2)$.

Case 2.2.α. If $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) \leq p(R_2)$, then $b_{\Phi(R)}(R_2) = \bar{\Phi}(R) = \bar{\Phi}(\bar{R}) = b_{\Phi(\bar{R})}(R_2) \leq p(R_2)$ and $w_{\Phi(R)}(R_2) = \underline{\Phi}(R) < \underline{\Phi}(\bar{R}) = w_{\Phi(\bar{R})}(R_2) < p(R_2)$. By single-peakedness, $\Phi(\bar{R})\ P_2\ \Phi(R)$.

If $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) > p(R_2)$, then $b_{\Phi(R)}(R_2) = b_{\Phi(\bar{R})}(R_2) = p(R_2)$, $w_{\Phi(R)}(R_2) \in \{\underline{\Phi}(R), \bar{\Phi}(R)\}$, and $w_{\Phi(\bar{R})}(R_2) \in \{\underline{\Phi}(\bar{R}), \bar{\Phi}(\bar{R})\}$. Then, $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) < p(R_2) < \bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ and single-peakedness imply $\Phi(\bar{R})\ R_2\ \Phi(R)$.

Case 2.2.β. Notice that $b_{\Phi(R)}(R_2) = \{\bar{\Phi}(R)\}$ and $w_{\Phi(R)}(R_2) = \{\underline{\Phi}(R)\}$.

If $\bar{\Phi}(\bar{R}) \leq p(R_2)$, then $\bar{\Phi}(\bar{R}) \in b_{\Phi(\bar{R})}(R_2)$ and $\underline{\Phi}(\bar{R}) \in w_{\Phi(\bar{R})}(R_2)$. Since then $\underline{\Phi}(R) \leq \bar{\Phi}(R) < \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R}) \leq p(R_2)$, by single-peakedness, $\Phi(\bar{R})\ P_2\ \Phi(R)$.

If $\bar{\Phi}(\bar{R}) > p(R_2)$, then $b_{\Phi(\bar{R})}(R_2) = \{p(R_2)\}$ and $w_{\Phi(\bar{R})}(R_2) \subseteq \{\underline{\Phi}(\bar{R}), \bar{\Phi}(\bar{R})\}$. Hence, $b_{\Phi(\bar{R})}(R_2)\ P_2\ b_{\Phi(R)}(R_2)$. Since $\underline{\Phi}(R) < \underline{\Phi}(\bar{R}) < p(R_2)$, by single-peakedness, $\underline{\Phi}(\bar{R})\ P_2\ \underline{\Phi}(R) = w_{\Phi(R)}(R_2)$.

If $\underline{\Phi}(\bar{R}) \in w_{\Phi(\bar{R})}(R_2)$, then $w_{\Phi(\bar{R})}(R_2)\ P_2\ w_{\Phi(R)}(R_2)$ and $\Phi(\bar{R})\ P_2\ \Phi(R)$.

Finally, if $\underline{\Phi}(\bar{R}) \notin w_{\Phi(\bar{R})}(R_2)$, then $w_{\Phi(\bar{R})}(R_2) = \{\bar{\Phi}(\bar{R})\}$. Note that $\bar{\Phi}(\bar{R}) \in \mathrm{Conv}(R_{-1})$. By Lemma 6 (i), $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R_{-1})$. Consider the change from profile R to R_{-1}. By *population-monotonicity* and Lemma 1, for each agent $i \in N \setminus \{1\}$, $\Phi(R_{-1})\ R_i\ \Phi(R)$. In particular, $\Phi(R_{-1})\ R_2\ \Phi(R)$ and $w_{\Phi(R_{-1})}(R_2)\ R_2\ w_{\Phi(R)}(R_2)$. Since agent 2 has the smallest peak at profile R_{-1}, *efficiency* and single-peakedness imply that $\bar{\Phi}(R_{-1}) \in w_{\Phi(R_{-1})}(R_2)$. Hence,

$\bar{\Phi}(\bar{R}) \in w_{\Phi(R_{-1})}(R_2)$ and $w_{\Phi(\bar{R})}(R_2) = \bar{\Phi}(\bar{R})\ R_2\ w_{\Phi(R)}(R_2)$. Therefore, $\Phi(\bar{R})\ R_2\ \Phi(R)$. □

D Proof of Theorem 1

Before proceeding with the proof of Theorem 1, we first prove some implications of *efficiency* and *(one-sided) replacement-dominance*. The first implication is *peak-monotonicity*, introduced by Ching (1994). The definition follows.

An fp-choice correspondence satisfies *peak-monotonicity* if whenever an agent's preferences change such that his peak moves to the left (right), the chosen set moves to the left (right). We formulate *peak-monotonicity* for fp-choice correspondences but as discussed in Remark 3, it easily extends to choice correspondences.

Peak-monotonicity. Let fixed population $N \in \mathcal{P}$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } p(\bar{R}_j) \leq p(R_j), \text{ then } \begin{cases} \text{minimum } \underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R) \\ \qquad\qquad \text{and} \\ \text{maximum } \bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R). \end{cases}$$

Lemma 7 (Efficiency and one-sided replacement-dominance ⇒ peak-monotonicity). *If a fixed population consists of at least 3 agents, then an associated fp-choice correspondence that satisfies efficiency and one-sided replacement-dominance also satisfies peak-monotonicity.*

Proof. Let fixed population $N \in \mathcal{P}$ such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{-j} = \bar{R}_{-j}$ and $p(\bar{R}_j) \leq p(R_j)$. By *efficiency*, $\Phi(R) \in \text{PE}(R)$ and $\Phi(\bar{R}) \in \text{PE}(\bar{R})$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

Case 1. Let agent j be a middle agent or have the smallest peak at profile R. Hence, $\underline{p}(\bar{R}) \leq \underline{p}(R) \leq \bar{p}(\bar{R}) = \bar{p}(R)$ and $\text{Conv}(R) \subseteq \text{Conv}(\bar{R})$. By *one-sided replacement-dominance* and Lemma 2, for each agent $i \in N \setminus \{j\}$, $\Phi(R)\ R_i\ \Phi(\bar{R})$. Finally, let agent $n \in N \setminus \{j\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R) = \bar{p}(\bar{R})$. By $\Phi(R)\ R_n\ \Phi(\bar{R})$ and *efficiency*, $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R) \leq p(R_n)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R) \leq p(R_n)$.

Case 2. Let agent j have the unique largest peak at profile R.

Case 2.1. Let agent j have the unique largest peak at profile R and be a middle agent at profile $\bar{R}$. Hence, $\underline{p}(\bar{R}) = \underline{p}(R) \leq \bar{p}(\bar{R}) < \bar{p}(R)$. By the symmetric argument of Case 1 (with agent n being a middle agent at profile $\bar{R}$ instead of agent 1 being a middle agent at profile R, and with agent n's peak moving to the right instead of agent 1's peak moving to the left), $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R)$.

Case 2.2. Let agent j have the unique largest peak at profile R and the unique smallest peak at profile $\bar{R}$. Hence, $\underline{p}(\bar{R}) < \underline{p}(R) \leq \bar{p}(\bar{R}) < \bar{p}(R)$. Begin from profile R and construct profile R^1 by changing agent j's preferences to R^1_j such that his peak $p(R^1_j) = \underline{p}(R)$, i.e., $R^1 = (R_{-j}, R^1_j)$. Since agent j has the unique largest peak at profile R and is a middle agent at profile R^1, by Case 2.1, $\underline{\Phi}(R^1) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(R^1) \leq \bar{\Phi}(R)$. Finally, change agent j's preferences to $\bar{R}_j$ and notice that the new profile $(R^1_{-j}, \bar{R}_j) = \bar{R}$. Since agent j is a middle agent at profile R^1, by Case 1, $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R^1) \leq \underline{\Phi}(R)$ and $\bar{\Phi}(\bar{R}) \leq \bar{\Phi}(R^1) \leq \bar{\Phi}(R)$. $\square$

The second implication of *efficiency* and *(one-sided) replacement-dominance* is *uncompromisingness*, introduced by Border and Jordan (1983). The definition follows.

Loosely speaking, an fp-choice correspondence satisfies *uncompromisingness* if whenever an agent's preferences change such that his peaks, before and after this change, both lie on the same side of the minimum (maximum) point chosen, the minimum (maximum) point chosen does not change. We formulate *uncompromisingess* –and later *set-uncompromisigness*– for fp-choice correspondences but as discussed in Remark 3, they easily extend to choice correspondences.

Uncompromisigness. Let fixed population $N \in \mathcal{P}$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$.

For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } \begin{cases} p(R_j) < \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \underline{\Phi}(R) \\ \text{or} \\ p(R_j) > \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \underline{\Phi}(R), \end{cases} \quad \text{then minima } \underline{\Phi}(R) = \underline{\Phi}(\bar{R})$$

and

$$\text{if } \begin{cases} p(R_j) > \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \bar{\Phi}(R) \\ \text{or} \\ p(R_j) < \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \bar{\Phi}(R), \end{cases} \quad \text{then maxima } \bar{\Phi}(R) = \bar{\Phi}(\bar{R}).$$

Uncompromisingness immediately implies the following notion of *set-uncompromisingness.*

Set-uncompromisigness. Let fixed population $N \in \mathcal{P}$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$,

$$\text{if } \begin{cases} p(R_j) < \underline{\Phi}(R) \text{ and } p(\bar{R}_j) \leq \underline{\Phi}(R) \\ \text{or} \\ p(R_j) > \bar{\Phi}(R) \text{ and } p(\bar{R}_j) \geq \bar{\Phi}(R), \end{cases} \quad \text{then } \Phi(R) = \Phi(\bar{R}).$$

Lemma 8 (**Uncompromisingness $\Rightarrow$ set-uncompromisingness**). *Each fp-choice correspondence satisfying uncompromisingness also satisfies set-uncompromisingness.*

Proof. Follows trivially by the definitions of *uncompomisingness* and *set-uncompromisingness.* □

Before stating in Lemma 10 some conditions under which an fp-choice correspondence satisfies *uncompromisingness*, we first state a result for the domain of symmetric single-peaked preferences $\mathcal{S}$ (Lemma 9). This is the only result where we have to change the proof technique when dealing with domain $\mathcal{S}$.[23] Specifically, we prove Lemma 9 using a so-called "leapfrogging" argument. During each leapfrog we right (left) extend the convex hull of the peaks by some distance and if this distance is not enough we repeat this argument as many

[23]Recall that all steps in all other proofs are for domain $\mathcal{R}$ but they automatically apply to domain $\mathcal{S}$.

(finite) times as necessary. Notice that Lemma 9 also holds on the domain of single-peaked preferences $\mathcal{R}$.

Lemma 9. *Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy efficiency and one-sided replacement-dominance. For each agent $j \in N$ and each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $R_{-j} = \bar{R}_{-j}$ and* $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(\bar{R})$*,*

- **(i)** *if minimum $\underline{\Phi}(R) < \bar{p}(R) < p(\bar{R}_j)$, then minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$. Moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R)$, then $\Phi(\bar{R}) = \Phi(R)$,*
- **(ii)** *if maximum $\bar{\Phi}(R) > \underline{p}(R) > p(\bar{R}_j)$, then maxima $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R)$. Moreover, if also minimum $\underline{\Phi}(R) > \underline{p}(R)$, then $\Phi(\bar{R}) = \Phi(R)$,*

Proof. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. By Lemmas 5 (Appendix B) and 7, Φ satisfies *extreme-peaks-onliness* and *peak-monotonicity*.

Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{-j} = \bar{R}_{-j}$ and $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(\bar{R})$. By *efficiency*, $\Phi(R) \in \mathrm{PE}(R)$. By *extreme-peaks-onliness*, it is without loss of generality to assume that both profiles R and $\bar{R}$ are symmetric, i.e., $R, \bar{R} \in \mathcal{S}^N$.[24] In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

(i) Let minimum $\underline{\Phi}(R) < \bar{p}(R) < p(\bar{R}_j)$. Since $\bar{p}(R) < p(\bar{R}_j)$ and $\Phi(R) \in \mathrm{PE}(R)$, by Proposition 1 (i), $\underline{p}(R) \leq \underline{\Phi}(R) \leq \bar{\Phi}(R) \leq \bar{p}(R) < p(\bar{R}_j)$. Since also $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(\bar{R})$, agent j either [is a middle agent at profile R and has the unique largest peak at profile $\bar{R}$] or [has the unique largest peak at both profiles R and $\bar{R}$].

Case 1. Let agent j be a middle agent at profile R and have the unique largest peak at profile $\bar{R}$. Let agent $n \in N \setminus \{j\}$ have the largest peak at profile R, i.e., $p(R_n) = \bar{p}(R)$. Hence, minimum $\underline{\Phi}(R) < \bar{p}(R)$ and *efficiency* imply $\underline{\Phi}(R) < p(R_n)$ and $\bar{\Phi}(R) \leq p(R_n)$. By single-peakedness, $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$ and $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$.

Let the distance between minimum $\underline{\Phi}(R)$ and peak $p(R_n)$ be $\delta_0 = |\underline{\Phi}(R) - p(R_n)|$. Let point $x_1 \in \mathbb{R}$ be on the right side of peak $p(R_n)$, i.e., $x_1 > p(R_n) = \bar{p}(R)$, such that the

[24] For each agent $i \in N$, we can replace preferences $R_i, \bar{R}_i \in \mathcal{R}$ by preferences $R'_i, \bar{R}'_i \in \mathcal{S}$ such that $p(R_i) = p(R'_i)$ and $p(\bar{R}_i) = p(\bar{R}'_i)$. Then, by *extreme-peaks-onliness*, $\Phi(R) = \Phi(R')$ and $\Phi(\bar{R}) = \Phi(\bar{R}')$.

distance between minimum $\underline{\Phi}(R)$ and point x_1 is $\delta_1 = |\underline{\Phi}(R) - x_1| = \frac{3}{2}\delta_0$. Hence, distance $|p(R_n) - x_1| = |\underline{\Phi}(R) - x_1| - |\underline{\Phi}(R) - p(R_n)| = \frac{1}{2}\delta_0 = \frac{1}{2}|\underline{\Phi}(R) - p(R_n)|$ and point x_1 is closer to peak $p(R_n)$ than minimum $\underline{\Phi}(R)$ is.

Step 1. Begin from profile R and construct profile R^1 by changing agent j's preferences to $R^1_j \in \mathcal{S}$ such that his peak

$$p(R^1_j) = \begin{cases} p(\bar{R}_j) & \text{if } p(\bar{R}_j) \leq x_1 \\ x_1 & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-j}, R^1_j)$. Hence, $R^1_{-j} = R_{-j}$. By *efficiency* and Proposition 1 (i), $\underline{p}(R) = \underline{p}(R^1) \leq \underline{\Phi}(R^1) \leq \bar{\Phi}(R^1) \leq \bar{p}(R^1) = p(\bar{R}_j)$. Since $p(R^1_j) > p(R_j)$, by *peak-monotonicity*, minimum $\underline{\Phi}(R^1) \geq \underline{\Phi}(R)$ and maximum $\bar{\Phi}(R^1) \geq \bar{\Phi}(R)$. Hence, $\underline{\Phi}(R^1) \in [\underline{\Phi}(R), p(\bar{R}_j)]$ and $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(\bar{R}_j)]$. Since $\mathrm{Conv}(R) \subsetneq \mathrm{Conv}(R^1)$, by *one-sided replacement-dominance* and Lemma 2, agent n ends up at most as well off, $\Phi(R)\ R_n\ \Phi(R^1)$. Hence, $b_{\Phi(R)}(R_n)\ R_n\ b_{\Phi(R^1)}(R_n)$ and $w_{\Phi(R)}(R_n)\ R_n\ w_{\Phi(R^1)}(R_n)$.

If $\underline{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$, then $w_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$. The distance of agent n's worst point $\bar{\Phi}(R^1)$ to peak $p(R_n)$ is $|p(R_n) - \bar{\Phi}(R^1)| \leq |p(R_n) - p(R^1_j)| \leq |p(R_n) - x_1| = \frac{1}{2}\delta_0 = \frac{1}{2}|\underline{\Phi}(R) - p(R_n)|$, which is smaller than the distance of minimum $\underline{\Phi}(R)$ to peak $p(R_n)$. By symmetric single-peakedness, agent n prefers $w_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$ to $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$; a contradiction. Hence, $\underline{\Phi}(R^1) \in [\underline{\Phi}(R), p(R_n))$ and $w_{\Phi(R^1)}(R_n) = \underline{\Phi}(R^1)$. Since $\underline{\Phi}(R^1) < p(R_n)$, for agent n to find $w_{\Phi(R)}(R_n) = \underline{\Phi}(R)$ at least as desirable as $w_{\Phi(R^1)}(R_n) = \underline{\Phi}(R^1)$, then minimum $\underline{\Phi}(R) \geq \underline{\Phi}(R^1)$. Hence, minima $\underline{\Phi}(R^1) = \underline{\Phi}(R)$.

Moreover, let maximum $\bar{\Phi}(R) < \bar{p}(R) = p(R_n)$. Then, $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$. Recall that $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(\bar{R}_j)]$. If $\bar{\Phi}(R^1) \in [p(R_n), p(\bar{R}_j)]$, then agent n prefers $b_{\Phi(R^1)}(R_n) = p(R_n)$ to $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$; a contradiction. Hence, $\bar{\Phi}(R^1) \in [\bar{\Phi}(R), p(R_n))$ and $b_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$. Since $\bar{\Phi}(R^1) < p(R_n)$, for agent n to find $b_{\Phi(R)}(R_n) = \bar{\Phi}(R)$ at least as desirable as $b_{\Phi(R^1)}(R_n) = \bar{\Phi}(R^1)$, then maximum $\bar{\Phi}(R) \geq \bar{\Phi}(R^1)$. Hence, maxima $\bar{\Phi}(R^1) = \bar{\Phi}(R)$ and $\Phi(R^1) = \Phi(R)$.

If $p(R^1_j) = p(\bar{R}_j)$, then $\mathrm{Conv}(R^1) = \mathrm{Conv}(\bar{R})$ and by *extreme-peaks-onliness*, $\Phi(R^1) = \Phi(\bar{R})$ and we are done. If $p(R^1_j) \neq p(\bar{R}_j)$, then note that agent n is now a middle agent and agent j has the unique largest peak at profile R^1. We now explain the term "leapfrogging" in order to explain the proof technique: in Step 1, the peak of agent j moves to the right of agent

n's peak by figuratively leapfrogging over agent n. In Step 2, the roles of agents j and n reverse, and agent n leapfrogs over agent j to the right, etc.

Let point $x_2 \in \mathbb{R}$ be on the right side of peak $p(R_j^1)$, i.e., $x_2 > p(R_j^1) = \bar{p}(R^1)$, such that the distance between minimum $\underline{\Phi}(R)$ and point x_2 is $\delta_2 = |\underline{\Phi}(R) - x_2| = \frac{3}{2}\delta_1$. Hence, distance $|p(R_j^1) - x_2| = |\underline{\Phi}(R) - x_2| - |\underline{\Phi}(R) - p(R_j^1)| = \frac{1}{2}\delta_1 = \frac{1}{2}|\underline{\Phi}(R) - p(R_j^1)|$ and point x_2 is closer to peak $p(R_j^1)$ than minimum $\underline{\Phi}(R)$ is.

Step 2. Begin from profile R^1 and construct profile R^2 by changing agent n's preferences to $R_n^2 \in \mathcal{S}$ such that his peak

$$p(R_n^2) = \begin{cases} p(\bar{R}_j) & \text{if } p(\bar{R}_j) \leq x_2 \\ x_2 & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R_{-n}^1, R_n^2)$. Hence, $R_{-n}^2 = R_{-n}^1$. By the arguments described in the previous step (with profiles R and R^1 replaced by profiles R^1 and R^2 and with agent n in the role of agent j), minima $\underline{\Phi}(R^2) = \underline{\Phi}(R^1) = \underline{\Phi}(R)$.

Moreover, let maximum $\bar{\Phi}(R) < \bar{p}(R)$. Then, maximum $\bar{\Phi}(R) = \bar{\Phi}(R^1) < \bar{p}(R^1) = p(R_j^1)$ and by the arguments described in the previous step (with profiles R and R^1 replaced by profiles R^1 and R^2 and with agent n in the role of agent j), $\Phi(R^2) = \Phi(R^1) = \Phi(R)$.

If $p(R_n^2) = p(\bar{R}_j)$, then $\text{Conv}(R^2) = \text{Conv}(\bar{R})$ and by *extreme-peaks-onliness*, $\Phi(R^2) = \Phi(\bar{R})$ and we are done. If $p(R_j^2) \neq p(\bar{R}_j)$. Then, according to the reasoning described below, repeat the leapfrogging steps described above $\nu \in \mathbb{N}^+$ amount of times.

Recall that $\delta_1 = \frac{3}{2}\delta_0$ and $\delta_2 = \frac{3}{2}\delta_1$. Hence, $\delta_\nu = \frac{3}{2}\delta_{\nu-1} = \left(\frac{3}{2}\right)^\nu \delta_0$ and since $\delta_0 \neq 0$, in the limit, $\lim_{\nu\to\infty} \delta_\nu = \infty$. Thus, for each profile $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}_{-j} = R_{-j}$ and $p(\bar{R}_j) > p(R_j)$, there exists a finite $\nu \in \mathbb{N}^+$ such that the distance $\delta_\nu > |\underline{\Phi}(R) - p(\bar{R}_j)|$. Therefore, for each profile $\bar{R} \in \mathcal{R}^N$ such that $\bar{R}_{-j} = R_{-j}$ and $p(\bar{R}_j) > p(R_j)$, there exists a profile R^ν such that $\text{Conv}(R^\nu) = \text{Conv}(\bar{R})$ and the following holds. If minimum $\underline{\Phi}(R) < \bar{p}(R) = p(R_n) < p(\bar{R}_j)$, then minima $\underline{\Phi}(R^\nu) = \underline{\Phi}(R)$ and moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R)$, then $\Phi(R^\nu) = \Phi(R)$. Since $\text{Conv}(R^\nu) = \text{Conv}(\bar{R})$, by *extreme-peaks-onliness*, $\Phi(R^\nu) = \Phi(\bar{R})$ and we are done.

Case 2. Let agent $j = n$ have the unique largest peak at profiles R and $\bar{R}$. Let agent $k \in N\backslash\{j\}$ be a middle agent at profile R and construct profile R^1 by changing his preferences to R_k^1 such that his peak $p(R_k^1) = \bar{p}(R)$, i.e., $R^1 = (R_{-k}, R_k^1)$. Since $\text{Conv}(R^1) = \text{Conv}(R)$, by *extreme-peaks-onliness*, $\Phi(R^1) = \Phi(R)$. Therefore, since minimum $\underline{\Phi}(R) < \bar{p}(R) = \bar{p}(R^1) = p(R_k^1) < p(\bar{R}_j)$, by Case 1, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R^1) = \underline{\Phi}(R)$ and moreover, if also maximum $\bar{\Phi}(R) < \bar{p}(R) = \bar{p}(R^1)$, by Case 1, $\Phi(\bar{R}) = \Phi(R^1) = \Phi(R)$.

(ii) Symmetric proof to (i). □

Lemma 10 (**Efficiency and one-sided replacement-dominance ⇒ uncompromisingness**). *If a fixed population consists of at least 3 agents, then an associated fp-choice correspondence that satisfies efficiency and one-sided replacement-dominance also satisfies uncompromisingness.*

Proof. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. By Lemmas 5 (Appendix B) and 7, Φ satisfies *extreme-peaks-onliness* and *peak-monotonicity*. Let agent $j \in N$ and the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $R_{-j} = \bar{R}_{-j}$. In the following, we refer to agents who have neither the unique smallest peak nor the unique largest peak as *middle agents*.

(i) We show that if $[p(R_j) < \underline{\Phi}(R)$ and $p(\bar{R}_j) \leq \underline{\Phi}(R)]$ or $[p(R_j) > \underline{\Phi}(R)$ and $p(\bar{R}_j) \geq \underline{\Phi}(R)]$, then minima $\underline{\Phi}(R) = \underline{\Phi}(\bar{R})$. By *efficiency*, $\Phi(R) \in \text{PE}(R)$. Hence by Proposition 1 (i), $\Phi(R) \subseteq \text{Conv}(R)$. Notice that $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$ or $\text{Conv}(\bar{R}) \supseteq \text{Conv}(R)$.

Case 1. Let $p(R_j) < \underline{\Phi}(R)$ and $p(\bar{R}_j) \leq \underline{\Phi}(R)$. Hence, since $\Phi(R) \subseteq \text{Conv}(R)$, $p(R_j) \neq \bar{p}(R)$.

Case 1.1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme-peaks-onliness*, $\Phi(R) = \Phi(\bar{R})$.

Case 1.2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, agent j has the unique smallest peak at profile R and minimum $\underline{\Phi}(R) \geq p(\bar{R}_j) \geq \underline{p}(\bar{R}) > p(R_j)$. Begin from profile R and construct profile $\bar{R}$ by changing agent j's preferences to $\bar{R}_j$, i.e., $\bar{R} = (R_{-j}, \bar{R}_j)$. Since $p(\bar{R}_j) > p(R_j)$ and $\bar{R}_{-j} = R_{-j}$, by *peak-monotonicity*, minimum $\underline{\Phi}(\bar{R}) \geq \underline{\Phi}(R)$. If minimum $\underline{\Phi}(\bar{R}) > \underline{\Phi}(R) \geq p(\bar{R}_j)$, then $\Phi(\bar{R}) \neq \Phi(R)$ and minimum $\underline{\Phi}(\bar{R}) > \underline{p}(\bar{R}) > p(R_j)$. Since $\bar{R}_{-j} = R_{-j}$, by Lemma 9 (ii) (with the roles of R and $\bar{R}$ reversed), $\Phi(\bar{R}) = \Phi(R) \neq \Phi(\bar{R})$, a contradiction. Therefore, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

Case 1.3. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Hence, agent j has the unique smallest peak at profile $\bar{R}$ and minimum $\underline{\Phi}(R) > p(R_j) \geq \underline{p}(R) > p(\bar{R}_j)$. By Lemma 9 (ii), $\Phi(\bar{R}) = \Phi(R)$.

Case 2. Let $p(R_j) > \underline{\Phi}(R)$ and $p(\bar{R}_j) \geq \underline{\Phi}(R)$. Hence, since $\Phi(R) \subseteq \text{Conv}(R)$, $p(R_j) \neq \underline{p}(R)$.

Case 2.1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme-peaks-onliness*, $\Phi(R) = \Phi(\bar{R})$.

Case 2.2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$. Hence, agent j has the unique largest peak at profile R and minimum $\underline{\Phi}(R) \leq p(\bar{R}_j) \leq \bar{p}(\bar{R}) < p(R_j)$. Begin from profile R and construct profile $\bar{R}$ by changing agent j's preferences to $\bar{R}_j$, i.e., $\bar{R} = (R_{-j}, \bar{R}_j)$. Since $p(\bar{R}_j) < p(R_j)$ and $\bar{R}_{-j} = R_{-j}$, by *peak-monotonicity*, minimum $\underline{\Phi}(\bar{R}) \leq \underline{\Phi}(R)$. If minimum $\underline{\Phi}(\bar{R}) < \underline{\Phi}(R) \leq p(\bar{R}_j)$, then minimum $\underline{\Phi}(\bar{R}) < \bar{p}(\bar{R}) < p(R_j)$. Since $\bar{R}_{-j} = R_{-j}$, by Lemma 9 (i) (with the roles of R and $\bar{R}$ reversed), minimum $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R) \neq \underline{\Phi}(\bar{R})$, a contradiction. Therefore, minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

Case 2.3. Let $\text{Conv}(\bar{R}) \supsetneq \text{Conv}(R)$. Hence, agent j has the unique largest peak at profile $\bar{R}$ and minimum $\underline{\Phi}(R) < p(R_j) \leq \bar{p}(R) < p(\bar{R}_j)$. By Lemma 9 (i), minima $\underline{\Phi}(\bar{R}) = \underline{\Phi}(R)$.

(ii) The proof that if $[p(R_j) > \bar{\Phi}(R)$ and $p(\bar{R}_j) \geq \bar{\Phi}(R)]$ or $[p(R_j) < \bar{\Phi}(R)$ and $p(\bar{R}_j) \leq \bar{\Phi}(R)]$, then maxima $\bar{\Phi}(R) = \bar{\Phi}(\bar{R})$ is symmetric to the proof of (i). □

The next result is crucial in the proof of Theorem 1.

Lemma 11. *Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy efficiency and one-sided replacement-dominance. Let fp-target set correspondence $\Phi^{a,b} \in \mathcal{F}^N$. For each pair of profiles $R, \bar{R} \in \mathcal{R}^N$ such that $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$, if $\Phi(R) = \Phi^{a,b}(R)$, then $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.*

Proof. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. Let fp-target set correspondence $\Phi^{a,b} \in \mathcal{F}^N$. By Propositions 1 and 4, $\Phi^{a,b}$ satisfies *efficiency* and *one-sided replacement-dominance*. By Lemma 5 (Appendix B), Lemma 10, and Lemma 8, Φ and $\Phi^{a,b}$ satisfy *extreme-peaks-onliness*, *uncompromisingness*, and *set-uncompromisingness*.

Let the pair of profiles $R, \bar{R} \in \mathcal{R}^N$ be such that $\Phi(R) = \Phi^{a,b}(R)$ and $\text{Conv}(\bar{R}) \subseteq \text{Conv}(R)$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\underline{p}(R) = p(R_1) \leq \cdots \leq p(R_n) = \bar{p}(R)$. We show that $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 1. Let $\text{Conv}(\bar{R}) = \text{Conv}(R)$. By *extreme-peaks-onliness* and the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = \Phi(R) = \Phi^{a,b}(R) = \Phi^{a,b}(\bar{R})$.

Case 2. Let $\text{Conv}(\bar{R}) \subsetneq \text{Conv}(R)$ be such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)$. By *extreme-peaks-onliness*, it is without loss of generality to assume that at both profiles R and $\bar{R}$, agent

1 has the smallest peak and all other agents have the largest peak, i.e., $R = (R_1, R_n, \dots, R_n)$ such that $p(R_1) \leq p(R_n)$ and $\bar{R} = (\bar{R}_1, R_n, \dots, R_n)$ such that $p(\bar{R}_1) \leq p(R_n)$. Hence, $R_{-1} = \bar{R}_{-1}$ and $\underline{p}(R) < \underline{p}(\bar{R}) \leq \bar{p}(\bar{R}) = \bar{p}(R)$. By *efficiency* and Proposition 1 (i), $p(R_1) = \underline{p}(R) \leq \underline{\Phi}(R) \leq \bar{\Phi}(R) \leq \bar{p}(R)$ and $p(\bar{R}_1) = \underline{p}(\bar{R}) \leq \underline{\Phi}(\bar{R}) \leq \bar{\Phi}(\bar{R}) \leq \bar{p}(\bar{R})$.

Case 2.1. Recall that $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ is such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) \leq \underline{\Phi}(R)$. Then, $p(R_1) = \underline{p}(R) < \underline{\Phi}(R)$. By *set-uncompromisingness*, $\Phi(\bar{R}) = \Phi(R) = \Phi^{a,b}(R)$ and by the definition of $\Phi^{a,b}$, point $a \geq \underline{p}(\bar{R})$. If point $a \leq \bar{p}(R) = \bar{p}(\bar{R})$, then $\Phi^{a,b}(R) = [a,b] \cap \mathrm{Conv}(R) = [a,b] \cap \mathrm{Conv}(\bar{R}) = \Phi^{a,b}(\bar{R})$. If point $a > \bar{p}(R) = \bar{p}(\bar{R})$, then, $\Phi^{a,b}(R) = \{\bar{p}(R)\} = \Phi^{a,b}(\bar{R})$. Therefore, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 2.2. Recall that $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ is such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) > \underline{\Phi}(R)$ and $p(\bar{R}_1) = \underline{p}(R) \leq \bar{\Phi}(R)$. Then, $\underline{\Phi}(R) \neq \bar{\Phi}(R)$ and $p(R_1) < \bar{\Phi}(R)$. By *uncompromisingness*, maxima $\bar{\Phi}(\bar{R}) = \bar{\Phi}(R)$. Recall that by *efficiency* and Proposition 1 (i), minimum $\underline{\Phi}(\bar{R}) \geq \underline{p}(\bar{R}) = p(\bar{R}_1)$. Next, assuming that minimum $\underline{\Phi}(\bar{R}) > \underline{p}(\bar{R}) = p(\bar{R}_1) > \underline{\Phi}(R)$ results in a contradiction as follows: since $p(\bar{R}_1) < \underline{\Phi}(\bar{R})$ and $p(R_1) < \underline{\Phi}(\bar{R})$, by *uncompromisingness*, minimum $\underline{\Phi}(R) = \underline{\Phi}(\bar{R}) \neq \underline{\Phi}(R)$, a contradiction. Hence, minimum $\underline{\Phi}(\bar{R}) = \underline{p}(\bar{R})$ and thus, $\Phi(\bar{R}) = [\underline{p}(\bar{R}), \bar{\Phi}(R)]$. Since $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ and $\Phi(R) = [a,b] \cap \mathrm{Conv}(R)$, $\Phi(\bar{R}) = \Phi(R) \cap \mathrm{Conv}(\bar{R}) = [a,b] \cap \mathrm{Conv}(\bar{R})$. Therefore, by the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = [a,b] \cap \mathrm{Conv}(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 2.3. Recall that $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ is such that $[\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) = \bar{p}(R)]$ and in addition, let $p(\bar{R}_1) = \underline{p}(\bar{R}) > \bar{\Phi}(R) \geq \underline{\Phi}(R)$. By the definition of $\Phi^{a,b}$, points $a, b < \underline{p}(\bar{R})$. Next, assuming that maximum $\bar{\Phi}(\bar{R}) > \underline{p}(\bar{R}) = p(\bar{R}_1) > \bar{\Phi}(R)$ results in a contradiction as follows: since $p(\bar{R}_1) < \bar{\Phi}(\bar{R})$ and $p(R_1) < \bar{\Phi}(\bar{R})$, by *uncompromisingness*, maximum $\bar{\Phi}(R) = \bar{\Phi}(\bar{R}) \neq \bar{\Phi}(R)$, a contradiction. Hence, maximum $\bar{\Phi}(\bar{R}) = \underline{p}(\bar{R})$ and thus $\Phi(\bar{R}) = \{\underline{p}(\bar{R})\}$. Since point $b < \underline{p}(\bar{R})$, by the definition of $\Phi^{a,b}$, $\Phi(\bar{R}) = \{\underline{p}(\bar{R})\} = \Phi^{a,b}(\bar{R})$.

Case 3. Let $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ be such that $\underline{p}(\bar{R}) = \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$. By a symmetric proof to Case 2, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$.

Case 4. Let $\mathrm{Conv}(\bar{R}) \subsetneq \mathrm{Conv}(R)$ be such that $\underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(\bar{R}) < \bar{p}(R)$. Let profile $R^1 \in \mathcal{R}^N$ be such that $\underline{p}(R^1) = \underline{p}(\bar{R}) > \underline{p}(R)$ and $\bar{p}(R^1) = \bar{p}(R)$. By Case 2, $\Phi(R^1) = \Phi^{a,b}(R^1)$. Next, since $\underline{p}(\bar{R}) = \underline{p}(R^1)$ and $\bar{p}(\bar{R}) < \bar{p}(R^1)$, by Case 3, $\Phi(\bar{R}) = \Phi^{a,b}(\bar{R})$. $\square$

Proof of Theorem 1. ***If part.*** By Propositions 1 and 4, each fp-target set correspondence satisfies *efficiency* and *one-sided replacement-dominance*.

Only if part. Let fixed population $N \in \mathcal{P}$ be such that $|N| \geq 3$ and fp-choice correspondence $\Phi \in \mathcal{F}^N$ satisfy *efficiency* and *one-sided replacement-dominance*. By Lemma 5 (Appendix B), Lemma 10, and Lemma 8, Φ satisfies *extreme-peaks-onliness*, *uncompromisingness*, and *set-uncompromisingness*.

For each pair of points $\alpha, \beta \in \mathbb{R}$ such that $\alpha \leq \beta$, define a profile $R^{\alpha,\beta} \in \mathcal{R}^N$ to be such that $\underline{p}(R^{\alpha,\beta}) = \alpha$ and $\bar{p}(R^{\alpha,\beta}) = \beta$. Without loss of generality, assume that $N = \{1, \ldots, n\}$ and $\alpha = p(R_1^{\alpha,\beta}) \leq \ldots \leq p(R_n^{\alpha,\beta}) = \beta$. By *efficiency* and Proposition 1 (i), $\alpha \leq \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) \leq \beta$.

We prove that there exists an fp-target set correspondence $\Phi^{a,b} \in \mathcal{F}^N$ such that for each profile $R \in \mathcal{R}^N$, $\Phi(R) = \Phi^{a,b}(R)$.

There are four cases. Loosely speaking, in all but the last case the proof proceeds as follows. Given a profile $R^{\alpha,\beta} \in \mathcal{R}^N$ and for each profile $R \in \mathcal{R}^N$ we select a profile such that the convex hull of its peaks is a superset of both $\mathrm{Conv}(R^{\alpha,\beta})$ and $\mathrm{Conv}(R)$ and then, we apply Lemma 11 to show that $\Phi(R) = \Phi^{a,b}(R)$.

Case 1. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) < \beta$. Define points $a := \underline{\Phi}(R^{\alpha,\beta})$ and $b := \bar{\Phi}(R^{\alpha,\beta})$. Since $\Phi(R^{\alpha,\beta}) = [a,b] = [a,b] \cap \mathrm{Conv}(R^{\alpha,\beta})$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R_1^1 such that his peak

$$p(R_1^1) = \begin{cases} p(R_1^{\alpha,\beta}) & \text{if } p(R_1^{\alpha,\beta}) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $p(R_1^{\alpha,\beta}) < \underline{\Phi}(R^{\alpha,\beta})$ and $p(R_1^1) < \underline{\Phi}(R^{\alpha,\beta})$, by *set-uncompromisingness*, $\Phi(R^1) = \Phi(R^{\alpha,\beta}) = [a,b]$. Then, change agent n's preferences to R_n^2 such that his peak

$$p(R_n^2) = \begin{cases} p(R_n^1) & \text{if } p(R_n^1) \geq \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R_{-n}^1, R_n^2)$. Since $p(R_n^1) > \bar{\Phi}(R^1)$ and $p(R_n^2) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = [a,b]$. Since $\Phi(R^2) = [a,b] = [a,b] \cap \mathrm{Conv}(R^2)$, by the definition of

$\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since, $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 11, $\Phi(R) = \Phi^{a,b}(R)$.

Case 2. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \leq \bar{\Phi}(R^{\alpha,\beta}) < \beta$, and for each $\bar{\alpha} \leq \alpha$ and its associated $R^{\bar{\alpha},\beta} \in \mathcal{R}^N$, $\bar{\alpha} = \underline{\Phi}(R^{\bar{\alpha},\beta}) \leq \bar{\Phi}(R^{\bar{\alpha},\beta}) < \beta$.

Case 2.1. There exist $\alpha, \beta \in \mathbb{R}$ as specified in Case 2 and in addition, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) < \bar{\Phi}(R^{\alpha,\beta}) < \beta$. Define points $a := -\infty$ and $b := \bar{\Phi}(R^{\alpha,\beta})$. Since $\Phi(R^{\alpha,\beta}) = [\underline{p}(R^{\alpha,\beta}), b] = [a,b] \cap \text{Conv}(R^{\alpha,\beta})$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R_1^1 such that his peak

$$p(R_1^1) = \begin{cases} p(R_1^{\alpha,\beta}) & \text{if } p(R_1^{\alpha,\beta}) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R_{-1}^{\alpha,\beta}, R_1^1)$. Since $\underline{p}(R^1) \leq \alpha$ and $\bar{p}(R^1) = \beta$, as specified in Case 2 and by *extreme-peaks-onliness*, $\underline{p}(R^1) = \underline{\Phi}(R^1)$. Since $p(R_1^{\alpha,\beta}) < \bar{\Phi}(R^{\alpha,\beta})$ and $p(R_1^1) < \bar{\Phi}(R^{\alpha,\beta})$, by *uncompromisingness*, maxima $\bar{\Phi}(R^1) = \bar{\Phi}(R^{\alpha,\beta}) = b$. Hence, $\Phi(R^1) = [\underline{p}(R^1), b]$. Then, change agent n's preferences to R_n^2 such that his peak

$$p(R_n^2) = \begin{cases} p(R_n^1) & \text{if } p(R_n^1) \geq \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R_{-n}^1, R_n^2)$. Since $p(R_n^1) > \bar{\Phi}(R^1)$ and $p(R_n^2) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = [\underline{p}(R^2), b]$. Since $\Phi(R^2) = [\underline{p}(R^2), b] = [a,b] \cap \text{Conv}(R^2)$, by the definition of $\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 11, $\Phi(R) = \Phi^{a,b}(R)$.

Case 2.2. There exist $\alpha, \beta \in \mathbb{R}$, as specified in Case 2 and in addition, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) = \bar{\Phi}(R^{\alpha,\beta}) < \beta$, and for each $\bar{\alpha} \leq \alpha$ and its associated $R^{\bar{\alpha},\beta} \in \mathcal{R}^N$, $\bar{\alpha} = \underline{\Phi}(R^{\bar{\alpha},\beta}) = \bar{\Phi}(R^{\bar{\alpha},\beta}) < \beta$. Define points $a, b := -\infty$. Since $b < \underline{p}(R^{\alpha,\beta})$ and $\Phi(R^{\alpha,\beta}) = \{\underline{p}(R^{\alpha,\beta})\}$, by the definition of $\Phi^{a,b}$, $\Phi(R^{\alpha,\beta}) = \Phi^{a,b}(R^{\alpha,\beta})$. Let $R \in \mathcal{R}^N$. Begin from profile $R^{\alpha,\beta}$ and construct profile R^1 by changing agent 1's preferences to R_1^1 such that his peak

$$p(R_1^1) = \begin{cases} p(R_1^{\alpha,\beta}) & \text{if } p(R_1^{\alpha,\beta}) \leq \underline{p}(R) \\ \underline{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^1 = (R^{\alpha,\beta}_{-1}, R^1_1)$. Since $\underline{p}(R^1) \le \alpha$ and $\bar{p}(R^1) = \beta$, as specified in this case and by *extreme-peaks-onliness*, $\Phi(R^1) = \{\underline{p}(R^1)\}$. Then, change agent n's preferences to R^2_n such that his peak

$$p(R^2_n) = \begin{cases} p(R^1_n) & \text{if } p(R^1_n) \ge \bar{p}(R) \\ \bar{p}(R) & \text{otherwise,} \end{cases}$$

i.e., $R^2 = (R^1_{-n}, R^2_n)$. Since $p(R^1_n) > \bar{\Phi}(R^1)$ and $p(R^2_n) > \bar{\Phi}(R^1)$, by *set-uncompromisingness*, $\Phi(R^2) = \Phi(R^1) = \{\underline{p}(R^2)\}$. Since $b < \underline{p}(R^2)$, by the definition of $\Phi^{a,b}$, $\Phi(R^2) = \Phi^{a,b}(R^2)$. Since $\Phi(R^2) = \Phi^{a,b}(R^2)$ and $\text{Conv}(R) \subseteq \text{Conv}(R^2)$, by Lemma 11, $\Phi(R) = \Phi^{a,b}(R)$.

Case 3. There exist $\alpha, \beta \in \mathbb{R}$ such that for $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\beta}) \le \bar{\Phi}(R^{\alpha,\beta}) = \beta$, and for each $\bar{\beta} \ge \beta$ and its associated $R^{\alpha,\bar{\beta}} \in \mathcal{R}^N$, $\alpha < \underline{\Phi}(R^{\alpha,\bar{\beta}}) \le \bar{\Phi}(R^{\alpha,\bar{\beta}}) = \bar{\beta}$. The proof of this case is symmetric to Case 2.

Case 4. For each $\alpha, \beta \in \mathbb{R}$ such that $\alpha \le \beta$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \le \bar{\Phi}(R^{\alpha,\beta}) = \beta$. Define points $a := -\infty$ and $b := \infty$. Since for each $\alpha, \beta \in \mathbb{R}$ and its associated $R^{\alpha,\beta} \in \mathcal{R}^N$, $\alpha = \underline{\Phi}(R^{\alpha,\beta}) \le \bar{\Phi}(R^{\alpha,\beta}) = \beta$, by *extreme-peaks-onliness*, for each $R \in \mathcal{R}^N$, $\Phi(R) = \text{Conv}(R)$. Therefore, since $a < \underline{p}(R)$ and $b > \bar{p}(R)$, by the definition of $\Phi^{a,b}$, $\Phi(R) = \Phi^{a,b}(R)$. □

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