The core for housing markets with limited externalities*

Bettina Klaus[†] Claudia Meo[‡]

March 3, 2021

Abstract

We propose a variant of the housing market model à la Shapley and Scarf (1974) that incorporates a limited form of externality in consumption; that is, agents care both about their own consumption (demand preferences) and about the agent who receives their endowment (supply preferences).

We consider different domains of preference relations by taking demand and supply aspects of preferences into account. First, for markets with three agents who have (additive) separable preferences such that all houses and agents are acceptable, the strong core is nonempty; a result that can be neither extended to the unacceptable case nor to markets with a larger number of agents. Second, for markets where all agents have demand lexicographic preferences (or all of them have supply lexicographic preferences), we show that the strong core is nonempty, independent of the number of agents and the acceptability of houses or agents, and possibly multi-valued.

Keywords: Externalities; housing markets; weak core; strong core.

JEL codes: C70, C71, C78, D62, D64.

1 Introduction

We generalize Shapley and Scarf's (1974) famous model of trading indivisible objects to markets with limited externalities. In classical *Shapley-Scarf housing markets* each agent is endowed with an indivisible commodity, for instance a house, and wishes to consume

*We gratefully acknowledge financial support from the Swiss National Science Foundation (SNSF) through Project 100018_192583. We thank Martin Bullinger, Vikram Manjunath, Seckin Özbilen, and William Thomson for their valuable comments.

[†]Faculty of Business and Economics, University of Lausanne, Internef 538, 1015 Lausanne, Switzerland; *e-mail*: `Bettina.Klaus@unil.ch`.

[‡]Dipartimento di Scienze Economiche e Statistiche, Università di Napoli Federico II, Italy; *e-mail*: `claudia.meo@unina.it`

exactly one commodity. Agents have complete, reflexive, and transitive preferences over all existing houses and may be better off by trading houses: exchanges do not involve monetary compensations. An outcome for a Shapley-Scarf housing market is a permutation of the endowment allocation.

One of the best known solution concepts for barter economies is the *weak core*, based on the absence of coalitions that may reallocate their endowments among themselves and make all their members strictly better off (i.e., no coalition can *strongly block* a weak core allocation). The weak core for Shapley-Scarf housing markets is always nonempty (Shapley and Scarf, 1974). If strong blocking is weakened to only require that members of a blocking coalition are not worse off while at least one member is better off, then a stronger solution, the *strong core*, results (i.e., no coalition can *weakly block* a strong core allocation). In contrast to the weak core, the strong core for Shapley-Scarf housing markets can be empty, unless no agent is indifferent between any of the houses. Hence, when preferences are strict, also the strong core is nonempty and, in fact, coincides with the unique competitive allocation (Roth and Postlewaite, 1977). Using the so-called top trading cycles (TTC) algorithm (due to David Gale, see Shapley and Scarf, 1974), one can easily determine the unique strong core allocation for any Shapley-Scarf housing market with strict preferences.

In this paper, we build on Shapley and Scarf's classical model and change the assumption that agents care only about the object they receive. We extend Shapley and Scarf's (1974) housing markets by introducing a limited form of externality in consumption: each agent cares both about his own consumption (traditional "demand preferences") and about the agent who receives his endowment (less traditional "supply preferences"). This form of externality is modelled, for each agent, by a preference relation defined over pairs formed by the object assigned to the agent himself and the recipient of his own object. An example would be that of kidney exchange, the agents being formed by recipient-donor pairs and the objects being the kidneys that will be donated. It is clear that each agent (recipient) cares about the kidney he will receive but in addition, each agent (donor) might also care about the recipient of his kidney. More broadly, models with limited externalities also fit well with exchanges that are not permanent, i.e., where the endowments are only temporarily exchanged and eventually return to their original owners: vacation home exchanges such as InterVac or ThirdHome are examples of such temporary exchanges. Other examples are car-sharing platforms such as CAROSET allowing car owners to temporarily swap their vehicles (Dentsu launched this app, which initially is free of charge, in December 2019).[1] After having developed our model and

[1]Webpages for the exchange platforms mentioned are: https://www.intervac-homeexchange.com/, https://www.thirdhome.com/, and https://caroset.co.jp/.

having obtained first results, we discovered that Aziz and Lee (2020) had introduced the same problem as "temporary exchange problem". The results obtained by Aziz and Lee (2020) complement ours, as explained later on when we discuss our results in more detail.

General forms of externalities for Shapley-Scarf housing markets have been analyzed before. When (strict) preferences are defined over allocations rather than over own allotments (i.e., individual consumptions), Mumcu and Sağlam (2007) proved that multiple weak core allocations may exist and the weak core may be empty even for markets with just three agents. Hong and Park (2020) analyze various core notions in the presence of externalities. Furthermore, they address the various difficulties that then occur when applying and adjusting the TTC algorithm accordingly. In order to get positive results, they restrict attention to "local" and "weak global" externalities by introducing so-called *hedonic* and *order-preserving* preferences.[2] They then introduce further properties to guarantee the existence of a top trading cycles allocation and prove that it is a core allocation (with respect to various core notions) and, moreover, is so-called stable (see Roth and Postlewaite, 1977, for the definition of stable allocations). In the same vein, Graziano et al. (2020) focused on two preference domains capturing specific types of externalities over allocations and prove that stable allocations exist and form a stable set à la von Neumann and Morgenstern. The first domain, which is the same as the order-preserving preferences with respect to own-trading cycles domain in Hong and Park (2020), is the *egocentric preferences* domain where agents are primarily interested in the house they receive. The second class of preferences are called *allocentric* and preferences in that class can accommodate some altruism among agents that cannot be addressed via egocentric preferences; the most simple example being that of two agents who have the same preferences over the set of allocations.

The main focus of our analysis is on the existence and uniqueness of weak and strong core allocations for markets with limited externalities, depending on two elements: the number of agents and the acceptability of all houses and / or agents. After providing an empty weak core example for a market with three agents that resembles a well-known roommate market example, we restrict our attention to different subclasses of preferences, proceeding from bigger to smaller preference domains.

First, for markets with three agents who have (additive) separable preferences such that all houses and agents are acceptable, the strong core is nonempty; a result that can be neither extended to the unacceptable case nor to markets with a larger number of agents. Second, for markets where all agents have demand lexicographic preferences, we show that the strong core is nonempty, independent of the number of agents and the acceptability of houses or agents, and possibly multi-valued. We remark that all

[2] In Section 2, Remark 1, we provide further details concerning these preference domains.

results and examples obtained for markets with demand lexicographic preferences can be symmetrically obtained for supply lexicographic preferences.

Our results and the independent findings of Aziz and Lee (2020) for housing markets with limited externalities complement each other. Aziz and Lee (2020), adopt a computational viewpoint that is missing in our analysis and show , in general, checking whether weakly core stable or Pareto optimal allocations exist is NP-hard, while for separable preferences a polynomial-time algorithm to obtain Pareto optimal and individually rational allocations exist.

The paper is organized as follows. In Section 2 we introduce our housing market model with an emphasis on various preference domains capturing limited externalities, as well as the core solutions we consider. In Section 3 we focus on the domains of separable and additive separable preferences and in Section 4 we consider demand (supply) lexicographic preferences. Depending on the number of agents and the acceptability of houses and agents, for each subdomain we obtain results concerning the existence, multi-valuedness, and set inclusions of various core-allocation sets. We conclude in Section 5.

2 The model

We consider an exchange market with indivisibilities formed by n agents and by the same number of indivisible objects, say houses; let $\boldsymbol{N} = \{1, \ldots, n\}$ and $\boldsymbol{H} = \{h_1, \ldots, h_n\}$ denote the **set of agents** and **houses**, respectively. Each agent owns one distinct house when entering the market, desires exactly one house, and has the option to trade the initially owned house in order to get a better one. All trades are made with no transfer of money. We assume that **agent $\boldsymbol{i}$ owns house $\boldsymbol{h_i}$**.

An **allocation $\boldsymbol{a}$** is an assignment of houses to agents such that each agent receives exactly one house, that is, a bijection $a : N \to H$. Alternatively, we will denote an allocation a as a vector $a = (a_1, \ldots, a_n)$ with $a_i \in H$ denoting the house assigned to agent $i \in N$ under allocation a. $\boldsymbol{\mathcal{A}}$ denotes the **set of all allocations** and $\boldsymbol{h} = (h_1, \ldots, h_n)$ the **endowment allocation**. Hence, the set of allocations $\mathcal{A}$ is obtained by all permutations of H. A nonempty subset S of N is called a **coalition**. For any coalition $S \subseteq N$ and any allocation $a \in \mathcal{A}$, let $\boldsymbol{a(S)} = \{a_i \in H : i \in S\}$ be the **set of houses that coalition $\boldsymbol{S}$ receives at allocation $\boldsymbol{a}$.**

Up to now we have followed the description of a classical *Shapley-Scarf housing market model* as introduced by Shapley and Scarf (1974). Now, in contrast with that model, we assume that each agent cares not only about the house he receives but also about the recipient of his own house. That is, preferences capture limited externalities that are modelled as follows.

Given an allocation $a \in \mathcal{A}$, the **allotment of agent $\boldsymbol{i}$** is the pair $\boldsymbol{(a(i), a^{-1}(h_i))} \in H \times N$, formed by the house $a(i)$ assigned to agent i and the agent who receives agent i's house, i.e., agent $a^{-1}(h_i)$. Note that $a(i) = h_i$ if and only if $a^{-1}(h_i) = i$, i.e., either both elements of agent i's endowment allotment (h_i, i) occur in his allotment or none. $\boldsymbol{\mathcal{A}_i}$ denotes the **set of all the allotments of agent $\boldsymbol{i}$**.

The next table sums up all possible allocations and the associated allotments for each agent in a three-agents market. We will refer to Table 1 throughout the paper.

Allocations	$\boldsymbol{\mathcal{A}_1}$	$\boldsymbol{\mathcal{A}_2}$	$\boldsymbol{\mathcal{A}_3}$
$a_1 = (h_1, h_2, h_3)$	$(h_1, 1)$	$(h_2, 2)$	$(h_3, 3)$
$a_2 = (h_3, h_1, h_2)$	$(h_3, 2)$	$(h_1, 3)$	$(h_2, 1)$
$a_3 = (h_2, h_3, h_1)$	$(h_2, 3)$	$(h_3, 1)$	$(h_1, 2)$
$a_4 = (h_2, h_1, h_3)$	$(h_2, 2)$	$(h_1, 1)$	$(h_3, 3)$
$a_5 = (h_3, h_2, h_1)$	$(h_3, 3)$	$(h_2, 2)$	$(h_1, 1)$
$a_6 = (h_1, h_3, h_2)$	$(h_1, 1)$	$(h_3, 3)$	$(h_2, 2)$

Table 1: Allocations and associated allotments for a three-agents market.

Each agent $i \in N$ has a preference relation $\succeq_i$ over the set $\boldsymbol{\mathcal{A}_i}$, that is, $\succeq_i$ is a *transitive*, *reflexive*, and *complete* binary relation. As usual, $\succ_i$ and $\sim_i$ denote the asymmetric and symmetric parts of $\succeq_i$, respectively. We also assume that preferences are **strict**, i.e., $\succeq_i$ is *antisymmetric* such that for all $(h, j),\ (h', k) \in \boldsymbol{\mathcal{A}_i}$, it holds that

$$(h, j) \succeq_i (h', k) \text{ and } (h', k) \succeq_i (h, j) \text{ if and only if } (h, j) = (h', k).$$

As a consequence,

$$(h, j) \sim_i (h', k) \text{ if and only if } (h, j) = (h', k)$$

and

$$(h, j) \succeq_i (h', k) \text{ if and only if } (h, j) \succ_i (h', k) \text{ or } (h, j) = (h', k).$$

We denote the general **domain of strict preferences** by $\boldsymbol{\mathcal{D}}$ and the **set of strict preference profiles** by $\boldsymbol{\mathcal{D}^N}$.

Throughout the paper several subdomains of $\boldsymbol{\mathcal{D}}$ will be analyzed where the agents' preferences over allotments are induced by their preferences over the houses and the other agents in the market. More specifically, we assume that each agent $i \in N$ has

a "**demand**" strict preference relation $\boldsymbol{\succeq_i^d}$ over the set H of houses and

a "**supply**" strict preference relation $\boldsymbol{\succeq_i^s}$ over the set N of agents.

We denote the set of demand preferences over H and the set of demand preference profiles by $\boldsymbol{\mathcal{D}_\mathbf{d}}$ and $\boldsymbol{\mathcal{D}_\mathbf{d}^N}$, respectively; and the set of supply preferences over N and the set of supply preference profiles by $\boldsymbol{\mathcal{D}_\mathbf{s}}$ and $\boldsymbol{\mathcal{D}_\mathbf{s}^N}$, respectively.

We say that **house $\boldsymbol{h}$** $\in H \setminus \{h_i\}$ **is acceptable** for agent $i \in N$ if $h \succ_i^d h_i$, otherwise it **is unacceptable**. Symmetrically, we say that **agent $\boldsymbol{j}$** $\in N \setminus \{i\}$ **is acceptable** for agent $i \in N$ if $j \succ_i^s i$, otherwise he **is unacceptable**.

We consider the following **subdomains of $\boldsymbol{\mathcal{D}}$**.

- The domain $\boldsymbol{\mathcal{D}_{\mathbf{sep}}}$ of **separable preferences**: an agent $i \in N$ has separable preferences $\succeq_i \in \mathcal{D}$ if for any two distinct houses $h, h' \in H \setminus \{h_i\}$ and any two distinct agents $j, k \in N \setminus \{i\}$,

$$j \succ_i^s k \text{ implies } (h, j) \succ_i (h, k),$$

$$h \succ_i^d h' \text{ implies } (h, j) \succ_i (h', j),$$

$$h \succ_i^d h_i \text{ and } j \succ_i^s i \text{ imply } (h, j) \succ_i (h_i, i),$$

  and

$$h_i \succ_i^d h \text{ and } i \succ_i^s j \text{ imply } (h_i, i) \succ_i (h, j).$$

- The domain $\boldsymbol{\mathcal{D}_{\mathbf{add}}}$ of **additive separable preferences**: an agent $i \in N$ has additive separable preferences $\succeq_i \in \mathcal{D}$ if his preferences $\succeq_i^d$ over houses as well as his preferences $\succeq_i^s$ over agents can be represented by utility functions $u_i^d : H \to \mathbb{R}$ and $u_i^s : N \to \mathbb{R}$ that induce cardinal utilities over allotments in an additive manner. Formally, $\succeq_i \in \mathcal{D}_{\text{add}}$ if for any $(h, j), (h', k) \in \mathcal{A}_i$,

$$(h, j) \succ_i (h', k) \text{ if and only if } u_i^d(h) + u_i^s(j) > u_i^d(h') + u_i^s(k).$$

- The domain $\boldsymbol{\mathcal{D}_{\mathbf{dlex}}}$ of **demand lexicographic preferences**: an agent $i \in N$ has demand lexicographic preferences $\succeq_i \in \mathcal{D}$ if he primarily cares about the house he receives and only secondarily about who receives his house. Formally, $\succeq_i \in \mathcal{D}_{\text{dlex}}$ if for any $(h, j), (h', k) \in \mathcal{A}_i$,

$$(h, j) \succ_i (h', k) \text{ if and only if } h \succ_i^d h' \text{ or } [h = h' \text{ and } j \succ_i^s k].$$

- The domain $\boldsymbol{\mathcal{D}}_{\mathbf{slex}}$ of **supply lexicographic preferences**: an agent $i \in N$ has supply lexicographic preferences $\succeq_i \in \mathcal{D}$ if he primarily cares about who receives his house and only secondarily about the house he receives. Formally, $\succeq_i \in \mathcal{D}_{\text{slex}}$ if for any $(h, j), (h', k) \in \mathcal{A}_i$,

$$(h, j) \succ_i (h', k) \text{ if and only if } j \succ_i^s k \text{ or } [j = k \text{ and } h \succ_i^d h'].$$

$\boldsymbol{\mathcal{D}}^{\boldsymbol{N}}_{\mathbf{sep}}$, $\boldsymbol{\mathcal{D}}^{\boldsymbol{N}}_{\mathbf{add}}$, $\boldsymbol{\mathcal{D}}^{\boldsymbol{N}}_{\mathbf{dlex}}$, and $\boldsymbol{\mathcal{D}}^{\boldsymbol{N}}_{\mathbf{slex}}$ will denote the set of separable, additive separable, demand lexicographic, and supply lexicographic preference profiles, respectively.

The next proposition, which we prove in Appendix A, illustrates the relationships between the above preference domains (see also Figure 2 in Appendix A).

Proposition 1. *The following relationships hold between the preference domains:*

$$\mathcal{D} \supseteq \mathcal{D}_{\text{sep}} \supseteq \mathcal{D}_{\text{add}} \supseteq (\mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}}) \text{ and } \mathcal{D}_{\text{dlex}} \cap \mathcal{D}_{\text{slex}} = \emptyset.$$

Moreover, if $|N| = 3$, then

$$\mathcal{D}_{\text{sep}} = \mathcal{D}_{\text{add}},$$

and if in addition all houses and agents are acceptable,

$$\mathcal{D}_{\text{add}} = \mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}}.[3]$$

A **housing market with limited externalities**, or **market** for short, is now completely described by the triplet $\boldsymbol{(N, h, \succeq)}$, where N is the set of agents, h is the endowment allocation, and $\succeq \in \mathcal{D}^N$ is a preference profile. Since the set of agents and the endowment allocation are fixed, we often denote a **market** by its **preference profile** $\boldsymbol{\succeq}$.

For each agent $i \in N$, a preference relation $\trianglerighteq_i$ on the set of allocations $\mathcal{A}$ can be associated with his preferences $\succeq_i$ over $\mathcal{A}_i$. Consider two allocations $a, b \in \mathcal{A}$. Then, we have

$$\begin{aligned} a \vartriangleright_i b \quad & \text{if and only if} \quad (a(i), a^{-1}(h_i)) \succ_i (b(i), b^{-1}(h_i)) \\ & \text{and} \\ a \sim_i b \quad & \text{if and only if} \quad (a(i), a^{-1}(h_i)) = (b(i), b^{-1}(h_i)). \end{aligned}$$

In the following, the symbols $\succeq$ and $\trianglerighteq$ will be used to denote generic preference relations over allotments and allocations, respectively.

[3]For $|N| = 2$, since there are only two allocations, all our preference domains coincide.

Remark 1 (Preference domains in Hong and Park (2020)). Hong and Park (2020) consider preferences over allocations with externalities as well. They define agent i's preferences $\trianglerighteq_i$ over $\mathcal{A}$ as

- *hedonic* if each agent just cares about his own *trading cycle*;[4] that is, for all allocations $a, b \in \mathcal{A}$ such that $\mathcal{S}_i^{a,h} = \mathcal{S}_i^{b,h}$, where $\mathcal{S}_i^{a,h}$ and $\mathcal{S}_i^{b,h}$ are agent i's trading cycles at allocations a and b, respectively, we have $a \sim_i b$;
- *order preserving with respect to own-allotments* if each agent is primarily interested in the house (or, the *allotment*, according to Hong and Park's terminology) that he receives; formally, for all $a, b \in \mathcal{A}$ with $a(i) \neq b(i)$, it holds that

$$a \rhd_i b \text{ implies for all } a', b' \in \mathcal{A} \text{ with } [a'(i) = a(i) \text{ and } b'(i) = b(i)] \text{ that } a' \rhd_i b'.$$

For our model, since for any two allocations $a, b \in \mathcal{A}$, $\mathcal{S}_i^{a,h} = \mathcal{S}_i^{b,h}$ implies $(a(i), a^{-1}(h_i)) = (b(i), b^{-1}(h_i))$, a preference relation $\trianglerighteq_i$ over allocations that is derived from strict preferences $\succeq_i \in \mathcal{D}$ over allotments is hedonic. Thus, throughout this paper, preferences over allocations are hedonic.

Preferences $\trianglerighteq$ over allocations that are induced by demand lexicographic preferences $\succeq \in \mathcal{D}_{\text{dlex}}$ are order preserving with respect to own-allotments. On the other hand, our domain of supply lexicographic preferences induces preferences over allocations that cannot be compared with order preserving preferences with respect to own-allotments. For instance, the following supply lexicographic preferences for agent 1

$$(h_3, 3) \succ_1 (h_2, 3) \succ_1 (h_1, 1) \succ_1 (h_3, 2) \succ_1 (h_2, 2)$$

induce preferences over allocations (see Table 1)

$$a_5 \rhd_1 a_3 \rhd_1 a_1 \sim_1 a_6 \rhd_1 a_2 \rhd_1 a_4$$

that are not order preserving with respect to own-allotments because $a_3 \rhd_1 a_1$ but $a_4 \not\rhd_1 a_6$. Vice versa, the following preferences over allocations

$$a_4 \rhd_1 a_3 \rhd_1 a_5 \rhd_1 a_2 \rhd_1 a_1 \sim_1 a_6$$

are order preserving with respect to own-allotments; however, the induced preferences

[4] For each allocation $a \in \mathcal{A}$, the set N of agents can be partitioned into trading cycles: a trading cycle is a sequence of agents $(j_0, j_1, ..., j_{K-1})$ such that for each $i = 0, 1, ..., K-1$, $a(j_i) = h_{j_{i+1}}$ (where indices are modulo K).

over allotments

$$(h_2, 2) \succeq_1 (h_2, 3) \succeq_1 (h_3, 3) \succeq_1 (h_3, 2) \succeq_1 (h_1, 1)$$

are not supply lexicographic because $(h_2, 2) \succeq_1 (h_2, 3)$ but $(h_3, 2) \not\succeq_1 (h_3, 3)$. □

We next introduce a voluntary participation and a mild efficiency requirement for allocations.

Definition 1 (**Individually rational and Pareto optimal allocations**).
Let $\succeq \in \mathcal{D}^N$ and $a \in \mathcal{A}$. Then, allocation a is

individually rational if for all agents $i \in N$, $(a_i, a^{-1}(h_i)) \succeq_i (h_i, i)$;

Pareto dominated by allocation $b \in \mathcal{A}$ if for all agents $i \in N$,

$$(b_i, b^{-1}(h_i)) \succeq_i ((a_i, a^{-1}(h_i))$$

and for some agent $j \in N$,

$$(b_j, b^{-1}(h_j)) \succ_j ((a_j, a^{-1}(h_j));$$

Pareto optimal if it is not Pareto dominated by another allocation.

We denote the **set of all individually rational** and **Pareto optimal allocations** for market $\succeq$ by $\boldsymbol{IR(\succeq)}$ and $\boldsymbol{PO(\succeq)}$, respectively.

Now, we introduce our two main solution concepts that represent the idea of "stable exchange" based on the absence of coalitions that can improve their allotments by reallocating their endowments among themselves.

Definition 2 (**Weak and strong core allocations**).
Let $\succeq \in \mathcal{D}^N$ and $a \in \mathcal{A}$. Then, **coalition $\boldsymbol{S} \subseteq N$ strongly blocks allocation $\boldsymbol{a}$** if there exists an allocation $b \in \mathcal{A}$ such that

(a) at allocation b agents in S reallocate their endowments, i.e., $b(S) = h(S)$, and

(b) all agents in S are strictly better off, i.e., for all agents $i \in S$,

$$(b_i, b^{-1}(h_i)) \succ_i ((a_i, a^{-1}(h_i)).$$

Allocation $\boldsymbol{a}$ is a (weak) core allocation if it is not strongly blocked by any coalition. We denote the **set of (weak) core allocations** for market $\succeq$ by $\boldsymbol{C(\succeq)}$.

Coalition S weakly blocks allocation a if there exists an allocation $b \in \mathcal{A}$ such that

(a) at allocation b agents in S reallocate their endowments, i.e., $b(S) = h(S)$, and

(b') all agents in S are weakly better off with at least one of them being strictly better off, i.e., for all agents $i \in S$,

$$(b_i, b^{-1}(h_i)) \succeq_i ((a_i, a^{-1}(h_i))$$

and for some agent $j \in S$,

$$(b_j, b^{-1}(h_j)) \succ_j ((a_j, a^{-1}(h_j)).$$

Allocation a is a strong core allocation if it is not weakly blocked by any coalition. We denote the **set of strong core allocations** for market $\succeq$ by $\boldsymbol{SC(\succeq)}$.

It follows from the definitions that, for any market $\succeq \in \mathcal{D}^N$, $SC(\succeq) \subseteq C(\succeq)$. When there are only three agents in the market, then the weak and strong core coincide.

Proposition 2. *Consider a housing market $(N, h, \succeq)$ where $|N| \leq 3$ and $\succeq \in \mathcal{D}^N$. Then, $SC(\succeq) = C(\succeq)$. Furthermore, if $|N| \leq 2$, then $SC(\succeq) \neq \emptyset$.*

Proof. Let $(N, h, \succeq)$ be such that $N \subseteq \{i, j, k\}$ and $\succeq \in \mathcal{D}^N$. Let $a \in C(\succeq)$ and suppose, by way of contradiction, that $a \notin SC(\succeq)$. Then, there exists a minimal coalition $S \subseteq N$ that weakly but not strongly blocks a through an allocation b. Hence, for some agent $i \in S$, $(b_i, b^{-1}(h_i)) = (a_i, a^{-1}(h_i))$. The latter implies that $S = \{i\}$ is not possible.

If $S = \{i, j\}$, then agents i and j block by swapping their endowments such that $(b_i, b^{-1}(h_i)) = (h_j, j)$ and $(b_j, b^{-1}(h_j)) = (h_i, i)$. Then, $(b_i, b^{-1}(h_i)) = (a_i, a^{-1}(h_i)) = (h_j, j)$ implies that $(b_j, b^{-1}(h_j)) = (a_j, a^{-1}(h_j)) = (h_i, i)$, contradicting that S weakly blocks a through b.

If $S = \{i, j, k\}$, then agents i, j, and k block by exchanging their endowments in a circular way such that $(b_i, b^{-1}(h_i)) = (h_j, k)$, $(b_j, b^{-1}(h_j)) = (h_k, i)$, and $(b_k, b^{-1}(h_k)) = (h_i, j)$. Then, $(b_i, b^{-1}(h_i)) = (a_i, a^{-1}(h_i)) = (h_j, k)$ implies that $(b_j, b^{-1}(h_j)) = (a_j, a^{-1}(h_j)) = (h_k, i)$ and $(b_k, b^{-1}(h_k)) = (a_k, a^{-1}(h_k)) = (h_i, j)$, contradicting that S weakly blocks a through b.

If $|N| \leq 2$, then it is easy to show that $SC(\succeq)$ either consists of the endowment allocation or the allocation that is obtained by pairwise trade. □

For Shapley-Scarf housing markets where agents only care about the house they receive, Shapley and Scarf (1974) showed that a core allocation always exists. Furthermore, Roth and Postlewaite (1977) proved that, when preferences are strict, the set of strong core allocations for any Shapley-Scarf housing market is a singleton. Using the so-called top trading cycles (TTC) algorithm (due to David Gale, see Shapley and Scarf, 1974) one can easily determine this unique strong core allocation for any classical housing market. We end this section with an example showing that, contrary to the classical Shapley-Scarf model, the core may be empty when preferences exhibit limited externalities.

Example 1 (A "roommate market" with an empty core). We describe a market that is in character very similar to the famous roommate market that shows that the core in one-to-one (so-called roommate) markets may be empty (see Gale and Shapley, 1962, Example 3). In a roommate market, the "objects" that agents can trade and consume are the companionship that they provide when sharing a room (when agents share a room, they consume the others' companionship; when they stay alone, they only consume their own solitude). Let $N = \{1, 2, 3\}$ and $H = \{h_1, h_2, h_3\}$ be the set of agents and the set of objects that represent social interaction or company, respectively; $h = a_1 = (h_1, h_2, h_3)$ is the endowment allocation. Table 2 specifies agents' preferences. The symbol $\ldots$ denotes that any strict ordering of the two remaining allotments can be considered.

Agent 1:	$(h_2, 2)$	$\succ_1$	$(h_3, 3)$	$\succ_1$	$\cdots$	$\succ_1$	$(h_1, 1)$
Agent 2:	$(h_3, 3)$	$\succ_2$	$(h_1, 1)$	$\succ_2$	$\cdots$	$\succ_2$	$(h_2, 2)$
Agent 3:	$(h_1, 1)$	$\succ_3$	$(h_2, 2)$	$\succ_3$	$\cdots$	$\succ_3$	$(h_3, 3)$

Table 2: Example 1 preferences $\succeq \in \mathcal{D}^N$.

We show that the core for market $(N, h, \succeq)$ is empty. Referring to Table 1 for the labeling of allocations and allotments, we first show that all allocations resulting from pairwise trades can be blocked by a pairwise trade that includes the residual agent: allocation $a_4 = (h_2, h_1, h_3)$ is blocked by coalition $S = \{2, 3\}$ via $a_6 = (h_1, h_3, h_2)$, allocation $a_6 = (h_1, h_3, h_2)$ is blocked by coalition $S = \{1, 3\}$ via $a_5 = (h_3, h_2, h_1)$, and allocation $a_5 = (h_3, h_2, h_1)$ is blocked by coalition $S = \{1, 2\}$ via $a_4 = (h_2, h_1, h_3)$. Furthermore, all non-pairwise trades can be blocked by a pairwise trade: allocation $a_2 = (h_3, h_1, h_2)$ is blocked by coalition $S = \{1, 3\}$ via $a_5 = (h_3, h_2, h_1)$ and allocation $a_3 = (h_2, h_3, h_1)$ is blocked by coalition $S = \{1, 2\}$ via $a_4 = (h_2, h_1, h_3)$. Finally, the no-trade allocation a_1 can be blocked by various pairwise trades, e.g., it is blocked by coalition $S = \{1, 2\}$ via $a_4 = (h_2, h_1, h_3)$. $\square$

The fact that core allocations do not exist for markets such as the one in Example 1 is not particularly surprising since the core is frequently observed to be empty in a framework with more than two agents and externalities in consumption (see for example Graziano et al., 2020; Mumcu and Sağlam, 2007). A different (five-agent) example of a market with limited externalities and an empty core has recently been provided by Aziz and Lee (2020).

Given this negative result, in the following sections we will focus on several preference subdomains of $\mathcal{D}^N$ and explore for each of them the existence of both weak and strong core allocations. The analysis will proceed from bigger to smaller domains and will distinguish, whenever necessary, between the case where all houses and agents are acceptable (the "**acceptable case**") and the case where acceptability of houses and agents is not required (the "**unacceptable case**").

3 Separable and additive separable markets

In this section we present three results for the separable domain $\mathcal{D}_{\text{sep}}$ and its subdomain of additive separable preferences, $\mathcal{D}_{\text{add}}$. First, we prove that the strong core is nonempty for the three agents case when preferences are separable and all houses and agents are acceptable (Section 3.1). Then, we show that this existence result can neither be extended to the unacceptable case (Section 3.2) nor to a market with a larger number of agents (Section 3.3).

Throughout this section, it will be helpful to represent an allocation and blocking coalitions as directed graphs where each agent is a node in the graph and a directed edge from agent i to agent j ($i \to j$) means that agent i consumes the house h_j of agent j. A directed edge from agent i to himself (a loop $i \circlearrowleft$) represents the case where agent i consumes his own house h_i.

3.1 Non-emptiness of the strong core for separable three-agents markets: the acceptable case

We show that the strong core for a separable three-agents market where all houses and agents are acceptable is nonempty.

Proposition 3. *Consider a housing market $(N, h, \succeq)$ where $|N| = 3$, $\succeq \in \mathcal{D}^N_{\text{sep}}$, and all houses and agents are acceptable. Then, $SC(\succeq) \neq \emptyset$.*

We prove Proposition 3 in Appendix B. In the proof, since there are four possible types for each agent, we consider a total of 64 cases that we group into three main sets for which we then construct corresponding strong core allocations.

3.2 Possible emptiness of the core for additive separable three-agents markets: the unacceptable case

We now present an additive separable three-agents market $\mathcal{H}_1$ where some agents have unacceptable houses / agents and show that the core is empty.

The next two tables (Tables 3 and 4) specify demand and supply preferences of each agent $i \in N$ (together with associated utilities u_i^d and u_i^s) and the resulting additive separable utilities, respectively.

Agent 1				**Agent 2**				**Agent 3**			
$\succeq_1^d$	u_1^d	$\succeq_1^s$	u_1^s	$\succeq_2^d$	u_2^d	$\succeq_2^s$	u_2^s	$\succeq_3^d$	u_3^d	$\succeq_3^s$	u_3^s
h_3	3	3	2	h_1	5	3	1	h_1	1	2	5
h_2	1	2	1	h_3	1	2	0	h_3	0	1	1
h_1	0	1	0	h_2	0	1	-2	h_2	-2	3	0

Table 3: Market $\mathcal{H}_1$ demand and supply preferences and utilities.

Agent 1	**Agent 2**	**Agent 3**
$u_1(h_3, 3) = 5$	$u_2(h_1, 3) = 6$	$u_3(h_1, 2) = 6$
$u_1(h_3, 2) = 4$	$u_2(h_1, 1) = 3$	$u_3(h_2, 2) = 3$
$u_1(h_2, 3) = 3$	$u_2(h_3, 3) = 2$	$u_3(h_1, 1) = 2$
$u_1(h_2, 2) = 2$	$u_2(h_2, 2) = 0$	$u_3(h_3, 3) = 0$
$u_1(h_1, 1) = 0$	$u_2(h_3, 1) = -1$	$u_3(h_2, 1) = -1$

Table 4: Market $\mathcal{H}_1$ additive separable utilities.

The core of the housing market $\mathcal{H}_1$ is empty; in Table 5 we list for each possible allocation for $\mathcal{H}_1$ how a subset of agents can block it.

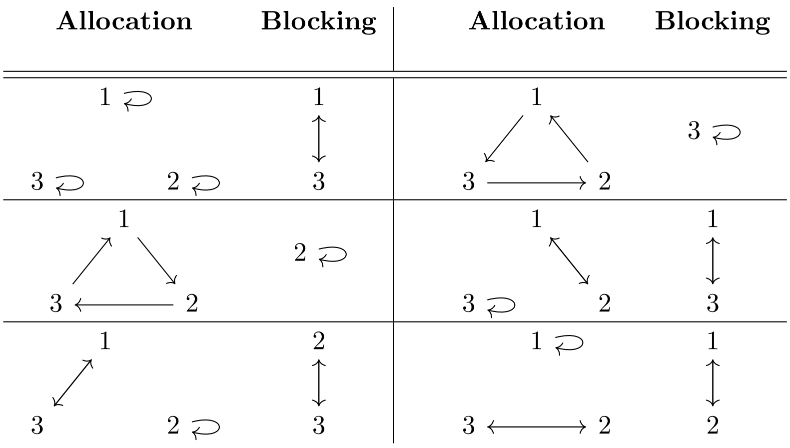


Table 5: Market $\mathcal{H}_1$ blocking of all possible allocations.

3.3 Possible emptiness of the core for additive separable larger markets: the acceptable case

We now present an additive separable four-agent market $\mathcal{H}_2$ with all acceptable agents and houses and an empty core.

The next two tables (Tables 6 and 7) specify demand and supply preferences of each agent $i \in N$ (together with associated utilities u_i^d and u_i^s) and the resulting additive separable utilities, respectively. For our purpose, the only relevant information regarding Agent 4 is that all the houses and agents are acceptable for him; his demand and supply preferences are not relevant and are therefore omitted.

Agent 1				**Agent 2**				**Agent 3**			
$\succeq_1^d$	u_1^d	$\succeq_1^s$	u_1^s	$\succeq_2^d$	u_2^d	$\succeq_2^s$	u_2^s	$\succeq_3^d$	u_3^d	$\succeq_3^s$	u_3^s
h_3	10	3	9	h_1	100	3	40	h_1	40	2	100
h_2	8	2	6	h_3	50	4	20	h_4	20	1	50
h_4	1	4	1	h_4	40	1	5	h_2	5	4	40
h_1	0	1	0	h_2	0	2	0	h_3	0	3	0

Table 6: Market $\mathcal{H}_2$ demand and supply preferences and utilities.

Agent 1	**Agent 2**	**Agent 3**
$u_1(h_3, 3) = 19$	$u_2(h_1, 3) = 140$	$u_3(h_1, 2) = 140$
$u_1(h_2, 3) = 17$	$u_2(h_1, 4) = 120$	$u_3(h_4, 2) = 120$
$u_1(h_3, 2) = 16$	$u_2(h_1, 1) = 105$	$u_3(h_2, 2) = 105$
$u_1(h_2, 2) = 14$	$u_2(h_3, 3) = 90$	$u_3(h_1, 1) = 90$
$u_1(h_3, 4) = 11$	$u_2(h_4, 3) = 80$	$u_3(h_1, 4) = 80$
$u_1(h_4, 3) = 10$	$u_2(h_3, 4) = 70$	$u_3(h_4, 1) = 70$
$u_1(h_2, 4) = 9$	$u_2(h_4, 4) = 60$	$u_3(h_4, 4) = 60$
$u_1(h_4, 2) = 7$	$u_2(h_3, 1) = 55$	$u_3(h_2, 1) = 55$
$u_1(h_4, 4) = 2$	$u_2(h_4, 1) = 45$	$u_3(h_2, 4) = 45$
$u_1(h_1, 1) = 0$	$u_2(h_2, 2) = 0$	$u_3(h_3, 3) = 0$

Table 7: Market $\mathcal{H}_2$ additive separable utilities.

The core of the housing market $\mathcal{H}_2$ is empty; in Table 8 we list for each possible allocation for $\mathcal{H}_2$ how a subset of agents can block it (note that for the construction of blocking coalitions it suffices to consider the associated ordinal separable preferences of $\mathcal{H}_2$).

4 Demand (supply) lexicographic markets

Recall that in Definition 2 we have defined the weak and the strong core for markets $(N, h, \succeq)$ with limited externalities. However, if we apply this definition to classical Shapley-Scarf housing markets $(N, h, \succeq^d)$ based on demand preferences $\succeq^d$, then we obtain the corresponding classical Shapley-Scarf (weak) core $C(\succeq^d)$ and strong core $SC(\succeq^d)$. Furthermore, instead of using demand preferences $\succeq^d$ we could use supply preferences $\succeq^s$ to define the corresponding (weak) core $C(\succeq^s)$ and strong core $SC(\succeq^s)$.

We consider markets where all agents have demand lexicographic preferences, that is $\succeq \in \mathcal{D}^N_{\text{dlex}}$. First, we show that a strong core allocation always exists, i.e., $SC(\succeq) \neq \emptyset$. This existence result does neither depend on the number of agents, nor on the acceptability of houses or agents; it is obtained by linking the strong core $SC(\succeq)$ of the original market to the strong core $SC(\succeq^d)$ of the Shapley-Scarf market that is based on the agents' demand preferences. It turns out that the strong core of any associated Shapley-Scarf

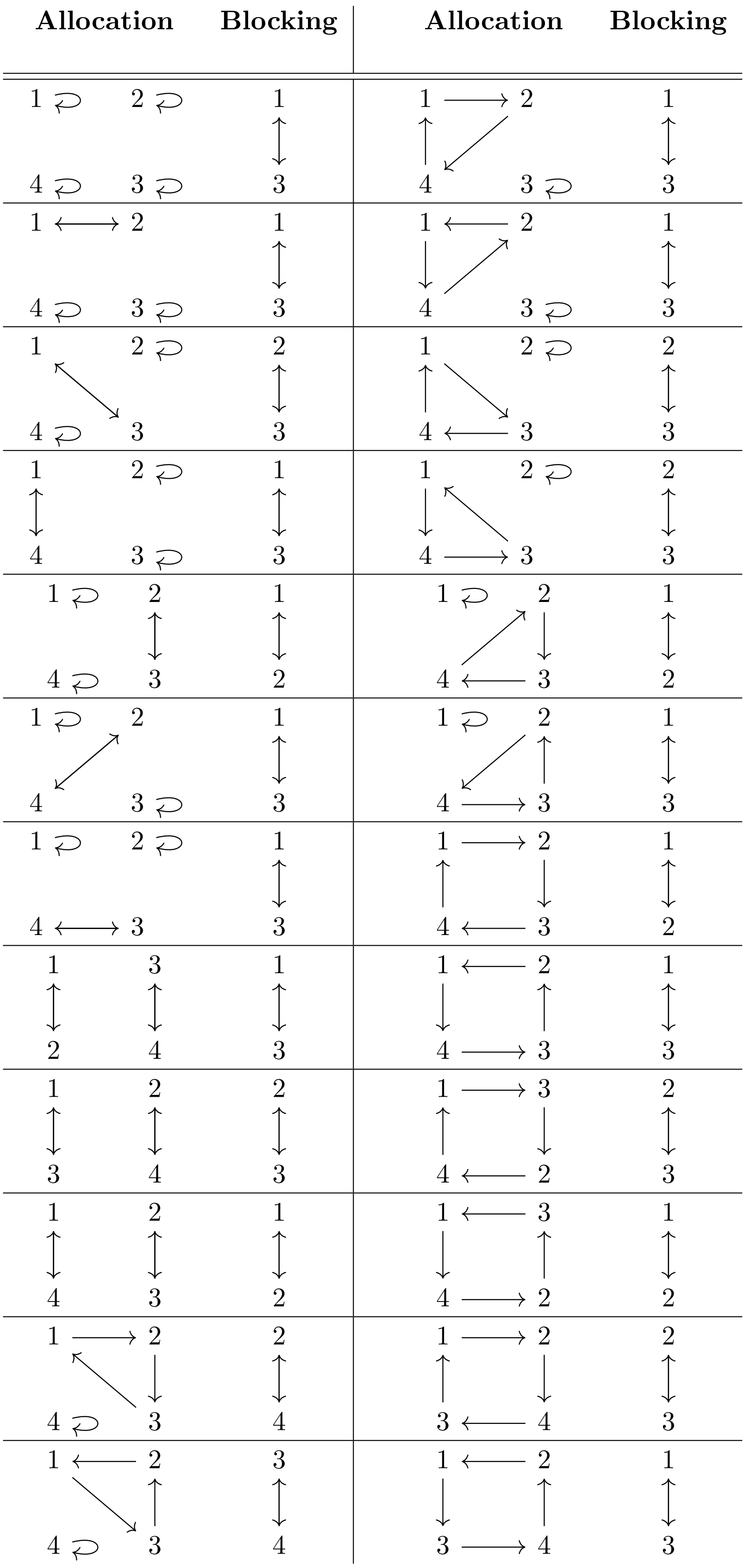


Table 8: Market $\mathcal{H}_2$ blocking of all possible allocations.

market is a subset of the strong core of the original market. Furthermore, we analyze the relationship between the weak core $C(\succeq)$ of the original market and the weak core $C(\succeq^d)$ of the associated Shapley-Scarf market.

Since we subsequently could symmetrically switch the roles of agents and houses, all results and examples obtained for demand lexicographic markets can be transcribed into corresponding results and examples for supply lexicographic markets.

4.1 The weak and strong core for demand lexicographic markets

Proposition 4. *Consider a housing market $(N, h, \succeq)$, $\succeq \in \mathcal{D}^N_{\text{dlex}}$, and its associated Shapley-Scarf market $(N, h, \succeq^d)$. Then, $SC(\succeq) \supseteq SC(\succeq^d) \neq \emptyset$.*

Proof. Let $(N, h, \succeq)$ be such that $\succeq \in \mathcal{D}^N_{\text{dlex}}$ and $(N, h, \succeq^d)$ be the associated Shapley-Scarf market. Hence, $SC(\succeq^d) \neq \emptyset$.

Next, we prove that $SC(\succeq) \supseteq SC(\succeq^d)$. Let $a \in SC(\succeq^d)$ and assume, by contradiction, that $a \notin SC(\succeq)$. Then, there exist a coalition $S \subseteq N$ and an allocation $b \in \mathcal{A}$ such that

(a) $b(S) = h(S)$ and

(b') for all agents $i \in S$,

$$(b(i), b^{-1}(h_i)) \succeq_i (a(i), a^{-1}(h_i))$$

and for some agent $j \in S$,

$$(b(j), b^{-1}(h_j)) \succ_j (a(j), a^{-1}(h_j)).$$

Let $S_1 = \{i \in S : b(i) \succ^d_i a(i)\}$ and $S_2 = \{i \in S : b(i) = a(i)\}$.

It cannot be the case that $S_2 = S$ since that would imply that for all agents $i \in S$, $b(i) = a(i)$ and $b^{-1}(h_i) = a^{-1}(h_i)$, contradicting (b'). Thus, for all agents $i \in S$, $b(i) \succeq^d_i a(i)$, and for some agent $j \in S$, $b(j) \succ^d_j a(j)$.

Hence, S weakly blocks a through b, which contradicts $a \in SC(\succeq^d)$. □

Our next example illustrates that the set inclusion in Proposition 4 may be strict and that multiple strong core allocations may exist.

Example 2. Let $N = \{1, 2, 3\}$ and $h = (h_1, h_2, h_3)$. We assume that $\succeq \in \mathcal{D}^N_{\text{dlex}}$ with demand and supply preferences as specified in Table 9. The empty column means that any linear order $\succeq^s_3$ can be considered. We refer to Table 1 for the labelling of allocations and allotments.

Agent 1		Agent 2		Agent 3	
$\succeq_1^d$	$\succeq_1^s$	$\succeq_2^d$	$\succeq_2^s$	$\succeq_3^d$	$\succeq_3^s$
h_2	3	h_1	1	h_2	$\cdot$
h_3	2	h_3	3	h_1	$\cdot$
h_1	1	h_2	2	h_3	$\cdot$

Table 9: Example 2 demand and supply preferences.

First, the strong core for the Shapley-Scarf market $\succeq^d$ is formed by the unique allocation $a_4 = (h_2, h_1, h_3)$, which can easily be computed by Gale's top trading cycles (TTC) algorithm based on $\succeq^d$.

Second, for market $\succeq$, allocation $a_3 = (h_2, h_3, h_1) \in SC(\succeq)$ because agent 1 gets his most preferred allotment $(h_2, 3)$ and coalition $S = \{2, 3\}$ cannot block by $a_6 = (h_1, h_3, h_2)$ (agent 2 would be worse off since $(h_3, 1) \succ_2 (h_3, 3)$). However, $a_3 \notin SC(\succeq^d)$ because it is weakly blocked by $S = \{2, 3\}$ through $a_6 = (h_1, h_3, h_2)$ (agent 2 receives the same house and agent 3 a better house). □

Proposition 5. *Consider a housing market $(N, h, \succeq)$, $\succeq \in \mathcal{D}^N_{\text{dlex}}$, and its associated Shapley-Scarf market $(N, h, \succeq^d)$. Then, $\emptyset \neq C(\succeq) \subseteq C(\succeq^d)$.*

Proof. Let $(N, h, \succeq)$ be such that $\succeq \in \mathcal{D}^N_{\text{dlex}}$ and $(N, h, \succeq^d)$ be the associated Shapley-Scarf market. First, $\emptyset \neq C(\succeq)$ follows from $C(\succeq) \supseteq SC(\succeq) \neq \emptyset$ (Proposition 4).

Second, we prove $C(\succeq) \subseteq C(\succeq^d)$ by contradiction. Let $a \in C(\succeq)$ and assume that $a \notin C(\succeq^d)$. Then, there exist a coalition $S \subseteq N$ and an allocation b such that

(a) $b(S) = h(S)$ and

(b) for all agents $i \in S$,

$$b(i) \succ_i^d a(i).$$

Since $\succeq \in \mathcal{D}^{\text{dlex}}$, (b) also implies that for all agents $i \in S$,

$$(b(i), b^{-1}(h_i)) \succ_i (a(i), a^{-1}(h_i)).$$

Thus, $a \notin C(\succeq)$. □

Our next example illustrates that the set inclusion in Proposition 5 may be strict.

Example 3. Let $N = \{1, 2, 3\}$ and $h = (h_1, h_2, h_3)$. We assume that $\succeq \in \mathcal{D}^N_{\text{dlex}}$ with demand and supply preferences as specified in Table 10. The empty column means that any linear order $\succeq^s_2$ can be considered. We refer to Table 1 for the labelling of allocations and allotments.

Agent 1		**Agent 2**		**Agent 3**	
$\succeq^d_1$	$\succeq^s_1$	$\succeq^d_2$	$\succeq^s_2$	$\succeq^d_3$	$\succeq^s_3$
h_3	2	h_3	·	h_1	1
h_2	3	h_1	·	h_2	2
h_1	1	h_2	·	h_3	3

Table 10: Example 3 demand and supply preferences.

For market $\succeq$, allocation $a_3 = (h_2, h_3, h_1) \in C(\succeq^d)$ because agents 2 and 3 receive their favorite house. However, $a_3 \notin C(\succeq)$ because it is strongly blocked by $S = \{1, 3\}$ through $a_5 = (h_3, h_2, h_1)$ (since $(h_3, 3) \succ_1 (h_2, 3)$ and $(h_1, 1) \succ_3 (h_1, 2)$, both agents in S are better off). $\square$

By Proposition 2, for any housing market $(N, h, \succeq)$ where $|N| = 3$ and $\succeq \in \mathcal{D}^N_{\text{dlex}}$, $C(\succeq) = SC(\succeq)$. Our next example shows that the distinction between the strong and the weak core matters in a demand lexicographic market when there are more than three agents.

Example 4. Let $N = \{1, 2, 3, 4\}$ and $h = (h_1, h_2, h_3, h_4)$. We assume that $\succeq \in \mathcal{D}^N_{\text{dlex}}$ with demand and supply preferences as specified in Table 11. The partially empty columns mean that any consistent linear orders $\succeq^d_2$, $\succeq^s_2$, $\succeq^d_4$, and $\succeq^s_4$ can be considered.

Agent 1		**Agent 2**		**Agent 3**		**Agent 4**	
$\succeq^d_1$	$\succeq^s_1$	$\succeq^d_2$	$\succeq^s_2$	$\succeq^d_3$	$\succeq^s_3$	$\succeq^d_4$	$\succeq^s_4$
h_2	3	h_3	1	h_1	2	h_1	1
h_3	4	·	·	h_4	1	·	·
h_4	1	·	·	h_2	4	·	·
h_1	2	h_2	2	h_3	3	h_4	4

Table 11: Example 4 demand and supply preferences.

For market $\succeq$, allocation $a = (h_2, h_3, h_4, h_1)$ is weakly blocked by $S = \{1, 2, 3\}$ through $b = (h_2, h_3, h_1, h_4)$ since agent 2 gets the same allotment at allocations a and b, while agents 1 and 3 are both better off. However, a cannot be strongly blocked by any coalition. Hence, $a \in C(\succeq)$ and $a \notin SC(\succeq)$. □

Our results for demand lexicographic markets can be summarized by Figure 1.[5]

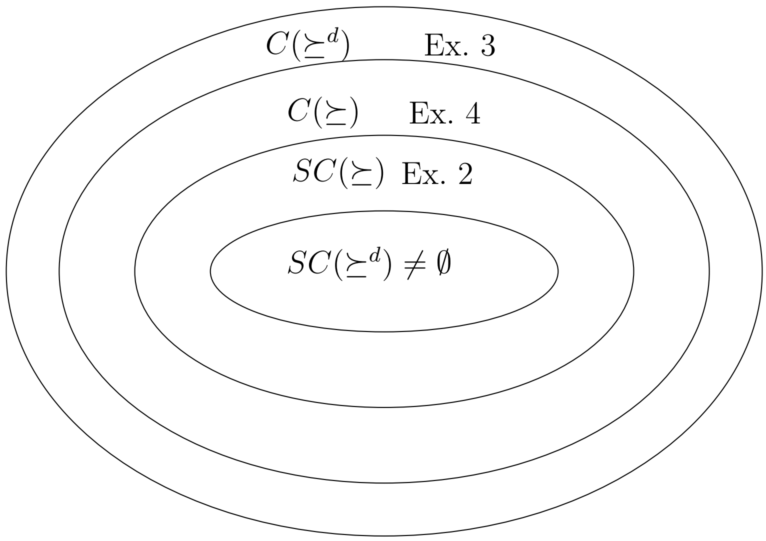


Figure 1: Set inclusions for weak and strong cores for a demand lexicographic market $\succeq$ and its associated Shapley-Scarf market $\succeq^d$.

5 Conclusions

Consumption externalities can occur in many economic models and housing markets are no exception. On one hand, they may easily occur in many real-life applications (e.g., vacation home exchanges); on the other hand, they are problematic because many (existence) results break down when externalities are present. The study of externalities in matching markets started with Sasaki and Toda (1996), who modeled them via preference relations that are defined over the set of all possible matchings. For housing markets with this general form of externalities, Mumcu and Sağlam (2007) prove that the core may be empty and multi-valued. Graziano et al. (2020) and Hong and Park (2020) introduce various classes of preferences accounting for different degrees of externalities and analyze several core-like solution concepts. The former paper mainly focuses on the existence of some of these solution concepts and their characterization as stable sets à la von Neumann and Morgenstern, while the latter one also adapts the TTC algorithm to some of

[5] A similar figure to summarize results for supply lexicographic markets $\succeq \in \mathcal{D}^N_{\text{slex}}$ can be obtained by replacing demand preferences $\succeq^d$ with supply preferences $\succeq^s$.

these preference domains with externalities and provides results for the then obtained TTC allocations.

The distinguishing feature of our paper is its focus on a very limited but natural form of externality: each agent cares about his own consumption, as in classical Shapley-Scarf housing markets, and about the agent who receives his endowment. Such limited externalities fit, for instance, situations where the agents' endowments are only temporarily traded and eventually return to their original owners. The modeling can easily be done via preferences that are defined over received-object – endowment-recipient pairs. For this class of markets with limited externalities, we analyze various natural preference domains and investigate the existence of weak and strong core allocations, depending on two factors: the number of agents and the acceptability of houses and / or agents. Our main findings are:

a. For the (additive) separable preference domain, the strong core in nonempty for three-agents markets when all houses and agents are acceptable; however, this existence result can be extended neither to markets with a larger number of agents nor to markets with unacceptable agents or houses.

b. For the demand lexicographic preference domain (as well as for the supply lexicographic preference domain), the strong core is nonempty, independently of the number of agents and the acceptability of houses or agents, and it can be multi-valued.

Appendix

A Relations between preference domains

In this appendix we first prove Proposition 1 and then provide examples showing that set inclusions between preference domains may be strict.

Proof of Proposition 1. Set inclusions $\mathcal{D} \supseteq \mathcal{D}_{\text{sep}} \supseteq \mathcal{D}_{\text{add}}$, as well as $\mathcal{D}_{\text{dlex}} \cap \mathcal{D}_{\text{slex}} = \emptyset$ follow easily from the corresponding preference domain definitions.

To prove that $\mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}} \subseteq \mathcal{D}_{\text{add}}$, without loss of generality, let $\succeq_i \in \mathcal{D}_{\text{dlex}}$. Then, it can easily be checked that the following utility functions, which are reminiscent of Borda scores, represent demand lexicographic preferences. For each $j \in N$, let $B_i(j) = \{k \in N : j \succeq_i^s k\}$ and $u_i(j) = |B_i(j)|$. For each $h \in H$, let $B_i(h) = \{h' \in H : h \succeq_i^d h'\}$ and $u_i(h) = n|B_i(h)|$.

Next, we show that when $|N| = 3$, $\mathcal{D}_{\text{sep}} = \mathcal{D}_{\text{add}}$. Let $N = \{i, j, k\}$ and $\{h_j, h_k\} = \{a, b\}$. Since in general we have $\mathcal{D}_{\text{sep}} \supseteq \mathcal{D}_{\text{add}}$, we only need to show that $\mathcal{D}_{\text{sep}} \subseteq \mathcal{D}_{\text{add}}$. Let $\succeq_i \in \mathcal{D}_{\text{sep}}$ with corresponding $\succeq_i^d \in \mathcal{D}_{\text{d}}$ and $\succeq_i^s \in \mathcal{D}_{\text{s}}$. Without loss of generality, suppose that $a \succ_i^d b$ and $j \succ_i^s k$. Then, $\succeq_i$ may rank the allotments different from (h_i, i) according to the following two types:

$$\text{Type I:} \quad (a, j) \succ_i (a, k) \succ_i (b, j) \succ_i (b, k)$$

$$\text{Type II:} \quad (a, j) \succ_i (b, j) \succ_i (a, k) \succ_i (b, k)$$

Note that, depending on how h_i and i rank in $\succ_i^d$ and $\succ_i^s$, any position for the endowment (h_i, i) is a priori possible (we consider the endowment in more detail later on).

Consider the following utility values u_i^d and u_i^s:

	$u_i^d(a)$	$u_i^d(b)$	$u_i^s(j)$	$u_i^s(k)$
Type I:	4	1	3	2
Type II:	3	2	4	1

It can now be easily verified that utility function $u(h, t) = u_i^d(h) + u_i^s(t)$ is an additive separable representation of $\succeq_i$ for allotments (a, j), (a, k), (b, j), and (b, k). Concerning the endowment allotment (h_i, i), it is always possible to find values $u_i^d(h_i)$ and $u_i^s(i)$ such that the corresponding additive separable preferences are respected. For example, consider Type I preferences and suppose that (h_i, i) is the most preferred allotment. These preferences over allotments can in fact result from three different demand / supply preferences as listed below. We give corresponding utility values $u_i^d(h_i)$ and $u_i^s(i)$ to show that $\succeq_i$ is additive separable.

1.	h_i	$\succ_i^d$	a	$\succ_i^d$	b
$\boldsymbol{u_i^d}$:	$\boxed{4.5}$		4		1
	i	$\succ_i^s$	j	$\succ_i^s$	k
$\boldsymbol{u_i^s}$:	$\boxed{3.5}$		3		2

2.	h_i	$\succ_i^d$	a	$\succ_i^d$	b
$\boldsymbol{u_i^d}$:	$\boxed{5.5}$		4		1
	j	$\succ_i^s$	i	$\succ_i^s$	k
$\boldsymbol{u_i^s}$:	3		$\boxed{2.5}$		2

3.	a	$\succ_i^d$	h_i	$\succ_i^d$	b
$\boldsymbol{u_i^d}$:	4		$\boxed{3}$		1
	i	$\succ_i^s$	j	$\succ_i^s$	k
$\boldsymbol{u_i^s}$:	$\boxed{5}$		3		2

We conclude that $\succeq_i \in \mathcal{D}_{\text{add}}$.

Moreover, when all houses and agents are acceptable, note that $\succeq_i \in \mathcal{D}_{\text{dlex}}$ when preferences are of Type I, while $\succeq_i \in \mathcal{D}_{\text{slex}}$ when preferences are of Type II. Hence, $\succeq_i \in \mathcal{D}_{\text{add}}$ implies $\succeq_i \in \mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}}$ and thus $\mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}} \supseteq \mathcal{D}_{\text{add}}$. □

The next four examples complete our analysis of relations between preference domains by showing that the following set inclusions are strict:

1. $\mathcal{D} \supsetneq \mathcal{D}_{\text{sep}}$ (Example 5),

2. $\mathcal{D}_{\text{sep}} \supsetneq \mathcal{D}_{\text{add}}$ for $|N| > 3$ (Example 6),

3. $\mathcal{D}_{\text{add}} \supsetneq \mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}}$ for $|N| > 3$ when all houses and agents are acceptable (Example 7), and

4. $\mathcal{D}_{\text{add}} \supsetneq \mathcal{D}_{\text{dlex}} \cup \mathcal{D}_{\text{slex}}$ for $|N| = 3$ when some houses or agents are unacceptable (Example 8).

Example 5 (**Strict preferences that are not separable**)**.** Consider the three agent market described in Example 1, Table 2. We show that agent 1's ranking of allotments, $\succeq_1 \in \mathcal{D}$, cannot result from separable preferences:

$$(h_3, 3) \succ_1 (h_2, 2) \succ_1 (h_1, 1) \succ_1 (h_2, 3) \succ_1 (h_3, 2)\ .$$

If $3 \succ_1^s 2$, then separability would imply $(h_2, 3) \succ_1 (h_2, 2)$, which is not true. However, if $2 \succ_1^s 3$, then separability would imply $(h_3, 2) \succ_1 (h_3, 3)$, which is not true. □

Example 6 (**Strict and separable preferences that are not additive separable**)**.** Consider a four agent market with $N = \{1, 2, 3, 4\}$, $H = \{h_1, h_2, h_3, h_4\}$, and agent 1's demand and supply preferences

$$\begin{array}{ccccccc} h_2 & \succ_1^d & h_3 & \succ_1^d & h_4 & \succ_1^d & h_1 \\ 2 & \succ_1^s & 3 & \succ_1^s & 4 & \succ_1^s & 1 \end{array}$$

It is easy to check that the following preferences $\succeq_1$ are separable:

$$(h_2, 2) \succ_1 (h_3, 2) \succ_1 (h_2, 3) \succ_1 (h_2, 4) \succ_1 (h_4, 2) \succ_1 (h_3, 3) \succ_1 (h_4, 3) \succ_1 (h_3, 4) \succ_1 (h_4, 4) \succ_1 (h_1, 1).$$

By way of contradiction, suppose that $\succeq_1$ is additive separable. Then, there exist two utility functions u_1^d and u_1^s such that for all $(h, j), (h', k) \in \mathcal{A}_1$,

$$(h, j) \succ_1 (h', k) \text{ if and only if } u_1^d(h) + u_1^s(j) > u_1^d(h') + u_1^s(k).$$

Since $(h_3, 2) \succ_1 (h_2, 3)$ and $(h_2, 4) \succ_1 (h_4, 2)$, we get $u_1^d(h_3) + u_1^s(2) > u_1^d(h_2) + u_1^s(3)$ and $u_1^d(h_2) + u_1^s(4) > u_1^d(h_4) + u_1^s(2)$. By adding up these two inequalities, we obtain $u_1^d(h_3) + u_1^s(4) > u_1^d(h_4) + u_1^s(3)$, which contradicts $(h_4, 3) \succ_1 (h_3, 4)$. We conclude that $\succeq_1$ is not additive separable. □

Example 7 (**Strict and additive preferences that are neither demand lexicographic nor supply lexicographic ($|N| = 4$)).** Consider a four agent market with $N = \{1,2,3,4\}$, $H = \{h_1, h_2, h_3, h_4\}$, and agent 1's preferences $\succeq_1$

$$(h_2,2) \succ_1 (h_2,3) \succ_1 (h_2,4) \succ_1 (h_3,2) \succ_1 (h_4,2) \succ_1 (h_3,3) \succ_1 (h_4,3) \succ_1 (h_3,4) \succ_1 (h_4,4) \succ_1 (h_1,1)$$

that are additive separable (all houses and agents are acceptable) with corresponding utility functions

$$\begin{array}{lccccccc} & h_2 & \succ_1^d & h_3 & \succ_1^d & h_4 & \succ_1^d & h_1 \\ \boldsymbol{u_1^d}: & 4 & & 1 & & 0.5 & & \text{-1} \\ & 2 & \succ_1^s & 3 & \succ_1^s & 4 & \succ_1^s & 1 \\ \boldsymbol{u_1^s}: & 3 & & 2 & & 1 & & \text{-1} \end{array}$$

Because $(h_3,2) \succ_1 (h_4,2) \succ_1 (h_3,3)$, preferences $\succeq_1$ are not demand lexicographic and because $(h_2,2) \succ_1 (h_2,3) \succ_1 (h_3,2)$, preferences $\succeq_1$ are not supply lexicographic. $\square$

Example 8 (**Strict and additive preferences that are neither demand lexicographic nor supply lexicographic ($|N| = 3$)).** Consider a three agent market with $N = \{1,2,3\}$, $H = \{h_1, h_2, h_3\}$, and agent 1's preferences $\succeq_1$

$$(h_3,2) \succ_1 (h_1,1) \succ_1 (h_3,3) \succ_1 (h_2,2) \succ_1 (h_2,3)$$

that are additive separable (house h_2 and agent 3 are not acceptable) with corresponding utility functions

$$\begin{array}{lccccc} & h_3 & \succ_1^d & h_1 & \succ_1^d & h_2 \\ \boldsymbol{u_1^d}: & 6 & & 5 & & 1 \\ & 2 & \succ_1^s & 1 & \succ_1^s & 3 \\ \boldsymbol{u_1^s}: & 5 & & 4 & & 2 \end{array}$$

Because $(h_3,2) \succ_1 (h_1,1) \succ_1 (h_3,3)$, preferences $\succeq_1$ are not demand lexicographic and because $(h_3,3) \succ_1 (h_2,2) \succ_1 (h_2,3)$, preferences $\succeq_1$ are not supply lexicographic. $\square$

Figure 2 illustrates the relationships between the preference domains.

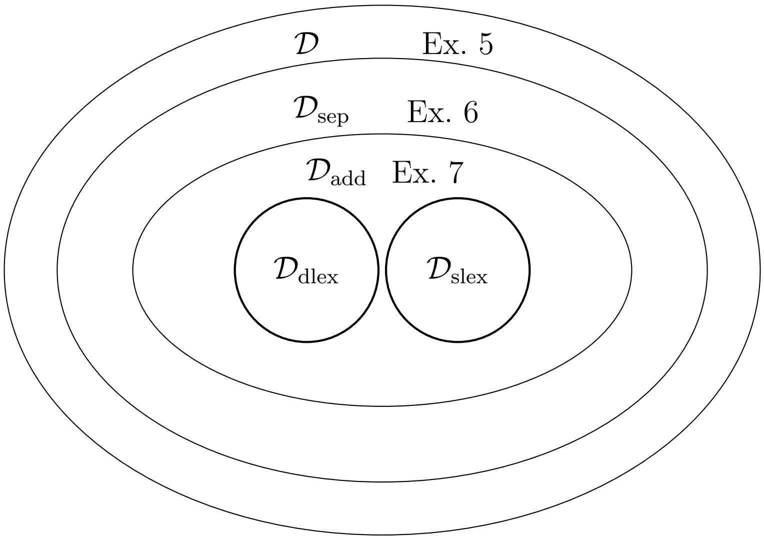


Figure 2: Preference domain set inclusions.

B Proof of Proposition 3

Proof of Proposition 3. Let $(N, h, \succeq)$ be such that $|N| = 3$, $\succeq \in \mathcal{D}^N_{\text{sep}}$, and all houses and agents are acceptable. By Proposition 1, $\mathcal{D}^N_{\text{sep}} = \mathcal{D}^N_{\text{add}}$. For each agent $i \in N$, let FC^d_i and FC^s_i denote his most preferred (***F***irst ***C***hoice) house and agent, respectively. Depending on FC^d_i and FC^s_i, there are four possible preference types for each agent, listed in Table 12.

Note that for each preference type of an agent in Table 12, there are two possible separable preferences; in the sequel, we refer to the specific associated preferences only when necessary. There are $4^3 = 64$ possible cases for separable preference profiles that we will denote by triplets indicating the preference type for each agent; for instance, the type-triplet $(t3, t2, t2)$ indicates that agent 1 is of type 3 while the other two agents are both of type 2.

By Proposition 2, it suffices to prove that the core is nonempty. To this aim, we group the possible type-triplets into three sets.

Set 1: two agents prefer to pairwise trade (12 cases). For two agents $i, j \in N$,

$$(FC^d_i, FC^s_i) = (h_j, j) \text{ and } (FC^d_j, FC^s_j) = (h_i, i).$$

(a) **Agent 1:**

Type 1		**Type 2**		**Type 3**		**Type 4**	
$\succeq_1^d$	$\succeq_1^s$	$\succeq_1^d$	$\succeq_1^s$	$\succeq_1^d$	$\succeq_1^s$	$\succeq_1^d$	$\succeq_1^s$
h_2	2	h_3	3	h_2	3	h_3	2
h_3	3	h_2	2	h_3	2	h_2	3
h_1	1	h_1	1	h_1	1	h_1	1

(b) **Agent 2:**

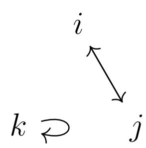
Type 1		**Type 2**		**Type 3**		**Type 4**	
$\succeq_2^d$	$\succeq_2^s$	$\succeq_2^d$	$\succeq_2^s$	$\succeq_2^d$	$\succeq_2^s$	$\succeq_2^d$	$\succeq_2^s$
h_3	3	h_1	1	h_3	1	h_1	3
h_1	1	h_3	3	h_1	3	h_3	1
h_2	2	h_2	2	h_2	2	h_2	2

(c) **Agent 3:**

Type 1		**Type 2**		**Type 3**		**Type 4**	
$\succeq_3^d$	$\succeq_3^s$	$\succeq_3^d$	$\succeq_3^s$	$\succeq_3^d$	$\succeq_3^s$	$\succeq_3^d$	$\succeq_3^s$
h_1	1	h_2	2	h_1	2	h_2	1
h_2	2	h_1	1	h_2	1	h_1	2
h_3	3	h_3	3	h_3	3	h_3	3

Table 12: Proof of Proposition 3 preference types for $\succeq \in \mathcal{D}^N_{\text{sep}}$.

The type-triplets contained in Set 1 are $(t1, t2, *)$, $(t2, *, t1)$, and $(*, t1, t2)$, where the symbol $* \in \{t1, t2, t3, t4\}$ can be any possible type for the corresponding agent (hence, we have $3 \cdot 4 = 12$ cases). The following allocation that results from the pairwise trade between agents i and j



belongs to the core.

Set 2: all agents compete for different houses (16 cases in total). Let

$$FC_i^d \neq FC_j^d \neq FC_k^d.$$

We partition Set 2 into the following subsets.

Set 2.1 (8 cases): $FC_1^d = h_3$, $FC_2^d = h_1$, and $FC_3^d = h_2$. The type-triplets included in this subset are $(t2, t2, t4)$, $(t2, t2, t2)$, $(t2, t4, t4)$, $(t2, t4, t2)$, $(t4, t2, t4)$, $(t4, t2, t2)$, $(t4, t4, t4)$, and $(t4, t4, t2)$.

Set 2.2 (8 cases): $FC_1^d = h_2$, $FC_2^d = h_3$, and $FC_3^d = h_1$. The type-triplets included in this subset are $(t3, t3, t1)$, $(t3, t3, t3)$, $(t3, t1, t1)$, $(t3, t1, t3)$, $(t1, t3, t1)$, $(t1, t3, t3)$, $(t1, t1, t1)$, and $(t1, t1, t3)$.

For Set 2.1, allocation (h_3, h_1, h_2) belongs to the core in seven of the cases (except possibly for $(t2, t2, t2)$) because at least one agent is of type 4 and gets his most preferred allotment and the other two agents cannot block because a pairwise trade would not be advantageous for at least one of them. For type-profile $(t2, t2, t2)$, if at least two agents have the following preferences

$$(h_3, 2) \succeq_1 (h_2, 3),$$
$$(h_1, 3) \succeq_2 (h_3, 1),$$
$$(h_2, 1) \succeq_3 (h_1, 2),$$

then allocation (h_3, h_1, h_2) belongs to the core. Otherwise, allocation (h_2, h_3, h_1) belongs to the core.

Symmetrically, for Set 2.2, allocation (h_2, h_3, h_1) belongs to the core in seven of the cases (except possibly for $(t1, t1, t1)$). For type-profile $(t1, t1, t1)$, if at least two agents have the following preferences

$$(h_2, 3) \succeq_1 (h_3, 2),$$
$$(h_3, 1) \succeq_2 (h_1, 3),$$
$$(h_1, 2) \succeq_3 (h_2, 1),$$

then allocation (h_2, h_3, h_1) belongs to the core. Otherwise, (h_3, h_1, h_2) belongs to the core.

Set 3: two agents compete over the same house without preferred pairwise trades (36 cases in total). There exist $i, j \in N$ such that

$$FC_i^d = FC_j^d = h_k \text{ and } (FC_k^d, FC_k^s) \neq \begin{cases} (h_i, i), \text{ if } k \succ_i^s j, \\ (h_j, j), \text{ if } k \succ_i^s i. \end{cases}$$

We partition Set 3 into the following subsets.

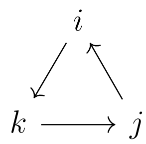
Set 3.1 (12 cases): $FC_1^d = FC_2^d = h_3$, that is, agents 1 and 2 compete for h_3. The type-triplets included in this subset are $(t2, t1, *)$ with $* \in \{t3, t4\}$, $(t2, t3, *)$ with $* \in \{t2, t3, t4\}$, $(t4, t1, *)$ with $* \in \{t1, t3, t4\}$, and $(t4, t3, *)$ with $* \in \{t1, t2, t3, t4\}$.

Set 3.2 (12 cases): $FC_1^d = FC_3^d = h_2$, that is, agents 1 and 3 compete for h_2. The type-triplets included in this subset are $(t1, *, t2)$ with $* \in \{t3, t4\}$, $(t3, *, t2)$ with $* \in \{t2, t3, t4\}$, $(t1, *, t4)$ with $* \in \{t1, t3, t4\}$, and $(t3, *, t4)$ with $* \in \{t1, t2, t3, t4\}$.

Set 3.3 (12 cases): $FC_2^d = FC_3^d = h_1$; that is, agents 2 and 3 compete for h_1. The type-triplets included in this subset are $(*, t2, t1)$ with $* \in \{t3, t4\}$, $(*, t2, t3)$ with $* \in \{t2, t3, t4\}$, $(*, t4, t1)$ with $* \in \{t1, t3, t4\}$, and $(*, t4, t3)$ with $* \in \{t1, t2, t3, t4\}$.

We show that the allocation where h_k is allotted to the agent preferred by the non-competing agent k, that is FC_k^s, belongs to the core. We distinguish two cases.

Case 1: $i \succ_k^s j$. We prove that the allocation

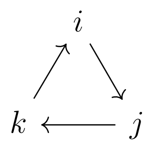


belongs to the core.

If $h_j \succ_k^d h_i$, then agent k gets his most preferred allotment and coalition $\{i, j\}$ cannot block through a pairwise trade because by separability of $\succeq_i$, $(h_k, j) \succ_i (h_j, j)$.

If, on the contrary, $h_i \succ_k^d h_j$, then $j \succ_i^s k$ (otherwise, the type profile would be in Set 1). Hence, agent i gets his most preferred allotment and coalition $\{j, k\}$ cannot block through a pairwise trade because by separability of $\succeq_k$, $(h_j, i) \succ_k (h_j, j)$.

Case 2: $j \succ_k^s i$. We prove that the allocation



belongs to the core.

If $h_i \succ_k^d h_j$, then agent k gets his most preferred allotment and coalition $\{i, j\}$ cannot block through a pairwise trade because by separability of $\succeq_i$, $(h_k, i) \succ_i (h_i, i)$.

If, on the contrary, $h_j \succ_k^d h_i$, then $i \succ_j^s k$ (otherwise, the type profile would be in Set 1). Hence, agent j gets his most preferred allotment and coalition $\{i, k\}$ cannot block through a pairwise trade because by separability of $\succeq_k$, $(h_i, j) \succ_k (h_i, i)$. □

References

Aziz, H. and Lee, E. J. (2020): "The temporary exchange problem." AAMAS2020 Conference Presentation.

Gale, D. and Shapley, L. S. (1962): "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69: 9–15.

Graziano, M. G., Meo, C., and Yannelis, N. C. (2020): "Shapley and Scarf housing markets with consumption externalities." *Journal of Public Economic Theory*, 22: 1481–1514.

Hong, M. and Park, J. (2020): "Core and top trading cycles in a market with indivisible goods and externalities." Working Paper.

Mumcu, A. and Sağlam, I. (2007): "The core of a housing market with externalities." *Economics Bulletin*, 55: 1–5.

Roth, A. E. and Postlewaite, A. (1977): "Weak versus strong domination in a market with indivisible goods." *Journal of Mathematical Economics*, 4: 131–137.

Sasaki, H. S. and Toda, M. (1996): "Two-sided matching problems with externalities." *Journal of Economic Theory*, 70: 93–108.

Shapley, L. S. and Scarf, H. (1974): "On cores and indivisibility." *Journal of Mathematical Economics*, 1: 23–37.