Normative properties for object allocation problems: Characterizations and trade-offs*

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1 Introduction

We consider the allocation of indivisible objects among agents when monetary transfers are not allowed. Agents have strict preferences over the objects (possibly about not getting any object) and are assigned at most one object. We will also assume that each object comes with capacity one.

How should one allocate offices to faculty members at a university when a department moves into a new building or when the current office allocation is not considered optimal anymore? Ideally, an allocation rule would be (1) fair / equitable, (2) efficient, and (3) incentive robust. Of course, as we will see in this chapter, our three objectives might find different formulations depending on the exact allocation situation. Unfortunately, often the most natural properties to reflect (1) - (3) are not compatible and thus, an ideal allocation method usually does not exist.

In the first part of our chapter (Section 3), each agent is endowed with one object, has strict preferences over the set of objects, and agents can trade objects. For these so-called Shapley-Scarf exchange problems, the strong core consists of a unique allocation that can be computed by the top trading cycles algorithm. We start with positive news for this model: the so-called *top trading cycles rule* is characterized by (1) voluntary participation (*individual rationality*), (2) efficiency (*Pareto optimality*), and (3) incentive robustness (*strategy-proofness*).

In the second part (Sections 4 and 5), the set of objects is commonly owned by the agents and all objects are acceptable. Then, *serial dictatorship rules* are characterized by (1) *neutrality* and (3) *group strategy-proofness* (or alternatively, *strategy-proofness* and *non-bossiness*); *serial dictatorship rules* also satisfy (2) *Pareto optimality*. One could interpret this result as positive news since again objectives (1) - (3) are compatible, but *serial dictatorship rules*

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might be criticised as unfair due to their hierarchical spirit (e.g., the agent with the highest priority will always receive his best object). If one does not require *neutrality*, then a much larger class of rules emerges: instead of requiring the same order of agents for all objects (which determines the sequence of dictators the *serial dictatorship rule* is based on), one could allow for a different order of agents per object, which can be considered as improving fairness aspect (1). We present this class of rules, the so-called *endowment inheritance rules*, in Section 5 and just note without a proof that they are part of a much larger class of rules, the *hierarchical exchange rules*, that are characterized by (2) *Pareto optimality* and (3) *reallocation-proofness* and *group strategy-proofness* (or alternatively, *strategy-proofness* and *non-bossiness*).

In the third part (Section 6), each object is (possibly) endowed with a priority ordering over agents capturing the rights of agents of receiving the object. For this setup, one of the most widely applied rules nowadays is the agent-proposing *deferred-acceptance rule*, which reflects (1) fairness in the form of *stability* (or *absence of justified envy*). *Deferred acceptance rules* tick boxes (1) and (3) but unfortunately not (2) (see Exercise 6). For *deferred acceptance rules* we present a characterization where the priority orderings are derived from the normative properties (for the setup of the second part): *deferred acceptance rules* are the only (variable population) rules satisfying (1) *$\emptyset$-individual rationality*, *population-monotonicity*, (2) *weak non-wastefulness*, and (3) *strategy-proofness*.

Finally (Section 7), we discuss how *top trading cycles rules*, *serial dictatorship rules*, *endowment inheritance rules*, *deferred acceptance rules*, and their properties are related. We then conclude by briefly discussing the embedding of the presented results into the literature (Section 8).

2 The basic model

We present common notation for all three parts of the chapter. Let $N = \{1, \ldots, n\}$ denote the finite *set of agents*. A nonempty subset $S \subseteq N$ is called a *coalition*. Let $O = \{o_1, \ldots, o_m\}$ denote the finite *set of (real) objects*, and by $\emptyset$ denote the *null object*. Each real object can only be assigned to one agent, whereas the null object can be assigned to an arbitrary number of agents. An *allocation (for N)* is a mapping $a : N \to O \cup \{\emptyset\}$ such that for all $i \neq j$, $a(i) = a(j)$ implies $a(i) = a(j) = \emptyset$. We denote the *set of all allocations (for N)* by $\mathcal{A}_N$. For allocation a, *agent i's allotment* is denoted by a_i (instead of $a(i)$).

We assume that each agent has a *strict preference relation* R_i over all possible allotments. A strict preference relation R_i is a complete, antisymmetric, and transitive relation on $O \cup \{\emptyset\}$. Let $i \in N$. Then, for any two allotments a_i, b_i, a_i is *weakly better than* b_i if $a_i \; R_i \; b_i$, and a_i is *strictly better than* b_i, denoted by $a_i \; P_i \; b_i$, if $[a_i \; R_i \; b_i$ and not $b_i \; R_i \; a_i]$. Finally, since preferences are strict, a_i is indifferent to b_i, denoted by $a_i \; I_i \; b_i$, only if $a_i = b_i$. The *set of all (strict) preferences* is denoted by $\mathcal{R}$. Sometimes we write $R_i : o \; o' \; \cdots$ to denote strict preferences where o is the most R_i-preferred object in $O \cup \{\emptyset\}$, o' is the most R_i-

preferred object in $(O \cup \{\emptyset\}) \setminus \{o\}$, and the remaining preferences are arbitrary. For agent i with preferences R_i, we say that *object* $o \in O \cup \{\emptyset\}$ *is acceptable* if $o\, R_i\, \emptyset$ and it is *unacceptable* if $\emptyset\, P_i\, o$. Let $A(R_i)$ denote the *set of acceptable objects under* R_i. Let $\underline{\mathcal{R}}$ be the subdomain of $\mathcal{R}$ where all real objects are acceptable; $\underline{\mathcal{R}}$ denotes the *set of all acceptable (strict) preferences*.

A *preference profile* is a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We assume that agents only care about their own allotments. Hence, agents' preferences over allocations are determined by preferences over allotments, i.e., for each $i \in N$ and for any $a, b \in \mathcal{A}_N$, $a_i\, P_i\, b_i$ implies $a\, P_i\, b$ and $a_i = b_i$ implies $a\, I_i\, b$. Following standard notation, for any coalition $S \subseteq N$, (R'_S, R_{-S}) is the preference profile that is obtained from R when all agents $i \in S$ change their preferences from R_i to R'_i. In particular, $R_{-i} = R_{N \setminus \{i\}}$.

We next introduce a voluntary participation and two efficiency requirements for allocations. Let $R \in \mathcal{R}^N$ and $a \in \mathcal{A}_N$. Then,

(i) a is *$\emptyset$-individually rational* if for all $i \in N$, $a_i\, R_i\, \emptyset$;

(ii) a is *weakly non-wasteful* if for all $i \in N$ and all $o \in O$, $[o\, P_i\, \emptyset$ and $a_i = \emptyset]$ implies that for some $j \in N$, $a_j = o$;

(iii) a is *Pareto dominated* by $b \in \mathcal{A}_N$ if for all $i \in N$, $b_i\, R_i\, a_i$ and for some $j \in N$, $b_j\, P_j\, a_j$;

(iv) a is *Pareto optimal* if it is not Pareto dominated by another allocation.

A *(allocation) problem* is now fully described by a triplet (N, O, R), which we simply denote by R (only in Section 6 we will allow the set of agents to vary). A *rule* φ is a function that associates with each problem $R \in \mathcal{R}^N$ an allocation $\varphi(R)$; for each $i \in N$, $\varphi_i(R)$ denotes agent i's allotment under rule φ.

Let $\mathbb{P}$ denote a property of an allocation. Then, a rule satisfies property $\mathbb{P}$ if it only assigns allocations with property $\mathbb{P}$. Typically, (1) fairness / equitability properties and (2) efficiency properties are modeled this way, i.e., "locally" via properties of allocations. However, (3) incentive robustness properties are typically modeled "globally" by considering and linking changes of problems due to changes in preference profiles:

Strategy-proofness requires that no agent can ever benefit from misrepresenting his preferences, i.e., rule φ is *strategy-proof* if for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R)\, R_i\, \varphi_i(R'_i, R_{-i})$.

Group strategy-proofness requires that no coalition can ever benefit from jointly misrepresenting their preferences, i.e., rule φ is *group strategy-proof* if for each $R \in \mathcal{R}^N$, there does not exist a coalition $S \subseteq N$ and preferences $R'_S \in \mathcal{R}^S$ such that for all $i \in S$, $\varphi_i(R'_S, R_{-S})\, R_i\, \varphi_i(R)$ and for at least one $j \in S$, $\varphi_j(R'_S, R_{-S})\, P_j\, \varphi_j(R)$.

Non-bossiness requires that no agent can ever change the allocation without changing his allotment by misrepresenting his preferences, i.e., rule φ is *non-bossy* if for each $R \in \mathcal{R}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $\varphi_i(R) = \varphi_i(R'_i, R_{-i})$ implies $\varphi(R) = \varphi(R'_i, R_{-i})$.

In the sequel, sometimes rules are defined on the subdomain of acceptable preferences, $\underline{\mathcal{R}}^N$, and *(group) strategy-proofness* and *non-bossiness* are then adjusted accordingly by requiring that agents can only deviate by reporting preferences belonging to $\underline{\mathcal{R}}$. The following should be proven in Exercise 1.

Proposition 1 (Pápai, 2000)**.** *A rule $\varphi : \mathcal{R}^N \to \mathcal{A}_N$ ($\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$) satisfies strategy-proofness and non-bossiness if and only if it satisfies group strategy-proofness.*

3 Top trading cycles rules

We first consider *(Shapley-Scarf exchange) problems* (Shapley and Scarf, 1974), i.e., we consider exchange markets formed by n agents and by the same number of indivisible objects: $N = \{1, \ldots, n\}$ and $O = \{o_1, \ldots, o_n\}$. Each agent $i \in N$ owns one object, say o_i, desires exactly one object, has the option to trade the initially owned object in order to get a better one, and ranks all objects acceptable. Hence, the set of problems equals $\underline{\mathcal{R}}^N$ and a rule is given by $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$.

A rule φ is *individually rational* if for each $R \in \underline{\mathcal{R}}^N$ and each $i \in N$, $\varphi_i(R)\, R_i\, o_i$. This property reflects the fact that agent i owns object o_i and models voluntary participation. Furthermore, it implies that at any chosen allocation all real objects are assigned and no agent ever receives the null object.

A central solution concept is the *strong core solution* (the *weak core solution* is the topic of Exercise 2). For $R \in \underline{\mathcal{R}}^N$, *coalition S weakly blocks allocation a* if there exists an allocation $b \in \mathcal{A}_N$ such that (a) $\cup_{i\in S}\{b_i\} = \cup_{i\in S}\{o_i\}$ and (b) for all $i \in S$, $b_i\ R_i\ a_i$ and for some $j \in S$, $b_j\ P_j\ a_j$. Allocation a is a *strong core allocation* if it is not weakly blocked by any coalition. Shapley and Scarf (1974) showed that a strong core allocation always exists. Furthermore, Roth and Postlewaite (1977) proved that, when preferences are strict, the set of strong core allocations equals a singleton. Using the so-called top-trading cycles algorithm (due to David Gale, see Shapley and Scarf, 1974) one can easily calculate the unique strong core allocation.

Top trading cycles algorithm:

Input. A problem $R \in \underline{\mathcal{R}}^N$ (which implicitly includes the initial endowment allocation).

Step 1. Let $N_1 := N$ and $O_1 := O$. We construct a (directed) graph with the set of nodes $N_1 \cup O_1$. For each agent $i \in N_1$ we add a directed edge to his most preferred object in O_1. For each directed edge (i, o), we say that agent i *points to* object o. For each object $o \in O_1$ we add a directed edge to its owner.

A *trading cycle* is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists for this graph. We assign to each agent in a trading cycle the object he points to and remove all trading cycle agents and objects. We define N_2 to be the set of remaining agents and O_2 to be the set of remaining objects and, if $N_2 \neq \emptyset$ (equivalently, $O_2 \neq \emptyset$), we continue with Step 2. Otherwise we stop. In general at Step t we have the following:

Step t. We construct a (directed) graph with the set of nodes $N_t \cup O_t$ where $N_t \subseteq N$ is the set of agents that remain after Step $t-1$ and $O_t \subseteq O$ is the set of objects that remain after Step $t-1$.

For each agent $i \in N_t$ we add a directed edge to his most preferred object in O_t. For each Object $o \in O_t$ we add a directed edge to its owner.

At least one trading cycle exists for this graph and we assign to each agent in a trading cycle the object he points to and remove all trading cycle agents and objects. We define N_{t+1} to be the set of remaining agents and O_{t+1} to be the set of remaining objects and, if $N_{t+1} \neq \emptyset$ (equivalently, $O_{t+1} \neq \emptyset$), we continue with Step $t+1$. Otherwise we stop.

Output. The top trading cycles algorithm terminates when each agent in N is assigned an object in O (it takes at most $|N|$ steps). We denote the object in O that agent i obtained by $\mathrm{TTC}_i(R)$.

The *top trading cycles rule* chooses for each problem R the allocation $\mathrm{TTC}(R)$.

Theorem 2 (Ma, 1994)**.** *The top trading cycles rule is the only rule* $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ *satisfying individual rationality, Pareto optimality, and strategy-proofness.*

The proof that a rule φ that satisfies *individual rationality*, *Pareto optimality*, and *strategy-proofness* equals the *top trading cycles rule* essentially uses the following insights and steps.

At Step 1 of the top trading cycles algorithm, each agent trading receives his favorite object. One now shows that rule φ does the same as follows.

Any agent whose favorite object is his endowment, by *individual rationality*, trades with himself.

For any real trading cycle (with more than one agent), if trading agents would all rank their favorite object first and their endowment second, by *individual rationality* they would either receive their endowment or their favorite object. However, if one trading cycle agent receives his endowment, he would interrupt all possible trade and hence all agents in the cycle would also receive their endowments. But that would be Pareto dominated by trading cycle agents receiving their favorite objects, a contradiction to *Pareto optimality*. Hence, each trading cycle agent receives his favorite object.

Of course, we just assumed preferences for trading cycle agents to have a specific form, which needs generally not be the case. This is when *strategy-proofness* enters the picture. One can now show that from the specific preference profile we assumed, one can go back to the original preference profile by changing trading cycle agents' preferences one by one. Each time, due to *strategy-proofness*, the agent who changes his preferences still gets his favorite object (as do the others). Thus, at the end, the trading cycle agents in Step 1 trade the same way under TTC and φ.

This proof step can now be repeated for each step of the top trading cycles algorithm and the proof that $\varphi = \mathrm{TTC}$ is complete.

Proof. Exercise 3 is dedicated to showing that the *top trading cycles rule* satisfies all the properties in Theorem 2.

Let rule $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ satisfy *individual rationality*, *Pareto optimality*, and *strategy-proofness*. Let $R \in \underline{\mathcal{R}}^N$. We show that $\mathrm{TTC}(R) = \varphi(R)$. Note that it suffices to consider the first step of the top trading cycles algorithm: once we have shown that for each first step trading cycle that forms for R, each agent in the cycle obtains the same allotment under TTC and φ, we can use the same arguments to show that for each second step trading cycle that forms for R, each agent in the cycle obtains the same allotment under TTC and φ, etc.

Thus, consider a trading cycle that forms in the first step of the top trading cycles algorithm for R. Note that an agent who points at his own object will receive it due to *individual rationality*. Hence, consider a trading cycle consisting of agents $i_0, \ldots, i_K$, $K \geq 1$, and objects $o_{i_0}, \ldots, o_{i_K}$. Note that each agent $i_k \in \{i_0, \ldots, i_K\}$, according to his preferences R_{i_k}, prefers object $o_{i_{k+1}}$ most among objects in O.

For every $i_k \in \{i_0, \ldots, i_K\}$ we define preferences $R'_{i_k} \in \underline{\mathcal{R}}^N$ such that $o_{i_{k+1}}\ o_{i_k}\ \cdots$ (modulo K), e.g., by moving o_{i_k} just after $o_{i_{k+1}}$ (without changing the ordering of other objects). We omit the mention of "modulo K" in the sequel.

Consider the preference profile $R^0 = (R'_{\{i_0,\ldots,i_K\}}, R_{-\{i_0,\ldots,i_K\}})$. For each $0 \leq k \leq K$, $\mathrm{TTC}_{i_k}(R^0) = o_{i_{k+1}}$. Furthermore, by *individual rationality*, for each $0 \leq k \leq K$, $\varphi_{i_k}(R^0) \in \{o_{i_k}, o_{i_{k+1}}\}$. So, the set of objects allocated to agents in $\{i_0, \ldots, i_K\}$ at allocation $\varphi(R^0)$ equals $\{o_{i_0}, \ldots, o_{i_K}\}$. Hence, by *Pareto optimality*, for each $i_k \in \{i_0, \ldots, i_K\}$, $\varphi_{i_k}(R^0) = o_{i_{k+1}} = \mathrm{TTC}_{i_k}(R^0)$.

Next, let $l_1 \in \{i_0, \ldots, i_K\}$ be such that $\mathrm{TTC}_{l_1}(R^0) = o_{l_1+1}$. Consider the preference profile $R^1 = (R'_{\{i_0,\ldots,i_K\}\setminus\{l_1\}}, R_{-(\{i_0,\ldots,i_K\}\setminus\{l_1\})}) = (R_{l_1}, R^0_{-l_1})$. Assume that starting from R^0, agent l_1 changes his preferences from R'_{l_1} to R_{l_1}. Then, since agent l_1's trading cycle did not change, $\mathrm{TTC}_{l_1}(R^0) = \mathrm{TTC}_{l_1}(R^1) = o_{l_1+1}$. Considering the same preference change under rule φ, by *strategy-proofness* we have $\varphi_{l_1}(R^1)\ R_{l_1}\ \varphi_{l_1}(R^0)$. Recall that $\varphi_{l_1}(R^0) = o_{l_1+1}$ is agent l_1's favorite object. Hence, $\varphi_{l_1}(R^1) = o_{l_1+1}$. Next, by *individual rationality*, for each $i_k \in \{i_0, \ldots, i_K\} \setminus \{l_1\}$, $\varphi_{i_k}(R^1) \in \{o_{i_k}, o_{i_{k+1}}\}$. So, the set of objects allocated to agents in $\{i_0, \ldots, i_K\}$ at allocation $\varphi(R^1)$ equals $\{o_{i_0}, \ldots, o_{i_K}\}$. Then, by *Pareto optimality*, for each $i_k \in \{i_0, \ldots, i_K\}$, $\varphi_{i_k}(R^1) = o_{i_{k+1}} = \mathrm{TTC}_{i_k}(R^1)$.

Now, let $l_2 \in \{i_0, \ldots, i_K\} \setminus \{l_1\}$ be such that $\mathrm{TTC}_{l_2}(R^1) = o_{l_2+1}$. Consider the preference profile $R^2 = (R'_{\{i_0,\ldots,i_K\}\setminus\{l_1,l_2\}}, R_{-(\{i_0,\ldots,i_K\}\setminus\{l_1,l_2\})}) = (R_{l_2}, R^1_{-l_2})$. Assume that starting from R^1, agent l_2 changes his preferences from R'_{l_2} to R_{l_2}. Then, since agent l_2's trading cycle did not change, $\mathrm{TTC}_{l_2}(R^1) = \mathrm{TTC}_{l_2}(R^2) = o_{l_2+1}$. Considering the same preference change under rule φ, by *strategy-proofness* we have $\varphi_{l_2}(R^2)\ R_{l_2}\ \varphi_{l_2}(R^1)$. Recall that $\varphi_{l_2}(R^1) = o_{l_2+1}$ is agent l_2's favorite object. Hence, $\varphi_{l_2}(R^2) = o_{l_2+1}$. Since the choice of agents $\{l_1, l_2\} \subseteq \{i_0, \ldots, i_K\}$ was arbitrary, by the same argument, changing the roles of l_1 and l_2, we obtain that $\varphi_{l_1}(R^2) = o_{l_1+1}$. Next, by *individual rationality*, for each $i_k \in \{i_0, \ldots, i_K\} \setminus \{l_1, l_2\}$, $\varphi_{i_k}(R^1) \in \{o_{i_k}, o_{i_{k+1}}\}$. So, the set of objects allocated to agents in $\{i_0, \ldots, i_K\}$ at allocation $\varphi(R^2)$ equals $\{o_{i_0}, \ldots, o_{i_K}\}$. Then, by *Pareto optimality*, for each $i_k \in \{i_0, \ldots, i_K\}$, $\varphi_{i_k}(R^2) = o_{i_{k+1}} = \mathrm{TTC}_{i_k}(R^2)$.

We continue to replace the preferences of agents in $\{i_0, \dots, i_K\} \setminus \{l_1, l_2\}$ one at a time as above until we reach the preference profile R with the conclusion that for each $i_k \in \{i_0, \dots, i_K\}$, $\varphi_{i_k}(R) = \mathrm{TTC}_{i_k}(R)$. $\square$

4 Serial dictatorship rules

We second consider *(object allocation) problems*, i.e., we consider problems formed by n agents and by the same number of indivisible objects: $N = \{1, \dots, n\}$ and $O = \{o_1, \dots, o_n\}$. In contrast to the previous section, all objects are commonly owned. Each agent desires exactly one object and considers all objects to be acceptable. Again, the set of problems equals $\underline{\mathcal{R}}^N$ and a rule is given by $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$.

We will consider rules satisfying *weak non-wastefulness*, which on $\underline{\mathcal{R}}^N$ implies that at any chosen allocation, all real objects are assigned, and no agent ever receives the null object.

In addition, we will also require that rules are immune to renaming objects: let $\sigma : O \to O$ be a bijection. Then, given $R_i \in \underline{\mathcal{R}}$, we define $\sigma(R_i)$ by [for each pair $o, o' \in O$, $o \, R_i \, o'$ if and only if $\sigma(o) \, \sigma(R_i) \, \sigma(o')$]. Furthermore, given $R \in \underline{\mathcal{R}}^N$, let $\sigma(R) = (\sigma(R_i))_{i \in N}$, and given allocation $a \in \mathcal{A}_N$, let $\sigma(a) \in \mathcal{A}_N$ be defined by [for each $i \in N$, if $a_i \in O$, then $(\sigma(a))_i = \sigma(a_i)$, and if $a_i = \emptyset$, then $(\sigma(a))_i = \emptyset$]. Now, a rule is *neutral* if for each $R \in \underline{\mathcal{R}}^N$ and each bijection $\sigma : O \to O$, $\varphi(\sigma(R)) = \sigma(\varphi(R))$.

The following algorithm uses a permutation $\pi : N \to N$ of the agents that is used as an order of (serial) dictators.

Serial dictatorship algorithm:

Input. A problem $R \in \underline{R}^N$ and an order of the agents π.

Step 1. Let $\mathrm{SD}^{\pi}_{\pi(1)}(R)$ be agent $\pi(1)$'s most $R_{\pi(1)}$-preferred object in O.

Step $t \geq 2$. Let $\mathrm{SD}^{\pi}_{\pi(t)}(R)$ be agent $\pi(t)$'s most $R_{\pi(t)}$-preferred object in $O \setminus \{\mathrm{SD}^{\pi}_{\pi(1)}(R), \dots, \mathrm{SD}^{\pi}_{\pi(t-1)}(R)\}$.

Output. After n steps, allocation $\mathrm{SD}^{\pi}(R)$ is determined.

A rule $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ is a *serial dictatorship rule* if there is an order of agents $\pi : N \to N$ such that for each $R \in \underline{\mathcal{R}}^N$, $\varphi(R) = \mathrm{SD}^{\pi}(R)$.

Theorem 3 (Svensson, 1999)**.** *Serial dictatorship rules are the only rules $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ satisfying weak non-wastefulness,*[1] *strategy-proofness, non-bossiness, and neutrality.*

The proof that a rule φ that satisfies *weak non-wastefulness*, *strategy-proofness*, *non-bossiness*, and *neutrality* is a *serial dictatorship rule* roughly proceeds as follows.

First, a "maximal conflict" preference profile where all agents rank all objects in the same order is considered. Since by *weak non-wastefulness* all objects are

[1] Svensson (1999) circumvents *weak non-wastefulness* by only considering allocations that do not assign the null object.

assigned, the allocation for these maximal conflict preferences induces an order π on the set of agents. The aim of the proof now is to show that this order induces a serial dictatorship with the agent who receives the most preferred object as first "dictator", the agent receiving the second most preferred object as second dictator, etc.

Thus, according to π, there is a candidate for first dictator, without loss of generality, agent 1. Now it is assumed by contradiction that agent 1 does not always receive his most preferred object. Then, by combining *neutrality*, *strategy-proofness*, and *non-bossiness*, a contradiction is reached. The proof then proceeds by induction to conclude that the rule is a serial dictatorship based on π.

Proof. Exercise 4 is dedicated to showing that *serial dictatorship rules* satisfy all the properties in Theorem 3.

Let rule $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ satisfy *weak non-wastefulness*, *strategy-proofness*, *non-bossiness*, and *neutrality*.

For each $i \in N$, let $\hat{R}_i \in \underline{\mathcal{R}}$ be such that $o_1 \; o_2 \; \cdots \; o_n \; \emptyset$. Let $\hat{R} = (\hat{R}_i)_{i \in N}$. By *weak non-wastefulness*, all objects are assigned at allocation $\varphi(\hat{R})$. Without loss of generality, for each $i \in N$, let $\varphi_i(\hat{R}) = o_i$. We show by induction that φ is the serial dictatorship rule with respect to the order $\pi : 1 \; 2 \; \cdots \; n$, i.e., for each $i \in N$, $\pi(i) = i$. First, we show that agent 1 always receives his most preferred object.

Induction basis: For all $R \in \underline{\mathcal{R}}^N$, $\varphi_1(R) = \mathrm{SD}_1^\pi(R)$, which we prove by contradiction. Suppose that there exists $R \in \underline{\mathcal{R}}^N$ such that $\varphi_1(R)$ is not the most R_1-preferred object. By *neutrality* (Exercise 4), we may assume that o_1 is the most R_1-preferred object. Because all objects are assigned, there exists an agent $j \in N \setminus \{1\}$ such that $\varphi_j(R) = o_1$. Now, again by *neutrality* (Exercise 4), we may assume that $\varphi_1(R) = o_j$. By another application of *neutrality* (Exercise 4), we may also assume that for each $i \in N \setminus \{1, j\}$, $\varphi_i(R) = o_i$. Note that by the above, we have $[\varphi_1(\hat{R}) = o_1$ and $\varphi_j(\hat{R}) = o_j]$, $[\varphi_1(R) = o_j$ and $\varphi_j(R) = o_1]$, and for each $i \in N \setminus \{1, j\}$, $\varphi_i(R) = o_i = \varphi_i(\hat{R})$.

Now let $\bar{R} \in \underline{\mathcal{R}}^N$ be such that $\bar{R}_1 : o_1 \; o_j \; \cdots$, $\bar{R}_j : o_1 \; o_j \; \cdots$, and for each $i \in N \setminus \{1, j\}$, $\bar{R}_i : o_i \; \cdots$. Then, starting from $\hat{R}$ and applying *strategy-proofness* and *non-bossiness* successively, we obtain $\varphi_1(\bar{R}) = o_1$ (from $\varphi_1(\hat{R}) = o_1$) whereas starting from R and applying *strategy-proofness* and *non-bossiness* successively, we obtain $\varphi_1(\bar{R}) \neq o_1$ (from $\varphi_1(R) \neq o_1$), which is a contradiction.

Induction hypothesis: For all $R \in \underline{\mathcal{R}}^N$ and each $i \in \{1, \ldots, k\}$, $\varphi_i(R) = \mathrm{SD}_i^\pi(R)$.

Induction step: We show that for any $R \in \underline{\mathcal{R}}^N$, agent $k+1$ receives his most preferred object in $O \setminus \{\mathrm{SD}_1^\pi(R), \ldots, \mathrm{SD}_k^\pi(R)\}$. Suppose not. Then there exists $R \in \underline{\mathcal{R}}^N$ such that $\varphi_{k+1}(R) \neq \mathrm{SD}_{k+1}^\pi(R)$. By the induction hypothesis, for each $i \in \{1, \ldots, k\}$, $\varphi_i(R) = \mathrm{SD}_i^\pi(R)$. We distinguish four cases.

In the first case, suppose that $R_{\{1,\ldots,k\}} = \hat{R}_{\{1,\ldots,k\}}$. Then, by the induction hypothesis, for each $i \in \{1, \ldots, k\}$, $\varphi_i(R) = o_i = \mathrm{SD}_i^\pi(R)$. Using the same arguments as in the induction basis (by fixing $\hat{R}_{\{1,\ldots,k\}}$) and the fact that

$\varphi_{k+1}(\hat{R}) = o_{k+1}$ it follows that agent $k+1$ is assigned the most R_{k+1}-preferred object among $O \setminus \{o_1, \dots, o_k\}$ at $\varphi(R)$, which is a contradiction.

In the second case, suppose that for each $i \in \{1, \dots, k\}$, $\varphi_i(R) = o_i = \mathrm{SD}^\pi_i(R)$. Then o_1 is the most R_1-preferred object, and for each $i \in \{2, \dots, k\}$, o_i is the most R_i-preferred object in $O \setminus \{o_1, \dots, o_{i-1}\}$. Let $R' = (\hat{R}_{\{1,\dots,k\}}, R_{N\setminus\{1,\dots,k\}})$. Applying *non-bossiness* and *strategy-proofness* successively to φ and SD^π, it follows that for each $i \in \{1, \dots, k\}$, $\varphi_i(R') = o_i = \mathrm{SD}^\pi_i(R')$ and $\varphi_{k+1}(R') = \varphi_{k+1}(R) \neq \mathrm{SD}^\pi_{k+1}(R) = \mathrm{SD}^\pi_{k+1}(R')$. Then R' belongs to the first case, which leads to a contradiction.

In the third case, suppose that the sets of allotted objects via φ and SD^π are the same, i.e., $\{\varphi_1(R), \dots, \varphi_k(R)\} = \{o_1, \dots, o_k\} = \{\mathrm{SD}^\pi_1(R), \dots, \mathrm{SD}^\pi_k(R)\}$. Now let $\sigma : O \to O$ be such that for each $i \in \{1, \dots, k\}$, $\sigma(\varphi_i(R)) = o_i$, and for each $o \in O\setminus\{o_1, \dots, o_k\}$, $\sigma(o) = o$. Then by *neutrality* of φ and SD^π, we have for each $i \in \{1, \dots, k\}$, $\varphi_i(\sigma(R)) = o_i = \mathrm{SD}^\pi_i(\sigma(R))$, and $\varphi_{k+1}(\sigma(R)) \neq \mathrm{SD}^\pi_{k+1}(\sigma(R))$. Then, $\sigma(R)$ belongs to the second case, which leads to a contradiction.

In the fourth case, suppose that the sets of allotted objects via φ and SD^π are not the same, i.e., $\{\varphi_1(R), \dots, \varphi_k(R)\} \neq \{o_1, \dots, o_k\}$. Then, similarly as in the previous case, let $\sigma : O \to O$ be such that for each $i \in \{1, \dots, k\}$, $\sigma(\varphi_i(R)) = o_i$, and for each $o \in O\setminus(\{o_1, \dots, o_k\} \cup \{\varphi_1(R), \dots, \varphi_k(R)\})$, $\sigma(o) = o$. Then by *neutrality* of φ and SD^π, we have for each $i \in \{1, \dots, k\}$, $\varphi_i(\sigma(R)) = o_i = \mathrm{SD}^\pi_i(\sigma(R))$, and $\varphi_{k+1}(\sigma(R)) \neq \mathrm{SD}^\pi_{k+1}(\sigma(R))$. Then, $\sigma(R)$ belongs to the second case again, which leads to a contradiction. □

5 Endowment inheritance rules

We now consider *(object allocation) problems* formed by n agents and an arbitrary number of indivisible objects: $N = \{1, \dots, n\}$ and $O = \{o_1, \dots, o_m\}$. We combine aspects of the two previous sections in that property rights over objects (as present in Shapley-Scarf exchange problems and used by the top trading cycles rule) will be modeled via permutations of the agents (as used by serial dictatorship rules).

Endowment inheritance rules allocate objects to agents using an iterative procedure that is similar to the top trading cycles algorithm, except that, apart from agents possibly owning multiple objects, it also specifies an order of inheritance of the objects in an iterative hierarchical manner. Each object is the individual "endowment" of an agent and we apply a round of top trading cycles exchange to these endowments. Given that multiple endowments are allowed, after the agents in top trading cycles are removed from the problem with only their allotted objects, their unallocated endowments are re-assigned as endowments to agents who are still present. In other words, these objects that are left behind are "inherited" as new endowments by agents who have not received their allotments yet. Notice that then each remaining object is the endowment of some remaining agent and the top trading cycles algorithm is well-defined at the second stage. We determine the allotments of agents who are in top trading cycles in this round, remove them with their allotted objects, and determine the endowments of the remaining agents for the next stage. And so on, until for each agent we have specified an allotment this way.

The initial endowments and the hierarchical endowments at later rounds are determined using object specific permutations of the agents that indicate the order of inheritance. Thus, each endowment inheritance rule is defined by an *endowment inheritance table* $\pi = (\pi_o)_{o \in O}$ such that for each object $o \in O$, π_o is a permutation of N.

Endowment Inheritance Algorithm:

Input. A problem $R \in \underline{\mathcal{R}}^N$ and an endowment inheritance table π.

For each Step $t \in \{1, \ldots, m\}$, we give recursive definitions of the associated *hierarchical endowments* $E_t(i, R)$, *top choices* $T_t(i, R)$, *trading cycles* $S_t(i, R)$, *assigned individuals* $W_t(R)$, and *assigned real objects* $F_t(R)$. For every problem $R \in \underline{\mathcal{R}}^N$ and Step t, let $W^t(R) \equiv \cup_{z=1}^t W_z(R)$ and $F^t(R) \equiv \cup_{z=1}^t F_z(R)$. Let $W^0(R) = \emptyset$ and $F^0(R) = \emptyset$.

Step t. If agent $i \in N \setminus W^{t-1}(R)$ is ranked highest with respect to object $o \in O \setminus F^{t-1}(R)$ among all agents in $N \setminus W^{t-1}(R)$, then o belongs to his hierarchical endowment at Step t. The null object $\emptyset$ is part of each agent's endowment.

t-th hierarchical endowments:

$$E_t(i, R) = \left\{ o \in O \setminus F^{t-1}(R) \mid i = \arg \min_{j \in N \setminus W^{t-1}(R)} \{\pi_o(j)\} \right\} \cup \{\emptyset\}.$$

Next, each agent $i \in N \setminus W^{t-1}(R)$ identifies his top choice in $(O \cup \{\emptyset\}) \setminus F^{t-1}(R)$.

Top choices:

$$T_t(i, R) = o \Leftrightarrow o \in (O \cup \{\emptyset\}) \setminus F^{t-1}(R) \text{ and for all } o' \in (O \cup \{\emptyset\}) \setminus F^{t-1}(R),\ oR_i o'.$$

A trading cycle consists of a set of agents in $N \setminus W^{t-1}(R)$ who would like to exchange objects from their hierarchical endowments in a "cyclical way" such that each of them receives his top choice.

Trading cycles:

$$S_t(i, R) \equiv \begin{cases} \{j_1, \ldots, j_g\} & \text{if } \{j_1, \ldots, j_g\} \subseteq N \setminus W^{t-1}(R) \text{ such that} \\ & |\{j_1, \ldots, j_g\}| = g \text{ and for all } v \in \{1, \ldots, g\}, \\ & T_t(j_v, R) \in E_t(j_{v+1}, R) \text{ where } i = j_1 = j_{g+1}, \\ & \\ \emptyset & \text{otherwise.} \end{cases}$$

Agents in a trading cycle are assigned their top choices from the set of objects that were not assigned yet.

Assigned individuals: $W_t(R) \equiv \{i \in N \mid S_t(i, R) \neq \emptyset\}$.

Assigned real objects: $F_t(R) \equiv \{T_t(i, R) \in O \mid i \in W_t(R)\}$.

Note that for all $R \in \underline{\mathcal{R}}^N$ there exists a last stage $t^* \leq m$ such that either $W^{t^*}(R) = N$ or $F^{t^*}(R) = O$ and for all $t < t^*$, $W^t(R) \neq N$ and $F^t(R) \neq O$.

Given endowment inheritance table π, for all $R \in \underline{\mathcal{R}}^N$ the allocation chosen by the *endowment inheritance rule* φ^π is defined as follows. For all $i \in N$,

$$\varphi_i^\pi(R) \equiv \begin{cases} T_t(i,R) & \text{if for some } t \in \{1,\ldots,m\},\ i \in W_t(R), \\ \emptyset & \text{otherwise.} \end{cases}$$

A rule φ is an *endowment inheritance rule* if there exists an endowment inheritance table π such that for each $R \in \underline{\mathcal{R}}^N$, $\varphi(R) = \varphi^\pi(R)$.

Example 4. Let $N = \{1,2,3,4,5\}$ and $O = \{a,b,c,d,e,f\}$. Consider the *endowment inheritance rule* defined by the following endowment inheritance table π. Associated with each object is a permutation of the agents (given by the column corresponding to the object).

π_a	π_b	π_c	π_d	π_e	π_f
2	1	2	1	2	2
1	2	1	2	1	1
3	3	3	3	3	3
4	4	4	5	5	4
5	5	5	4	4	5

R_1	R_2	R_3	R_4	R_5
a	b	b	d	a
f	f	e	a	b
d	a	c	b	f
e	c	a	c	c
c	d	d	e	d
b	e	f	f	e

For example, the first column shows that object a is agent 2's initial endowment, which is (possibly) inherited by 1, 3, 4, and 5, in this order. We illustrate the use of this table for the preference profile $R \in \underline{\mathcal{R}}^N$ given above, which shows the rankings of objects from the top down for each agent.

Step 1. The initial endowments are given by the first row of the endowment inheritance table. The endowments are $E_1(1,R) = \{b,d\}$ for agent 1 and $E_1(2,R) = \{a,c,e,f\}$ for agent 2, and $\emptyset$ for agents 3, 4, and 5. Then, $T_1(1,R) = a$, $T_1(2,R) = b$, $T_1(3,R) = b$, $T_1(4,R) = d$, and $T_1(5,R) = a$ are the top choices of the agents in $O \cup \{\emptyset\}$. Hence, $\{1,2\}$ is the only cycle under which 1 receives a from 2 and 2 receives b from 1, i.e., $S_1(1,R) = S_1(2,R) = \{1,2\}$, $W_1(R) = \{1,2\}$, and $F_1(R) = \{a,b\}$.

Step 2. Since agents 1 and 2 already received their allotments, objects c, d, e, and f are left behind from 1's and 2's endowments. These objects are inherited by agent 3, i.e., $E_2(3,R) = \{c,d,e,f\}$ and $E_2(4,R) = E_2(5,R) = \emptyset$. Then, 3 picks his top choice, object e, among the remaining objects. So, $S_2(3,R) = \{3\}$, $W_2(R) = \{3\}$, and $F_2(R) = \{e\}$.

Step 3: Now only agents 4 and 5 remain. Agent 4 inherits $\{c,f\}$ and 5 inherits $\{d\}$, i.e., $E_3(4,R) = \{c,f\}$ and $E_3(5,R) = \{d\}$. Because $T_3(4,R) = d$ and $T_3(5,R) = f$, 4 and 5 form a trading cycle and receive their top choices in $\{c,d,f\}$, i.e., $S_3(4,R) = S_3(5,R) = \{4,5\}$, $W_3(R) = \{4,5\}$, and $F_3(R) = \{d,f\}$. Then, $\varphi^\pi(R) = (a,b,e,d,f)$ are the allotments to $(1,2,3,4,5)$. □

The class of *endowment inheritance rules* is a subclass of Pápai's (Pápai, 2000) *hierarchical exchange rules*, which generalizes the idea of endowment inheritance from endowment inheritance tables to more general endowment inheritance structures (for instance by allowing for endowment inheritance trees). The next property was introduced by Pápai (2000) to exclude joint preference manipulations by two individuals who plan to swap objects ex-post under the condition that the collusion changed both their allotments and is self-enforcing in the sense that neither agent changes his allotment in case he misreports while the other agent reports the truth.

A rule φ is *reallocation-proof* if for each problem $R \in \underline{\mathcal{R}}^N$ and each pair of agents $i, j \in N$, there exist no preferences $\widetilde{R}_i, \widetilde{R}_j \in \underline{\mathcal{R}}$ such that

$$\varphi_j(\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}})\; R_i\; \varphi_i(R),$$

$$\varphi_i(\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}})\; P_j\; \varphi_j(R),$$

and for $k = i, j$,

$$\varphi_k(R) = \varphi_k(\widetilde{R}_k, R_{-k}) \neq \varphi_k(\widetilde{R}_i, \widetilde{R}_j, R_{-\{i,j\}}).$$

We state the following result without a proof.

Theorem 5 (Pápai, 2000)**.** *Hierarchical exchange rules are the only rules* $\varphi : \underline{\mathcal{R}}^N \to \mathcal{A}_N$ *satisfying Pareto optimality, strategy-proofness, non-bossiness, and reallocation-proofness.*

6 Deferred acceptance rules

We now consider *variable population (object allocation) problems*, i.e., (object allocation) problems formed by coalitions $N' \subseteq N$ and a set of real objects O. In other words, not necessarily all agents belonging to N are present. Let $\mathcal{R}^{N'}$ denote the set of preference profiles for coalition N' and similarly, $\mathcal{A}_{N'}$ the set of allocations for N'. Now a (variable population) rule is a mapping

$$\varphi : \cup_{\emptyset \neq N' \subseteq N} \mathcal{R}^{N'} \to \cup_{\emptyset \neq N' \subseteq N} \mathcal{A}_{N'}$$

such that for each $\emptyset \neq N' \subseteq N$ and each $R \in \mathcal{R}^{N'}$ we have $\varphi(R) \in \mathcal{A}_{N'}$.

When the sets of agents vary, a natural property is *population-monotonicity*, which is a so-called *solidarity property* requiring that those who are not responsible for a change in a problem (the initially present agents) are all affected in the same way by that change (the presence of additional agents). Since in this type of resource allocation problem, more agents competing for the same resources is bad news, the natural requirement is that as the set of agents becomes larger, the initially present agents all get (weakly) worse off. Formally, for any $\emptyset \neq N'' \subseteq N' \subseteq N$ and any $R \in \mathcal{R}^{N'}$, for each $i \in N''$, $\varphi_i(R_{N''})\; R_i\; \varphi_i(R)$.

Given object $o \in O$, let $\succ_o$ denote a *priority ordering on* N, e.g., $\succ_o$: $1\ 2\ \cdots\ n$ means that agent 1 has higher priority for object o than agent 2, who has higher priority for object o than agent 3, etc. Let $\succ \equiv (\succ_o)_{o\in O}$ denote a *priority structure*. Then, given a priority structure $\succ$ and a problem $R \in \mathcal{R}^{N'}$, we can interpret $(R, \succ)$ as a *marriage market* (Gale and Shapley, 1962) where the set of agents N', for instance, corresponds to the set of women, the set of objects O corresponds to the set of men, preferences at R correspond to women's preferences over available men, and the priority structure $(\succ_o)_{o\in O}$ specifies the men's preferences over women.

Stability is an important requirement for many real-life matching markets and it turns out to be essential in our context of allocating indivisible objects to agents as well. Given problem $R \in \mathcal{R}^{N'}$ and priority structure $\succ$, an allocation $a \in \mathcal{A}_{N'}$ is *stable under* $\succ$ if there exists no agent-object pair $(i, o) \in N' \times (O \cup \{\emptyset\})$ such that $o \, P_i \, a_i$ and either (a) $o = \emptyset$, or (b) there exists no $j \in N'$ such that $a_j = o$, or (c) there exists $j \in N'$ such that $a_j = o$ and $i \succ_o j$.[2] Furthermore, rule φ is *stable* if there exists a priority structure $\succ$ such that for each problem $R \in \mathcal{R}^{N'}$ (with $N' \subseteq N$), $\varphi(R)$ is *stable* under $\succ$. Note that *stability* implies $\emptyset$-*individual rationality* and *weak non-wastefulness*, but it does not imply *Pareto optimality* (see Exercise 6).

For any marriage market $(R, \succ)$, we denote by $\mathrm{DA}^{\succ}(R)$ the agent-optimal stable allocation that is obtained by using the agent-proposing deferred acceptance algorithm (Gale and Shapley, 1962). A rule $\varphi : \mathcal{R}^N \to \mathcal{A}$ is a *deferred acceptance rule* if there is a priority structure $\succ$ such that for each $R \in \mathcal{R}^N$, $\varphi(R) = \mathrm{DA}^{\succ}(R)$.

Theorem 6 (Ehlers and Klaus, 2016)**.** *Deferred acceptance rules are the only (variable population) rules satisfying $\emptyset$-individual rationality, weak non-wastefulness, strategy-proofness, and population-monotonicity.*

The proof that a rule φ that satisfies $\emptyset$-*individual rationality*, *weak non-wastefulness*, *strategy-proofness*, and *population-monotonicity* is a deferred acceptance rule roughly proceeds as follows.

First, for each real object $o \in O$, a "maximal conflict" preference profile where all agents declare the object o as the only acceptable object is considered. By $\emptyset$-*individual rationality* and *weak non-wastefulness*, the object is assigned to an agent. When this agent leaves, the residual maximal conflict preference profile for the remaining agents is considered and the sequence of maximal conflict preferences induces an order $\succ_o$ on the set of agents. Since this can be done for each real object, a priority structure $\succ$ is constructed.

The aim of the remainder of the proof then is to show that this priority ordering, together with the deferred acceptance algorithm, determines all outcomes of the rule.

Proof. First, note that all *deferred acceptance rules* are *stable* and that *stability* implies $\emptyset$-*individual rationality* and *weak non-wastefulness*. Dubins and Freed-

[2] A situation with an agent-object pair (i, o) such that (a) constitutes a violation of $\emptyset$-*individual rationality* while (b) can be interpreted as the allocation being *wasteful*, and if (c) is the case, then agent i has *justified envy* against agent j. Hence, *stability* is equivalent to the properties $\emptyset$-*individual rationality*, *non-wastefulness* and *no justified envy.*

man (1981) and Roth (1982) proved *strategy-proofness* of all *deferred acceptance rules*. Crawford (1991) studied comparative statics of *deferred acceptance rules* and from his results it follows that all *deferred acceptance rules* are *population-monotonic*. Hence, *deferred acceptance rules* satisfy all properties of Theorem 6.

Second, let φ be a rule satisfying *$\emptyset$-individual rationality*, *weak non-wastefulness*, and *population-monotonicity* (we will add *strategy-proofness* later on). We construct a priority structure using maximal conflict preference profiles.

Let $o \in O$ and $R^o \in \mathcal{R}$ be such that $A(R^o) = \{o\}$. We denote the set of all preference relations that have $o \in O$ as the unique acceptable object by $\mathcal{R}^o$. For any coalition $S \subseteq N$, let $R^o_S = (R^o_i)_{i \in S}$ be such that for each $i \in S$, $R^o_i = R^o$.

Consider the problem R^o_N. By *$\emptyset$-individual rationality* and *weak non-wastefulness*, for some $j \in N$, $\varphi_j(R^o_N) = o$, say $j = 1$. Then, for all $i \in N \setminus \{1\}$, we set $1 \succ_o i$.

Next consider the problem $R^o_{N \setminus \{1\}}$. By *$\emptyset$-individual rationality* and *weak non-wastefulness*, for some $j \in N \setminus \{1\}$, $\varphi_j(R^o_{N \setminus \{1\}}) = o$, say $j = 2$. Then, for all $i \in N \setminus \{1, 2\}$, we set $2 \succ_o i$.

By induction, we obtain $\succ_o$ for any real object o and thus a priority structure $\succ = (\succ_o)_{o \in O}$ for N. It is easy to show that *$\emptyset$-individual rationality*, *weak non-wastefulness*, and *population-monotonicity* imply that for any $o \in O$ and any $i, j \in N$ such that $i \succ_o j$, $\varphi_i(R^o, R^o) = o$ and $\varphi_j(R^o, R^o) = \emptyset$ (Exercise 6).

Let φ satisfy *strategy-proofness*. With the following step, Theorem 6 follows. Note that $\mathrm{DA}^{\succ}$ satisfies all properties of Theorem 6.

Let $S \subseteq N$, $R \in \mathcal{R}^S$, and denote by $Z(R) := |\{i \in S \mid |A(R_i)| \leq 1\}|$ the number of agents who find at most one object acceptable. Let $R \in \mathcal{R}^S$ be such that $\varphi(R) \neq \mathrm{DA}^{\succ}(R)$ and assume that $Z(R)$ is maximal, i.e., for all $R' \in \mathcal{R}^S$ such that $\varphi(R') \neq \mathrm{DA}^{\succ}(R')$, $Z(R) \geq Z(R')$.

We first show that for each $i \in S$ such that $\varphi_i(R) \neq \mathrm{DA}^{\succ}_i(R)$, $|A(R_i)| = 1$. If $\varphi_i(R) \, P_i \, \mathrm{DA}^{\succ}_i(R)$, by *$\emptyset$-individual rationality*, $\varphi_i(R) = o \in O$. If $|A(R_i)| = 1$, then we are done. If $|A(R_i)| > 1$, then consider $(R^o, R_{-i}) \in \mathcal{R}^S$ where R^o is as in the construction of $\succ$. By *strategy-proofness* and *$\emptyset$-individual rationality* of both φ and $\mathrm{DA}^{\succ}$, $\varphi_i(R^o, R_{-i}) = o$ and $\mathrm{DA}^{\succ}_i(R^o, R_{-i}) = \emptyset$. Hence, (R^o, R_{-i}) is such that $\varphi(R^o, R_{-i}) \neq \mathrm{DA}^{\succ}(R^o, R_{-i})$ and $Z(R^o, R_{-i}) > Z(R)$; contradicting our assumption that $Z(R)$ was maximal. If $\mathrm{DA}^{\succ}_i(R) \, P_i \, \varphi_i(R)$, then $\mathrm{DA}^{\succ}_i(R) := o \in O$ and the proof that $|A(R_i)| = 1$ proceeds as before (with $\mathrm{DA}^{\succ}$ in the role of φ and vice versa). Hence, for each $i \in S$ such that $\varphi_i(R) \neq \mathrm{DA}^{\succ}_i(R)$, $|A(R_i)| = 1$. By *strategy-proofness*, we can assume that for each $i \in S$ such that $\varphi_i(R) \neq \mathrm{DA}^{\succ}_i(R)$, $R_i = R^o$ where R^o is as in the construction of $\succ$.

If $o = \mathrm{DA}^{\succ}_i(R) \, P_i \, \varphi_i(R) = \emptyset$, then by *weak non-wastefulness* there exists $j \in S \setminus \{i\}$ such that $\varphi_j(R) = o$. Hence, $\varphi_j(R) \neq \mathrm{DA}^{\succ}_j(R)$ and $R_j = R^o$ where R^o is as in the construction of $\succ$. Thus, $(R_i, R_j) = (R^o, R^o)$ and $o = \varphi_j(R) \, P_j \, \mathrm{DA}^{\succ}_j(R) = \emptyset$. Now $\mathrm{DA}^{\succ}_i(R) = o$ and *population-monotonicity* (and *$\emptyset$-individual rationality*) for $\mathrm{DA}^{\succ}$ imply $\mathrm{DA}^{\succ}_i(R_i, R_j) = o$ and $\mathrm{DA}^{\succ}_j(R_i, R_j) = \emptyset$. Hence, $i \succ_o j$. Next, $\varphi_j(R) = o$ and *population-monotonicity* (and *$\emptyset$-individual rationality*) for φ imply $\varphi_j(R_i, R_j) = o$ and $\varphi_i(R_i, R_j) = \emptyset$. Hence, $j \succ_o i$; a contradiction.

If $o = \varphi_i(R) \; P_i \; \mathrm{DA}_i^{\succ}(R) = \emptyset$, then by *weak non-wastefulness* there exists $j \in S \setminus \{i\}$ such that $\mathrm{DA}_j^{\succ}(R) = o$. We obtain a contradiction as above. □

Theorem 6 implies the following additional result. We call a priority structure $\succ$ *acyclic* if there do not exist distinct $i, j, k \in N$ and distinct $o, o' \in O$ such that $i \succ_o j \succ_o k$ and $k \succ_{o'} i$. Ergin (2002) showed that acyclicity of the priority structure $\succ$ is necessary and sufficient for the deferred acceptance rule based on $\succ$ to be *Pareto optimal*.

Corollary 7 (Ehlers et al., 2002)**.** *Deferred acceptance rules with acyclic priority structures are the only (variable population) rules satisfying Pareto optimality, strategy-proofness, and population-monotonicity.*

The above corollary follows immediately from Theorem 6 and Ergin (2002) as *Pareto optimality* implies both *∅-individual rationality* and *weak non-wastefulness*.

7 Relationship between classes of rules

We have introduced four (five) classes of rules in our chapter: the *top trading cycles rule*, *serial dictatorship rules*, *endowment inheritance (hierarchical exchange) rules*, and *deferred acceptance rules*. These classes of rules and their properties are very closely related.

First, *endowment inheritance rules* are a natural extension of the *top trading cycles rule* from the model of Section 3, where each agent is endowed with exactly one object, to the model considered in Sections 4 and 5, where the number of objects is arbitrary: instead of explicit endowments, an inheritance table administers property rights and inheritance of those property rights whenever necessary. Specifically, the *top trading cycles rule* is an *endowment inheritance rule* for the special situation where the number of objects equals the number of agents and where the endowment inheritance table lists each agent as top agent for exactly one object, which then essentially turns into his endowment.

Next, an *endowment inheritance rule* is specified by an endowment inheritance table π while a *deferred acceptance rule* is specified by a priority structure $\succ$. Note that while inheritance tables and priority structures come with different background stories (property rights versus priority rankings) both are mathematically identical concepts: for each object, a permutation of agents is specified. So the inputs for these rules can be chosen to be the same and the difference lies solely in how that input is used to compute the output. This gives rise to the following question:

> "For which endowment inheritance tables / priority structures, do *endowment inheritance rules* and *deferred acceptance rules* produce the same results?"

Clearly, *serial dictatorship rules* are a class of rules in that intersection, but some more rules qualify.

Theorem 8 (Kesten, 2006)**.** *Endowment inheritance rules / deferred acceptance rules always produce the same outcomes if and only if endowment inheritance tables / priority structures used are acyclic.*

Finally, Table 1 shows the trade-offs of some of the key properties we discussed for the different classes of rules.

Table 1: Rules and their properties.

Rules	**PO**	**IR**	**SP**	**NB**	**ST**	**PMON**	**NEU**
serial dictatorship	✓	✓	✓	✓	✓	✓	✓
acyclic deferred acceptance / acyclic endowment inheritance	✓	✓	✓	✓	✓	✓	
deferred acceptance		✓	✓	✓	✓	✓	
endowment inheritance / hierarchical exchange	✓	✓	✓	✓			

Notation:
PO stands for *Pareto optimality*,
IR stands for *individual rationality*, respectively $\emptyset$-*individual rationality*
SP stands for *strategy-proofness*,
NB stands for *non-bossiness*,
ST stands for *stability*,
PMON stands for *population-monotonicity*, and
NEU stands for *neutrality*.

8 Notes on the literature

Theorem 3 (Svensson, 1999, Theorem 1) showed that a rule satisfies *neutrality*, *strategy-proofness*, and *non-bossiness* if and only if it is a *serial dictatorship rule*.[3] The following two research contributions show what happens when *neutrality* is dropped.

Pápai (2000) introduced *hierarchical exchange rules*, a class of rules that extends the way the *top trading cycles rule* works (Gale and Shapley, 1962) by specifying ownership rights for the objects in an iterative hierarchical manner and by allowing for associated iterative top trading cycles. Theorem 5 (Pápai, 2000, Theorem) showed that a rule satisfies *Pareto optimality*, *group strategy-proofness*, and *reallocation-proofness* if and only if it is a *hierarchical exchange rule*. Pycia and Ünver (2017, Theorem 1) extended this result by providing a full characterization of the class of *Pareto optimal* and *group strategy-proof* rules, called *trading cycles rules*. The set of *trading cycles rules* extends the set of *hierarchical exchange rules* by allowing agents to not only own objects

[3] As mentioned in Section 4, in our model we need to add *weak non-wastefulness* to assure that all objects are allocated while Svensson incorporated this assumption into his model.

throughout the iterative trading cycles allocation procedure but to also have a different "control right" called "brokerage" (a broker cannot necessarily consume a brokered object directly himself but he can trade it for another object).

Sönmez and Ünver (2010) considered a model with publicly as well as individually owned objects and characterized an important extension of the class of *endowment inheritance rules*, the *YRMH-IGYT (you request my house - I get your turn) rules* that were introduced by Abdulkadiroğlu and Sönmez (1999): a rule satisfies *individual rationality*, *Pareto optimality*, *strategy-proofness*, *weak neutrality*,[4] and *consistency*[5] if and only if it is a *YRMH-IGYT rule*. *YRMH-IGYT rules* are essentially *serial dictatorship rules* that adapt to individual ownership rights that agents have for objects. Karakaya et al. (2019, Theorems 2 and 3) showed that a rule is an *endowment inheritance rule* based on ownership-adapted acyclic priorities if and only if it satisfies *individual rationality*, *Pareto optimality*, *strategy-proofness*, *consistency*, and either *reallocation-proofness* or *non-bossiness*.

Note that we have simplified our object allocation models by assuming that only one copy of each object is available. Usually, a more general model of object allocation with quotas is studied: for each object type more than one object can be assigned, up to a fixed quota; e.g., in school choice problems, school seats at a specific school can be assigned until the school's specific capacity quota is reached. For object allocation with quotas, both Ergin (2002) and Kesten (2006) studied allocation rules and their properties in relation to acyclic inheritance tables / priority structures. Ergin (2002) and Kesten (2006) used different notions of acyclicity that coincide for our simplified model (see Section 6), we refer the interested reader to the original papers. Ergin (2002, Theorem 1) showed that for the *deferred acceptance rule* $DA^{\succ}$, the following are equivalent: $DA^{\succ}$ is *Pareto optimal*, $DA^{\succ}$ is *group strategy-proof*, $DA^{\succ}$ is *consistent*, and $\succ$ is Ergin acyclic. Kesten (2006, Theorems 1 and 2) strengthened Ergin's acyclicity condition and showed that an *endowment inheritance rule* based on an inheritance table π equals a *deferred acceptance rule* based on priority structure $\succ$ if and only if $\succ = \pi$ is Kesten acyclic.

Finally, we would also like to briefly mention that the presented characterizations of *deferred acceptance rules* were established for object allocation with quotas. More generally, Kojima and Manea (2010), Ehlers and Klaus (2014), and Ehlers and Klaus (2016) characterize so-called *choice-based deferred acceptance rules* which use responsive / substitutable choice functions to define the associated deferred acceptance algorithm. For yet another widely used class of rules, the *immediate acceptance rules* (see Exercise 7), characterization results based on responsive / substitutable choice functions were presented in Kojima and Ünver (2014) and Doğan and Klaus (2018).

[4]A rule is *weakly neutral* if it is independent of the names of the publicly owned objects.

[5]A rule is *consistent* if the following holds: suppose that after objects are allocated according to the rule, some agents leave with their assigned objects and no object that was individually owned by a remaining agent was removed. Then, if the remaining agents were to allocate the remaining objects according to the rule, each of them would receive the same object.

Exercises

1 (*Strategy-proofness* and *non-bossiness* $\Leftrightarrow$ *group strategy-proofness*)**.** Prove Proposition 1.

2 (Weak preferences and the weak / strong core)**.** Consider a Shapley-Scarf exchange problem where agent $i \in N$ owns object o_i with weak preferences in $\underline{\mathcal{W}}$: any $R_i \in \underline{\mathcal{W}}$ is transitive and complete and all real objects are strictly preferred to the null object.

For $R \in \underline{\mathcal{W}}^N$, *coalition S strictly blocks allocation a* if there exists an allocation $b \in \mathcal{A}_N$ such that (a) $\cup_{i\in S}\{b_i\} = \cup_{i\in S}\{o_i\}$ and (b) for all $i \in S$, $b_i\, P_i\, a_i$. Allocation a is a *weak core allocation* if it is not strictly blocked by any coalition. Furthermore, an allocation a is *competitive under R* if there exist prices $p : O \to \mathbb{R}_+$ such that for all $i \in N$, (i) $p(a_i) \leq p(o_i)$ and (ii) $x\, P_i\, a_i$ implies $p(x) > p(o_i)$. Let $\mathrm{SC}(R)$ denote the *set of strong core allocations of R*, $\mathrm{WC}(R)$ denote the *set of weak core allocations of R*, and $\mathrm{comp}(R)$ denote the *set of competitive allocations under R*. Show the following.

(a) The strong core might be empty for certain $R \in \underline{\mathcal{W}}^N$. In addition, show that if $\mathrm{SC}(R) \neq \emptyset$, then for all $a, b \in \mathrm{SC}(R)$ we have that for all $i \in N$, $a_i\, I_i\, b_i$.

(b) Let $R \in \underline{\mathcal{W}}^N$. Then, $\hat{R} \in \underline{\mathcal{R}}^N$ is a *strict resolution of R* if it breaks ties in R to obtain strict preferences, i.e., for all $x, y \in O \cup \{\emptyset\}$ and all $i \in N$, $x\, P_i\, y$ implies $x\, \hat{P}_i\, y$. Let $\mathrm{ST}(R)$ denote the *set of all strict resolutions of R*. Then,

$$\mathrm{comp}(R) = \cup_{\hat{R}\in \mathrm{ST}(R)}\{\mathrm{TTC}(\hat{R})\}.$$

(c) Show the following inclusion relations and also establish that they might be strict for some problems:

$$\mathrm{SC}(R) \subseteq \mathrm{comp}(R) \subseteq \mathrm{WC}(R).$$

3 (*Top trading cycles rules*)**.** Show that *top trading cycles rules* satisfy *individual rationality*, *Pareto optimality*, and *strategy-proofness*. Also show that when $|N| \geq 3$, a *top trading cycles rule* may not be *stable*.

4 (*Serial dictatorship rules*)**.** Show that *serial dictatorship rules* satisfy *Pareto optimality* (which implies *weak non-wastefulness*), *strategy-proofness*, *non-bossiness*, *neutrality*, and *population-monotonicity*. Work out the use of *neutrality* in the proof of Theorem 3's induction basis; in particular, derive the required renaming of objects σ.

5 (Properties of *endowment inheritance rules*)**.** Which of the properties, $\emptyset$-*individual rationality*, *Pareto optimality*, *strategy-proofness*, *non-bossiness*, *neutrality*, *population-monotonicity*, and *stability* do *endowment inheritance rules* satisfy and which ones do they not satisfy? Explain (prove) your answers. Note that no characterization of endowment inheritance rules has been established until now.

6 (*Deferred acceptance rules*)**.** Prove the statement in the proof of Theorem 6 that for any $o \in O$ and any $i, j \in N$ such that $i \succ_o j$, $\varphi_i(R^o, R^o) = o$ and $\varphi_j(R^o, R^o) = \emptyset$.

Let $\succ$ be a priority structure. Show that deferred acceptance rule $\mathrm{DA}^{\succ}$ is not *Pareto optimal* if $\succ$ violates acyclicity.

7 (*Immediate acceptance rules*)**.** *Immediate acceptance algorithm*:

Input. A problem $R \in \mathcal{R}^N$ and a priority structure $\succ$.

Step 1. Each agent applies to his favorite object in $O \cup \{\emptyset\}$. Each real object $o \in O$ accepts the applicant who has the highest priority and rejects all the other applicants. The null object $\emptyset$ accepts all applicants.

Step $t \geq 2$. Each applicant who was rejected at step $t - 1$ applies to his next favorite object in $O \cup \{\emptyset\}$. Each real object $o \in O$ that has not been assigned yet accepts the applicant who has the highest priority and rejects all the other applicants. Each real object that has been assigned already rejects all applicants and the null object $\emptyset$ accepts all applicants.

Output. The algorithm terminates when each agent is accepted by a real object or the null object. We denote the object in O that accepted agent i by $\mathrm{IA}_i^{\succ}(R)$.

A rule $\varphi : \mathcal{R}^N \to \mathcal{A}$ is an *immediate acceptance rule* if there is a priority structure $\succ$ such that for each $R \in \mathcal{R}^N$, $\varphi(R) = \mathrm{IA}^{\succ}(R)$.

Which of the properties *$\emptyset$-individual rationality*, *Pareto optimality*, *strategy-proofness*, *non-bossiness*, *neutrality*, *population-monotonicity*, and *stability* do *immediate acceptance rules* satisfy and which ones do they not satisfy? Explain (prove) your answers.

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