# Object Allocation via Immediate-Acceptance: Characterizations and an Affirmative Action Application*

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**Abstract**

Which mechanism to use to allocate school seats to students still remains a question of hot debate. Meanwhile, immediate acceptance mechanisms remain popular in many school districts. We formalize desirable properties of mechanisms when respecting the relative rank of a school among the students' preferences is crucial. We show that those properties, together with well-known desirable resource allocation properties, characterize immediate acceptance mechanisms. Moreover, we show that replacing one of the properties, *consistency*, with a weaker property, *non-bossiness*, leads to a characterization of a much larger class of mechanisms, which we call *choice-based immediate acceptance mechanisms*. It turns out that certain objectives that are not achievable with immediate acceptance mechanisms, such as affirmative action, can be achieved with a choice-based immediate acceptance mechanism.

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# 1 Introduction

Public School Choice programs let students express their preference rankings over schools and aim to assign school seats to the students in a way that *respects* their preferences. How to respect the preferences under capacity constraints, i.e., which mechanism to use to allocate a limited number of school seats, still remains a question of hot debate in public policy. Immediate acceptance (IA) mechanisms[1] are an essential part of the debate. Despite their various shortcomings (see Abdulkadiroğlu and Sönmez, 2003), IA mechanisms remain popular and they are currently being used in many school districts in the US (e.g., Minneapolis, Lee County of Florida, Denver, and Cambridge in Massachusetts) and in other countries (e.g., Spain, Calsamiglia and Güell, 2017).

A key feature of the IA mechanisms is that the relative rank of a school in the students' preferences is crucial in determining who receives a seat: seats are allocated to students who rank the school first, followed by those who rank it second *only when* seats are still available, and so on.[2] Here, we formalize desirable properties of mechanisms when respecting the relative rank of a school among the students' preferences is crucial. We show that those properties, together with well-known desirable properties, characterize IA mechanisms. Moreover, we show that weakening one of the properties characterizes a larger class of mechanisms which may be useful to reach objectives such as affirmative action.

We consider the model of allocating (object) types to agents, where each type has a certain number of copies (the capacity), each agent has a preference ranking over types, and he should receive a copy of one type without any monetary transfers. Public School Choice programs are applications of our model. Each IA mechanism is associated with a priority structure (for each type, a priority ordering over agents) and at each allocation problem, copies of each type are allocated to agents who rank the type first (if demand exceeds supply, rationing is based on the priority ordering); any remaining copies are allocated to those who rank the type second and who could not get their first choice (if demand exceeds supply, rationing is based on the priority ordering), and so on. As an important remark, a priority structure is not a primitive of our model, but it is derived from the properties together with a specific mechanism.[3]

[1]IA mechanisms are also known as Boston mechanisms because they were used in the Boston school district until 2005. We use the "immediate acceptance" terminology due to Thomson (2011).

[2]A pre-determined priority ordering is used to allocate the seats among students who assign the school to the same rank.

[3]One motivation for not starting with a given priority structure is that the priority structure itself is often

Our first result is that IA mechanisms are the only mechanisms satisfying *weak non-wastefulness*, *resource-monotonicity*, *consistency*, *favoring-higher-ranks*, and *rank-respecting unavailable-type-invariance* (Theorem 1).[4] The first three properties are well-known in the literature (see Section 3 for details). The last two properties are desirable in allocation problems where the relative rank of a type among the agents' preferences is crucial in determining who receives a copy. *Favoring-higher-ranks* has been introduced by Kojima and Ünver (2014) and requires that, for each agent, no type that he prefers to his allotment is assigned to an agent who ranks it lower than he does. *Rank-respecting unavailable-type-invariance* is a new property and requires that, when the positions of the unavailable types (the types which have zero capacity) are shuffled at a preference profile in such a way that the relative rank of each available type among the agents' preferences is preserved (i.e., for any two agents, say $i$ and $j$, $i$ now ranks an available type as high as $j$ if and only if $i$ used to rank the same type as high as $j$ used to), the allocation should not change. Our characterization of IA mechanisms in Theorem 1 remains valid if we replace *rank-respecting unavailable-type-invariance* with a much weaker tie-breaking property called *weak uniform tie-breaking*.

Our characterization result shows that a social planner who agrees on the desirability of our properties, one of which is a novel property, has to stick with an IA mechanism which relies on an *unconditional* priority structure, i.e., for each type, whenever demand exceeds supply, the remaining copies have to be allocated according to the same priority ordering, independent from other details of the problem such as who has received a copy before (in an earlier step). We observe that this restrictiveness is essentially driven by *consistency*.

For our second result, we weaken *consistency* to *non-bossiness*[5] and identify the class of mechanisms satisfying *weak non-wastefulness*, *resource-monotonicity*, *non-bossiness*, *favoring-higher-ranks*, and *rank-respecting unavailable-type-invariance*. To this end, we introduce *choice-based IA mechanisms* that operate in a similar fashion as the IA mechanisms, similar in the sense that types are allocated to agents who rank the type first; any remaining copies are allocated to those who rank the type second and who could not get their first choice, and so on. The difference is that at a step where demand exceeds supply, instead of relying on a

a design object for the social planner. For other advantages of our approach, see the discussion in Kojima and Ünver (2014).

[4] We are not the first to characterize IA mechanisms. In Section 4, we relate our characterization to Kojima and Ünver (2014), who provided the first characterization of IA mechanisms.

[5] *Non-bossiness* requires that whenever a change in an agent's preference relation does not bring about a change in his allotment, it does not bring about a change in anybody's allotment.

priority ordering, the agents who receive a copy are determined based on a choice function that does not necessarily induce a priority ordering and that possibly depends on who received a copy of the type before (in an earlier step). Also, we introduce two properties for choice functions, called *sequence-monotonicity* and *sequence-substitutability*, that limit the extent to which choice depends on what happened in an earlier step. We show that choice-based IA mechanisms whose choice structures satisfy *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability* are the only mechanisms satisfying *weak non-wastefulness*, *resource-monotonicity*, *non-bossiness*, *favoring-higher-ranks*, and *rank-respecting unavailable-type-invariance* (Theorem 2).[6]

The class of mechanisms characterized in Theorem 2 includes choice-based IA mechanisms where the choice may depend on who received what before, which may be desirable in some applications. In Section 6, we discuss an affirmative action example; we provide a class of mechanisms, which is a subset of the class characterized in Theorem 2, that are designed to "promote" a group of "minority" agents. Each of these mechanisms favor minority agents and how many minority agents will be favored depends on how many minority agents already received a copy throughout the steps of the IA algorithm: this clearly requires a choice function that depends on "who received what before." The class of choice-based IA mechanisms may particularly be useful for a social planner who has been using an IA mechanism and who is planning to incorporate new design objectives, such as affirmative action, without entirely abolishing the current principles. Our characterization results reveal that such a change essentially requires consenting to a weaker form of *consistency*.

Another interesting aspect of our analysis concerns the *substitutability* of the choice structure as a result of desirable resource allocation properties. Similar results have been obtained by Kojima and Manea (2010) and Ehlers and Klaus (2016): they characterize choice-based deferred acceptance (DA) mechanisms and part of their characterizations is that the underlying choice structure must be *substitutable*. *Substitutability* of the choice structure ensures that *stable* allocations exist (e.g., Kelso and Crawford, 1982; Roth, 1984), and hence a corresponding choice-based DA mechanism is guaranteed to indeed find a stable allocation. Choice-based IA mechanisms are not stable but surprisingly the properties that characterize them do imply *substitutability* and the existence of stable matchings.

The paper is organized as follows. In Section 2, we introduce our general object allocation model. In Section 3, we introduce well-known and desirable resource allocation properties

[6]The definitions of *acceptance*, *monotonicity*, and *substitutability*, which are borrowed from choice theory, are in Section 5.

and properties that concern respecting the relative rank of a school among the students' preferences. In Section 4, we characterize the set of IA mechanisms (Theorem 1). First, we show that a property that we call "uniform pairwise-tie-breaking" is satisfied. Second, we show that *favoring-higher-ranks* and *uniform pairwise-tie-breaking* pin down an IA mechanism. We conclude Section 4 by discussing the relation of our characterization of IA mechanisms with that of Kojima and Ünver (2014). In Section 5, we weaken *consistency* to *non-bossiness* and characterize the set of choice-based IA mechanisms (Theorem 2). Section 6 demonstrates the versatility of the set of choice-based IA mechanisms with an example of affirmative action IA mechanisms. In Section 7, we discuss the related literature. The proofs of all the results are in Appendix A. The independences of properties in our characterizations are in Appendix B.

## 2 The Model and Notation

Our allocation model is identical to that described in Ehlers and Klaus (2014) and Ehlers and Klaus (2016).

Let $N$ denote a finite *set of agents*, $|N| \geq 2$. Let $O$ denote a finite set of potential (real) object types or *types* for short. We assume that $O$ contains at least two elements and that $O$ is finite. Not receiving any real object is called "receiving the null object". Let $\emptyset$ represent the *null object*.

Each agent $i \in N$ is equipped with a *strict preference relation* $R_i$, which is a complete, transitive, and anti-symmetric binary relation over all types $O \cup \{\emptyset\}$. Given $x, y \in O \cup \{\emptyset\}$, $x \, R_i \, y$ means that either $x = y$ or $x \neq y$ and agent $i$ prefers $x$ to $y$. If agent $i$ prefers $x$ to $y$, we also write $x \, P_i \, y$. For any agent $i \in N$ and type $x \in O \cup \{\emptyset\}$, the *lower contour set of $R_i$ at $x$* is $L(R_i, x) := \{y \in O \cup \{\emptyset\} : x \, R_i \, y\}$. Let $\mathcal{R}$ denote the *set of all preference relations* over $O \cup \{\emptyset\}$, and $\mathcal{R}^N$ the *set of all (preference) profiles* $R = (R_i)_{i \in N}$ such that for all $i \in N$, $R_i \in \mathcal{R}$.

Given $R \in \mathcal{R}^N$ and $S \subseteq N$, let $R_S$ denote the profile $(R_i)_{i \in S}$; it is the restriction of $R$ to the set of agents $S$. We also use the notation $R_{-S} = R_{N \setminus S}$ and $R_{-i} = R_{N \setminus \{i\}}$. Given $O' \subseteq O \cup \{\emptyset\}$, let $R_i|_{O'}$ denote the restriction of $R_i$ to $O'$ and $R|_{O'} = (R_i|_{O'})_{i \in N}$. Given $i \in N$ and $R_i \in \mathcal{R}$, type $x \in O$ is *acceptable at $R_i$* if $x \, P_i \, \emptyset$. Let $A(R_i) = \{x \in O : x \, P_i \, \emptyset\}$ denote the *set of acceptable types at $R_i$*. Given $i \in N$, $R_i \in \mathcal{R}_i$, and $x \in O \cup \{\emptyset\}$, let $R_i(x) = |\{y \in O \cup \{\emptyset\} : y \, R_i \, x\}|$ be the *rank of type $x$ in $R_i$*, i.e., a type of rank 1 is the best type, a type of rank 2 is the second best type, etc. To conform with colloquial language, we say that for two types $x, y \in O$, if $R_i(x) < R_i(y)$ then agent $i$ *ranks type $x$ higher than type $y$*.

For each type $x \in O$, at most $\overline{q}_x \in \mathbb{N}$ copies are available in any economy, where $1 \leq \overline{q}_x \leq |N|$. By introducing, for each $x \in O$, a "global upper bound" $\overline{q}_x$, we can, for instance, specify the so-called *house allocation model* where at most one object of each type is available, i.e., for all $x \in O$, $\overline{q}_x = 1$. Let $q_x \in \{0, 1, \ldots, \overline{q}_x\}$ denote the number of available objects or the *capacity* of type $x$.[7] Let $q = (q_x)_{x \in O}$ denote a *capacity vector* and $\mathcal{Q} = \times_{x \in O}\{0, 1, \ldots, \overline{q}_x\}$ denote the *set of all capacity vectors*. The null object is always available without scarcity and therefore we set $q_\emptyset = \infty$. Given a capacity vector $q$, let $O_+(q) = \{x \in O : q_x > 0\}$ denote the *set of available real types at* $q$. The *set of available types at* $q$ is $O_+(q) \cup \{\emptyset\}$; it includes the null object.

An *(allocation) problem (with capacity constraints)* consists of a profile $R \in \mathcal{R}^N$ and a capacity vector $q$. We denote a problem by $(R, q)$ and the set of all problems by $\mathcal{R}^N \times \mathcal{Q}$.

Given $q \in \mathcal{Q}$, an *allocation* assigns to each agent exactly one object of a type in $O \cup \{\emptyset\}$ taking capacity constraints into account. Formally, an *allocation at* $q$ is a list $a = (a_i)_{i \in N}$ such that for all $i \in N$, $a_i \in O \cup \{\emptyset\}$, and no real type $x \in O$ is assigned more than $q_x$ times, i.e., for all $x \in O$, $|\{i \in N : a_i = x\}| \leq q_x$. The null object can be assigned to any number of agents and not all real objects have to be assigned. Let $a(x) = \{i \in S : a_i = x\}$ denote the set of agents who are assigned type $x$ at $a$. Let $\mathcal{A}(q)$ denote the *set of all allocations at* $q$ and $\mathcal{A} = \bigcup_{q \in \mathcal{Q}} \mathcal{A}(q)$ *the set of all allocations*. Given $q \in \mathcal{Q}$, $a \in \mathcal{A}(q)$, and $x \in O$, we say $x$ is *exhausted* if $|a(x)| = q_x$ and otherwise we say that $x$ is *not exhausted*. Note that the null object is never exhausted.

A *mechanism* is a function $\varphi : \mathcal{R}^N \times \mathcal{Q} \to \mathcal{A}$ such that for all $R \in \mathcal{R}^N$ and all $q \in \mathcal{Q}$, $\varphi(R, q) \in \mathcal{A}(q)$. Given $i \in N$, we call $\varphi_i(R, q)$ the *allotment* of agent $i$ at $\varphi(R, q)$.

# 3 Desirable Properties of Mechanisms

*Individual rationality* requires that no agent should prefer the null object (which may represent an outside option such as off-campus housing in the context of university housing allocation, or private schools or home schooling in the context of student placement in public schools) to his allotment.

**Individual Rationality:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and all $i \in N$, $\varphi_i(R, q) \; R_i \; \emptyset$.

[7]By allowing for $q_x = 0$ we implicitly model the removal of object types without introducing a full variable object type model (in which object types are removed and not listed in agents' preferences). In school choice $q_x = 0$ could reflect that school $x$ opted out of the centralized allocation scheme.

Next, we introduce two properties that require a mechanism to not waste any resources. First, *non-wastefulness* (Balinski and Sönmez, 1999) requires that no agent prefers a type that is not exhausted to his allotment.

**Non-Wastefulness:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in N$, if $x \; P_i \; \varphi_i(R, q)$, then $|\varphi(R, q)(x)| = q_x$.

Next, we weaken *non-wastefulness* by requiring that no agent receives the null object while he prefers a type that is not exhausted to the null object.

**Weak Non-Wastefulness:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O_+(q)$, and all $i \in N$, if $x \; P_i \; \varphi_i(R, q)$ and $\varphi_i(R, q) = \emptyset$, then $|\varphi(R, q)(x)| = q_x$.

*Weak non-wastefulness* is a mild efficiency requirement. For example, suppose that a central agency registers all unemployed agents (those who did not receive anything) and all those agents report all jobs (real types) which are acceptable to them. Then it should not be the case that an unemployed agent prefers one of the jobs with an unassigned position to being unemployed.

When the capacities of types vary, a natural requirement is *resource-monotonicity*; it is a widely used solidarity property introduced by Chun and Thomson (1988) and it describes the effect of a change in the available resources on the welfare of the agents. A mechanism is *resource-monotonic* if the availability of more copies of the real types has a (weakly) positive effect on all agents.

**Resource-Monotonicity:** For all $R \in \mathcal{R}^N$, and all $q, q' \in \mathcal{Q}$, if for all $x \in O$, $q_x \leq q'_x$, then for all $i \in N$, $\varphi_i(R, q') \; R_i \; \varphi_i(R, q)$.

*Consistency* is one of the key properties in many frameworks with variable population scenarios. Thomson (2015) provides an extensive survey of *consistency* in various contexts. *Consistency* requires that if some agents leave a problem with their allotments, then the mechanism should allocate the remaining objects among the remaining agents in the same way as in the original problem. We introduce a fixed population counterpart of *consistency* which requires that if some agents change their preferences in such a way that no real type is acceptable anymore, and if their allotments are removed, then the mechanism should allocate the remaining objects among the agents who did not change their preferences in the same way as in the original problem.

**Consistency:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $S \subseteq N$, all $R'_S \in \mathcal{R}^S$, if for each $i \in S$, $A(R'_i) = \emptyset$, then for each $j \in N \setminus S$, $\varphi_j(R, q) = \varphi_j((R'_S, R_{-S}), \tilde{q})$ where for all $x \in O$, $\tilde{q}_x = q_x - |\{i \in S : \varphi_i(R, q) = x\}|$.

A well-known property that is implied by *consistency*, called *non-bossiness*, requires that whenever a change in an agent's preference relation does not bring about a change in his allotment, it does not bring about a change in anybody's allotment (Satterthwaite and Sonnenschein, 1981).

**Non-Bossiness:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $i \in N$, and all $R'_i \in \mathcal{R}$, if $\varphi_i(R, q) = \varphi_i((R'_i, R_{-i}), q)$, then $\varphi(R, q) = \varphi((R'_i, R_{-i}), q)$.

A recent survey (Thomson, 2016) discusses many other logical relationships of *non-bossiness* with well-known normative or strategic properties. The following strengthening of *non-bossiness* will be convenient in some of our proofs and strengthen some of our results (Barberà and Jackson, 1995).

**Group Non-Bossiness:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $S \subseteq N$, and all $R'_S \in \mathcal{R}^S$, if for each $i \in S$, $\varphi_i(R, q) = \varphi_i((R'_S, R_{-S}), q)$, then $\varphi(R, q) = \varphi((R'_S, R_S), q)$.

We next introduce a property due to Kojima and Ünver (2014). For this property, the rank of a type in an agent's preference relation is essential. The property requires that, for each type $x$, if there is an agent $i \in N$ who prefers $x$ to his allotment, then type $x$ isn't assigned to an agent who ranks it lower.

**Favoring-Higher-Ranks:** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O$, and all $i \in N$, if $x \, P_i \, \varphi_i(R, q)$, then there is no agent $j \in N$ such that $\varphi_j(R, q) = x$ and $R_j(x) > R_i(x)$.[8]

Our next requirement is the following: suppose that the positions of the unavailable types are shuffled at a profile in such a way that for each available type $x$, the relative rank of $x$ among the agents is preserved (i.e., for any two agents $i$ and $j$, $i$ now ranks $x$ as high as $j$ if and only if $i$ used to rank $x$ as high as $j$), then the allocation should not change.

**Rank-Respecting Unavailable-Type-Invariance (RR-UTI):** For all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $R' \in \mathcal{R}^N$ such that $R|_{O_+(q) \cup \{\emptyset\}} = R'|_{O_+(q) \cup \{\emptyset\}}$, if [for all $x \in O_+(q)$ and for all $i, j \in N$, $R_i(x) \geq R_j(x)$ if and only if $R'_i(x) \geq R'_j(x)$], then $\varphi(R, q) = \varphi(R', q)$.

**Example 1** (*Rank-respecting unavailable-type-invariance* applies)**.** Let $N = \{1, 2, 3\}$, $O = \{x, y, z, \bar{z}\}$, $q_x = 1$, $q_y = 2$, and $q_z = q_{\bar{z}} = 0$, i.e., objects $x$ and $y$ are available while $z$ and $\bar{z}$ are unavailable. We compare problems $(R, q)$, $(R', q)$, and $(\bar{R}, q)$ with the following preferences (available types are listed in bold face):

[8] Kojima and Ünver (2014) introduced a somewhat stronger version of *favoring-higher-ranks*: for all $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, all $x \in O \cup \{\emptyset\}$, and all $i \in N$, if $x \, P_i \, \varphi_i(R, q)$, then $|\{k \in N : \varphi_k(R, q) = x\}| = q_x$ and for each $j \in \{k \in N : \varphi_k(R, q) = x\}$, $R_j(x) \leq R_i(x)$. Hence, their original definition of *favoring-higher-ranks* also included *individual rationality* and *non-wastefulness*.

| $R_1$ | $R_2$ | $R_3$ | | $R'_1$ | $R'_2$ | $R'_3$ | | $\bar{R}_1$ | $\bar{R}_2$ | $\bar{R}_3$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{x}$ | $\boldsymbol{y}$ | $z$ | | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $z$ | | $\boldsymbol{x}$ | $\boldsymbol{y}$ | $\bar{z}$ |
| $z$ | $\bar{z}$ | $\bar{z}$ | | $\bar{z}$ | $\boldsymbol{x}$ | $\boldsymbol{y}$ | | $\bar{z}$ | $z$ | $z$ |
| $\bar{z}$ | $\boldsymbol{x}$ | $\boldsymbol{y}$ | | $\boldsymbol{y}$ | $z$ | $\boldsymbol{x}$ | | $z$ | $\boldsymbol{x}$ | $\boldsymbol{y}$ |
| $\boldsymbol{y}$ | $z$ | $\boldsymbol{x}$ | | $z$ | $\bar{z}$ | $\bar{z}$ | | $\boldsymbol{y}$ | $\bar{z}$ | $\boldsymbol{x}$ |
| $\emptyset$ | $\emptyset$ | $\emptyset$ | | $\emptyset$ | $\emptyset$ | $\emptyset$ | | $\emptyset$ | $\emptyset$ | $\emptyset$ |

Note that the only difference when moving from $R$ to $R'$ is that all agents have moved their second ranked unavailable type to be the worst acceptable type. Furthermore, the only change when moving from $R$ to $\bar{R}$ is that all agents swap the rankings of the unavailable types $z$ and $\bar{z}$. These are two examples of situations in which, starting from $R$, the relative ranks of all available types are preserved and hence *rank-respecting unavailable-type-invariance* implies that the allocation at problems $(R, q)$, $(R', q)$, and $(\bar{R}, q)$ should be the same. □

*Rank-respecting unavailable-type-invariance* implies the following *tie-breaking* property: assume that at a problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ only one copy of type $x \in O$ is available and nothing else and only agents $i, j \in N$ find $x$ acceptable and rank it the same, while all other agents rank $x$ as the lowest (unacceptable) type. Then, if agent $i$ receives $x$ at $(R, q)$, agent $i$ also receives $x$ at $(R', q)$ where $R'$ is such that agents $i, j \in N$ find only $x$ acceptable, while all other agents have the same preferences as before.

**Weak Uniform Tie-Breaking:** For all $x \in O$, all $i, j \in N$, all $q \in \mathcal{Q}$ such that $q_x = 1$ and $q_y = 0$ for each $y \in O \setminus \{x\}$, all $R, R' \in \mathcal{R}^N$ such that $R_{-\{i,j\}} = R'_{-\{i,j\}}$, $R_i(x) = R_j(x)$, $x \in A(R_i) \cap A(R_j)$, $A(R'_i) = A(R'_j) = \{x\}$, and for each $k \in N \setminus \{i, j\}$, $R_k(x) = R'_k(x) = |O|+1$, if $\varphi_i(R, q) = x$, then $\varphi_i(R', q) = x$.

Note that for problems $(R, q)$ and $(R', q)$ as defined in the definition of *weak uniform tie-breaking* it follows easily that $R|_{O_+(q) \cup \{\emptyset\}} = R'|_{O_+(q) \cup \{\emptyset\}}$ and [for all $x \in O_+(q)$ and for all $i, j \in N$, $R_i(x) \geq R_j(x)$ if and only if $R'_i(x) \geq R'_j(x)$]. Hence, as mentioned above, *rank-respecting unavailable-type-invariance* implies *weak uniform tie-breaking*.

We use the following relationships between the above properties to prove our results (all proofs are in Appendix A).

**Lemma 1.** *Resource-monotonicity implies individual rationality.*

It has already been noted in the literature that *consistency* implies *non-bossiness* (Thomson, 2016).

**Lemma 2.** *Consistency implies group non-bossiness.*

**Lemma 3.** *Weak non-wastefulness, resource-monotonicity, and consistency together imply non-wastefulness.*

The following lemma holds with *non-bossiness* as well as with *group non-bossiness*; using *group non-bossiness* allows for a more compact proof but we mention in the proof how to proceed with the weaker property *non-bossiness*.

**Lemma 4.** *Weak non-wastefulness, resource-monotonicity, (group) non-bossiness, and favoring-higher-ranks together imply non-wastefulness.*

# 4 Immediate Acceptance Mechanisms

Given real type $x \in O$, let $\succ_x$ denote a *priority ordering on* $N$, e.g., $\succ_x$: $1\ 2\ \ldots\ |N|$ means that agent 1 has higher priority for type $x$ than agent 2, who has higher priority for type $x$ than agent 3, *etc.* Let $\succ \equiv (\succ_x)_{x \in O}$ denote a *priority structure for N.*

**Immediate $\succ$-Acceptance Algorithm:** Let $\succ$ be a priority structure. Then, for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, the algorithm runs as follows:

***Step 1:*** Each agent applies to his favorite type in $O \cup \{\emptyset\}$. Each real type $x \in O$ such that $q_x > 0$ accepts the $q_x$ applicants who have the highest priority (accepts all if there are fewer than $q_x$ applicants), and rejects all the other applicants. Each real type $x \in O$ such that $q_x = 0$ rejects all applicants and the null object $\emptyset$ accepts all applicants.

***Step $r \geq 2$:*** Each applicant who was rejected at step $r-1$ applies to his next favorite type in $O \cup \{\emptyset\}$. For each real type $x \in O$, let $q_x^r$ be $q_x$ minus the total number of agents accepted by $x$ before step $r$. Each real type $x \in O$ such that $q_x^r > 0$ accepts the $q_x^r$ applicants who have the highest priority (accepts all if there are fewer than $q_x^r$ applicants), and rejects all the other applicants. Each real type $x \in O$ such that $q_x^r = 0$ rejects all applicants and the null object $\emptyset$ accepts all applicants.

The algorithm terminates when each agent is accepted by a real type or the null object. The allocation where each agent is assigned the type that he was accepted by at the end of the algorithm is called the *immediate $\succ$-acceptance allocation at* $(R, q)$, denoted by $IA^{\succ}(R, q)$.

**Immediate $\succ$-Acceptance or IA Mechanisms:** A mechanism $\varphi : \mathcal{R}^N \to \mathcal{A}$ is an *IA mechanism* if there is a priority structure $\succ$ such that for each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = IA^{\succ}(R, q)$.

We prove, in two steps, that IA mechanisms are the only mechanisms satisfying *weak non-wastefulness*, *resource-monotonicity*, *consistency*, *favoring-higher-ranks*, and *weak uniform tie-breaking* (or *RR-UTI*): first, we show that *weak non-wastefulness*, *resource-monotonicity*, *consistency*, and *weak uniform tie-breaking* (or *RR-UTI*) imply a property that we call "uniform pairwise-tie-breaking"; second, we show that *favoring-higher-ranks* and *uniform pairwise-tie-breaking* pin down an immediate $\succ$-acceptance mechanism.

*Uniform pairwise-tie-breaking* requires that for each pair of agents and each type, there can't be two problems where the two agents assign the same rank to the type and in one problem one of the agents gets the type while the other would like to receive it and in the other problem the situation is reversed.

**Uniform Pairwise-Tie-Breaking:** There are no pair of problems $(R, q)$, $(R', q')$, no real type $x \in O$, and no pair of agents $i, j \in N$ such that

***i.*** $R_i(x) = R_j(x)$, $\varphi_i(R, q) = x$, $x \; P_j \; \varphi_j(R, q)$, and

***ii.*** $R'_i(x) = R'_j(x)$, $\varphi_j(R', q') = x$, and $x \; P'_i \; \varphi_i(R', q')$.

The following lemma holds with *weak uniform tie-breaking* as well as with the stronger property *RR-UTI* (this and further proofs for results in this section are in Appendix A).

**Lemma 5.** *Weak non-wastefulness, resource-monotonicity, consistency, and weak uniform tie-breaking (rank-respecting unavailable-type-invariance) together imply uniform pairwise-tie-breaking.*

The following theorem holds with *weak uniform tie-breaking* as well as with *RR-UTI*.

**Theorem 1.** *A mechanism satisfies weak non-wastefulness, resource-monotonicity, consistency, favoring-higher-ranks, and weak uniform tie-breaking (rank-respecting unavailable-type-invariance) if and only if it is an IA mechanism.*

We prove the independence of characterizing properties in Theorem 1 in Appendix B.

The proof of Theorem 1 does not require more than one copy of any real type. Hence, Theorem 1 is also valid for the well-known *house allocation model.*

**Corollary 1** (Theorem 1 for House Allocation)**.**
*In the house allocation model, a mechanism satisfies weak non-wastefulness, resource-monotonicity, consistency, favoring-higher-ranks, and weak uniform tie-breaking (rank-respecting unavailable-type-invariance) if and only if it is an IA mechanism.*

In a related study, Kojima and Ünver (2014) show that a mechanism satisfies *weak non-wastefulness*, *resource-monotonicity*, *consistency*, *"favoring-higher-ranks"*, and *"rank-respecting invariance"* if and only if it is an IA mechanism. Their set of characterizing properties differs from ours in two ways.

First, Kojima and Ünver (2014) study a stronger version of *favoring-higher-ranks* (although we use the same name), requiring that if an agent $i \in N$ prefers a type $x \in O \cup \{\emptyset\}$ to his assigned type (possibly the null object), then $x$'s capacity must be filled with agents who rank $x$ at least as high as $i$, which readily implies *non-wastefulness* (see Footnote 8). Note that we study a weaker version of *favoring-higher-ranks*, which does not imply any form of non-wastefulness. Yet, we include *weak non-wastefulness*, which together with *resource-monotonicity* and *consistency* implies *non-wastefulness*.

Second, and most importantly, we replace *"rank-respecting invariance"* with *weak uniform tie-breaking* (or *RR-UTI*). For the sake of completeness, we provide a formal definition of *rank-respecting invariance*. The property is based on the concept of a "monotonic transformation of a preference relation at a type". A preference relation $R'_i \in \mathcal{R}$ is a *monotonic transformation* of $R_i \in \mathcal{R}$ at $x \in O \cup \{\emptyset\}$ if $L(R_i, x) \subseteq L(R'_i, x)$. A profile $R' \in \mathcal{R}^N$ is a monotonic transformation of $R \in \mathcal{R}^N$ at an allocation $a \in \mathcal{A}$ if for each $i \in N$, $R'_i$ is a monotonic transformation of $R_i$ at $a_i$.

For each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, each $i \in N$, and each $a \in \mathcal{A}$, define the set of agents who rank type $a_i$ at least as high as agent $i$ does and at least as high as their own allotment

$$U_i(R, a) := \{j \in N : R_j(a_i) \leq R_i(a_i) \text{ and } R_j(a_i) \leq R_j(a_j)\}$$

and define the set of agents who rank type $a_i$ higher than agent $i$ does and at least as high as their own allotment

$$V_i(R, a) := \{j \in N : R_j(a_i) < R_i(a_i) \text{ and } R_j(a_i) \leq R_j(a_j)\}.$$

A profile $R' \in \mathcal{R}^N$ is a *rank-respecting monotonic transformation* of $R \in \mathcal{R}^N$ at an allocation $a \in \mathcal{A}$ if $R'$ is a monotonic transformation of $R$ at $a$ and for each $i \in N$ such that $a_i \in O$, $U_i(R', a) \subseteq U_i(R, a)$ and $V_i(R', a) \subseteq V_i(R, a)$.

**Rank-Respecting Invariance:** For all $(R, q), (R', q) \in \mathcal{R}^N \times \mathcal{Q}$, if $R'$ is a rank-respecting monotonic transformation of $R$ at $\varphi(R, q)$, then $\varphi(R', q) = \varphi(R, q)$.

*Rank-respecting invariance* and *RR-UTI* are logically independent, but *rank-respecting invariance* implies *weak uniform tie-breaking*.

**Lemma 6.** *Rank-respecting invariance implies weak uniform tie-breaking.*

Hence, Theorem 1 is a strengthening of Kojima and Ünver's (2014) characterization of IA mechanisms that is obtained by replacing their *rank-respecting invariance* with the weaker *weak uniform tie-breaking* (Lemma 6).

# 5 Choice-Based Immediate Acceptance Mechanisms

In this section, we consider a class of mechanisms that operate in a similar fashion as the IA mechanisms, similar in the sense that types are allocated to agents who rank the type first; any remaining copies are allocated to those who rank the type second and who could not get their first choice, and so on. The difference is that when there is a step at which there are fewer copies available than demanded, instead of relying on a priority ordering, the agents who will receive a copy are chosen based on a choice function that does not necessarily induce a priority ordering and that possibly depends on who received a copy of the type before (in an earlier step).

First, we formalize our notion of a choice function. Given any set of agents $A \subseteq N$, let $2^A$ denote the set of all subsets of $A$. For each real type $x \in O$, let

$$\mathcal{D}_x \equiv \big\{(B,S,l) \in 2^N \times 2^N \times \{1,\dots,\overline{q}_x\} : B \cap S = \emptyset, S \neq \emptyset, \text{ and } l \leq \overline{q}_x - |B|\big\}.$$

The role of a triple $(B,S,l) \in \mathcal{D}_x$ will be that $B$ is the set of agents who have received a copy of $x$ *before* the time when agents in $S$ simultaneously apply to receive a copy from $l$ remaining copies of $x$.

For any real type $x \in O$, let $\mathcal{C}_x : \mathcal{D}_x \to 2^N \setminus \{\emptyset\}$ denote a *choice function* such that for each $(B,S,l) \in \mathcal{D}_x$, $\mathcal{C}_x(B,S,l) \subseteq S$ and $0 < |\mathcal{C}_x(B,S,l)| \leq l$. We interpret $\mathcal{C}_x(B,S,l)$ as the agents in $S$ who receive a copy of $x$ when the agents in $B$ have received a copy of $x$ before and the agents in $S$ compete for $l$ available copies of $x$ in the same step of the IA algorithm. We next list some properties of choice functions. Let $x \in O$ and $\mathcal{C}_x$ be a choice function.

**Acceptance:** for each $(B,S,l) \in \mathcal{D}_x$,

$$|\mathcal{C}_x(B,S,l)| = \min\{|S|, l\}.$$

**Monotonicity:** for each $(B, S, l) \in \mathcal{D}_x$,

$$\mathcal{C}_x(B, S, l-1) \subseteq \mathcal{C}_x(B, S, l).$$

**Substitutability:** for each $(B, S, l) \in \mathcal{D}_x$ and each pair $i, j \in S$ such that $i \neq j$,

$$i \in \mathcal{C}_x(B, S, l) \text{ implies } i \in \mathcal{C}_x(B, S \backslash \{j\}, l).$$

**Remark 1.** A choice function $\mathcal{C}_x$ that satisfies *acceptance* and *substitutability* also satisfies *irrelevance of rejected agents* (IRA):[9] for all $(B, S, l), (B, S', l) \in \mathcal{D}_x$, $\mathcal{C}_x(B, S, l) \subseteq S' \subseteq S$ implies $\mathcal{C}_x(B, S, l) = \mathcal{C}_x(B, S', l)$ (see Ehlers and Klaus, 2016, Lemma 1). Furthermore, a choice function $\mathcal{C}_x$ is *substitutable* and satisfies *IRA* if and only if it is *path-independent*: for all $(B, S, l), (B, S', l) \in \mathcal{D}_x$, $\mathcal{C}_x(B, S \cup S', l) = \mathcal{C}_x(B, S \cup \mathcal{C}_x(B, S', l), l)$ (see Aizerman and Malishevski, 1981; Chambers and Yenmez, 2017). □

The above properties of choice functions are borrowed from the literature on choice functions that do not have the $B$ component, and for that reason none of the above properties imposes any restriction on how the choice changes when $B$ changes. To be more precise, let $\mathcal{AC}_x : 2^N \setminus \{\emptyset\} \times \{1, \ldots, \bar{q}_x\} \to 2^N \setminus \{\emptyset\}$ denote a choice function without the $B$ component, or simply an *auxiliary choice function* for our setup, such that for each $(S, l) \in 2^N \setminus \{\emptyset\} \times \{1, \ldots, \bar{q}_x\}$, $\mathcal{AC}_x(S, l) \subseteq S$ and $0 < |\mathcal{AC}_x(S, l)| \leq l$. Now, if we associate each $B \in 2^N$ with an auxiliary choice function $\mathcal{AC}_x^B$ that satisfies *acceptance*, *monotonicity*, and *substitutability*, then the choice function $\mathcal{C}_x$ induced by the auxiliary choice functions in the following straightforward way also satisfies the three properties: for each $(B, S, l) \in \mathcal{D}_x$, let $\mathcal{C}_x(B, S, l) \equiv \mathcal{AC}_x^B(S, l)$. In order for the three properties to be preserved, there is no restriction on which auxiliary choice function to associate with which $B \in 2^N$, provided that each auxiliary choice function satisfies the three properties.

Now, we introduce two new properties of choice functions that restrict how the choice changes when $B$ changes. Let $x \in O$ and $\mathcal{C}_x$ be a choice function.

The first property requires that, if an agent $i$ is chosen from $S$, then $i$ should still be chosen when an agent $j$ is removed from $B$ and the number of available copies for agents in $S$ increases from $l$ to $l + 1$. In our context, it amounts to requiring that a chosen agent is still chosen if an agent who was chosen before the competition step is not chosen before anymore and there is one more available copy.

[9] Irrelevance of Rejected Agents (IRA) is referred to as *irrelevance of rejected contracts* (IRC) by Aygün and Sönmez (2013) and as *consistency* by Chambers and Yenmez (2017).

**Sequence-Monotonicity:** for each $(B, S, l) \in \mathcal{D}_x$,

$$i \in \mathcal{C}_x(B, S, l) \text{ and } j \in B \text{ imply } i \in \mathcal{C}_x(B \setminus \{j\}, S, l+1).$$

The second property requires that, if an agent $i$ is chosen together with another agent $j$ from $S$, then $i$ should still be chosen when $j$ is moved to $B$ and the number of available copies for agents in $S \setminus \{j\}$ decreases from $l$ to $l-1$. In our context, this amounts to requiring that whether $j$ is chosen at a competition step or before does not affect whether $i$ is chosen at the competition step.

**Sequence-Substitutability:** for each $(B, S, l) \in \mathcal{D}_x$ such that $l \geq 2$,

$$i, j \in \mathcal{C}_x(B, S, l) \text{ and } i \neq j \text{ imply } i \in \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1).$$

Note that if we associate each $B \in 2^N$ with an auxiliary choice function in an arbitrary manner, then the choice function induced by the auxiliary choice functions does not necessarily satisfy any of the two properties that we have just introduced: the two new properties restrict how the choice changes when $B$ changes. On the other hand, if each $B \in 2^N$ is associated with the same auxiliary choice function that satisfies *acceptance*, *monotonicity*, and *substitutability*, or in other words if the choice does not vary with $B$ at all, then the induced choice function satisfies all the five properties introduced so far. Therefore, the two new properties restrict how much the choice varies with $B$, or in other words how much the choice depends on $B$.

A particular example of a choice function that satisfies all properties mentioned above is a "responsive choice function:" let $x \in O$ and $\succ_x$ be a priority ordering on $N$. If for each $(B, S, l) \in \mathcal{D}_x$, $\mathcal{C}_x(B, S, l) = \{i_1, \ldots, i_k\} \subseteq S$ such that $k = \min\{|S|, l\}$ and for each $l \in S \setminus \{i_i, \ldots, i_k\}$ we have $i_1, \ldots, i_k \succ_x l$, i.e., choice function $\mathcal{C}_x$ chooses the highest priority agents in set $S$ according to priority ordering $\succ_x$, then $\mathcal{C}_x$ is *responsive to priority ordering* $\succ_x$. We denote a choice function that is responsive to a priority ordering $\succ_x$ by $\mathcal{C}^{\succ_x}$.

The "choice-based IA mechanisms" that we introduce now are defined through "IA algorithms based on choice structures". A choice structure associates each real type with a list of choice functions, i.e., $\mathcal{C} = (\mathcal{C}_x)_{x \in O}$. We say that a choice structure $\mathcal{C} = (\mathcal{C}_x)_{x \in O}$ is *acceptant* / *monotonic* / *substitutable* / *sequence-monotonic* / *sequence-substitutable* if for all real types $x \in O$, $\mathcal{C}_x$ is *acceptant* / *monotonic* / *substitutable* / *sequence-monotonic* / *sequence-substitutable*. Next, we define an IA algorithm based on a choice structure.

**Immediate $\mathcal{C}$-Acceptance Algorithm:** Let $\mathcal{C}$ be a choice structure. Then, for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, the algorithm runs as follows:

***Step 1:*** Each agent applies to his favorite type in $O \cup \{\emptyset\}$. Each real type $x \in O$ such that $q_x > 0$ accepts the applicants in $\mathcal{C}_x(\emptyset, S_x, q_x)$ where $S_x$ is the set of agents who applied to $x$, and rejects all the other applicants. Each real type $x \in O$ such that $q_x = 0$ rejects all applicants and the null object $\emptyset$ accepts all applicants.

***Step $r \geq 2$:*** Each applicant who was rejected at step $r-1$ applies to his next favorite type in $O \cup \{\emptyset\}$. For each real type $x \in O$, let $S_{x,r}$ be the set of agents who applied to $x$ at step $r$, let $B_{x,r} \in 2^N$ be the set of agents who were accepted by $x$ before step $r$, and let $q_x^r = q_x - |B_{x,r}|$ denote $x$'s residual quota. Each real type $x \in O$ such that $q_x^r > 0$ accepts the applicants in $\mathcal{C}_x(B_{x,r}, S_{x,r}, q_x^r)$ and rejects all the other applicants. Each real type $x \in O$ such that $q_x^r = 0$ rejects all applicants and the null object $\emptyset$ accepts all applicants.

The algorithm terminates when each agent is accepted by a real type or the null object. The allocation where each agent is assigned the type that he was accepted by at the end of the algorithm is called the immediate $\mathcal{C}$-acceptance allocation at $(R, q)$, denoted by $IA^{\mathcal{C}}(R, q)$.

**Immediate $\mathcal{C}$-Acceptance or Choice-Based IA Mechanisms:** A mechanism $\varphi : \mathcal{R}^N \times \mathcal{Q} \to \mathcal{A}$ is a *choice-based IA mechanism* if there is a choice structure $\mathcal{C}$ such that for each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = IA^{\mathcal{C}}(R, q)$.

It is no wonder that whether a choice-based IA mechanism satisfies certain desirable properties is intimately related to whether its associated choice structure satisfies certain properties. In particular, the following result reveals that whether a choice-based IA mechanism is *resource-monotonic* is intimately related to whether its associated choice structure is *sequence-monotonic*; and whether it is *non-bossy* is intimately related to whether its associated choice structure is *sequence-substitutable*.

The following proposition holds with *non-bossiness* as well as with the stronger property *group non-bossiness*; we indicate this by writing *(group) non-bossiness*.

**Proposition 1.** *Let $\varphi$ be a choice-based IA mechanism based on a choice structure $\mathcal{C}$ that satisfies acceptance, monotonicity, and substitutability.*

***i.*** *Suppose that $|O| \geq 3$. The mechanism $\varphi$ is resource-monotonic if and only if $\mathcal{C}$ is sequence-monotonic.*[10]

[10]The assumption that $|O| \geq 3$ is only necessary to show that *resource-monotonicity* of $\varphi$ implies *sequence-monotonicity* of $\mathcal{C}$.

***ii.*** *The mechanism $\varphi$ is (group) non-bossy if and only if $\mathcal{C}$ is sequence-substitutable.*[11]

We are now ready to present the main result of this section. We prove, in two steps, that choice-based IA mechanisms, where the choice structure satisfies *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*, are the only mechanisms satisfying *weak non-wastefulness*, *resource-monotonicity*, *(group) non-bossiness*, *favoring-higher-ranks*, and *RR-UTI*: first, we show that *weak non-wastefulness*, *resource-monotonicity*, *(group) non-bossiness*, and *RR-UTI* imply a tie-breaking property that we call "uniform choice"; second, we show that *favoring-higher-ranks* and *uniform choice* pin down an immediate $\mathcal{C}$-acceptance mechanism (with $\mathcal{C}$ satisfying *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*). We should add that for the second result, we need at least three types (Example 2 presents a mechanism that is not an immediate $\mathcal{C}$-acceptance mechanism satisfying all the above mentioned properties in a model with at most two types).

In plain words, a mechanism $\varphi$ satisfies *uniform choice* if for each set of agents and each type, in every problem where the set of agents rank the type at the same rank, the same set of agents received a copy of the type before them (i.e., they ranked the type higher), and they compete for the same number of remaining copies, then the same subset of agents gets an objects of that type.

**Uniform Choice:** There is no pair of problems $(R, q)$, $(R', q') \in \mathcal{R}^N \times \mathcal{Q}$ and no real type $x \in O$ such that $q_x = q'_x$, no triple $(B, S, l) \in \mathcal{D}_x$, $l = q_x - |B|$, and no numbers $r, r' \in \mathbb{N}$ such that

***i.*** agents in $S$ rank $x$ the same in either of the problems and all agents in $S$ receive an allotment that is at most as desirable as $x$, i.e.,

$$S = \{i \in N : R_i(x) = r \text{ and } x\, R_i\, \varphi_i(R, q)\} = \{i \in N : R'_i(x) = r' \text{ and } x\, R'_i\, \varphi_i(R', q')\},$$

***ii.*** the set $B$ equals the set of agents who rank type $x$ higher than agents in $S$ in either of the problems and who have received it as allotments, i.e.,

$$B = \{i \in N : R_i(x) < r \text{ and } \varphi_i(R, q) = x\} = \{i \in N : R'_i(x) < r \text{ and } \varphi_i(R', q') = x\},$$

and

[11]Note that *monotonicity* of the choice structure is in fact not needed for this part of the proposition.

***iii.*** the subsets of agents in $S$ who receive $x$ in either of the problems as allotments differ, i.e.,

$$\{i \in S : \varphi_i(R, q) = x\} \neq \{i \in S : \varphi_i(R', q') = x\}.$$

The following lemma holds with *non-bossiness* as well as with *group non-bossiness*; using *group non-bossiness* allows for a more compact proof but we mention in the proof how to proceed with the weaker property *non-bossiness* (this and further proofs for results in this section are in Appendix A).

**Lemma 7.** *Weak non-wastefulness, resource-monotonicity, (group) non-bossiness, favoring-higher-ranks, and rank-respecting unavailable-type-invariance together imply uniform choice.*

The following theorem holds with *non-bossiness* as well as with *group non-bossiness*.

**Theorem 2.** *Let $|O| \geq 3$. A mechanism satisfies weak non-wastefulness, resource-monotonicity, (group) non-bossiness, favoring-higher-ranks, and rank-respecting unavailable-type-invariance if and only if it is a choice-based IA mechanism with a choice structure that satisfies acceptance, monotonicity, substitutability, sequence-monotonicity, and sequence-substitutability.*[12]

We prove the independence of characterizing properties in Theorem 2 in Appendix B.

Theorems 1 and 2 imply that any choice-based IA mechanisms that is not responsive must violate *consistency* (we illustrate that with an affirmative-action-target example in Section 6 (Example 4). The next example shows that Theorem 2 need not hold if there are at most two types.

**Example 2.** Let $O = \{x, y\}$ and $N = \{1, 2, 3, 4\}$ (we can easily generalize the example to more agents by assuming that additional agents have lower priorities and choices for them are based on priorities). Consider the following priority orderings

$$\succ_x\!: \ 1\ 2\ 3\ 4 \quad \text{and} \quad \succ'_x\!: \ 1\ 3\ 4\ 2;$$

$$\succ_y\!: \ 2\ 1\ 3\ 4 \quad \text{and} \quad \succ'_y\!: \ 2\ 3\ 4\ 1.$$

We define choice functions based on them as follows.

[12] The assumption that $|O| \geq 3$ is only necessary to show the only if part of the characterization.

For each $(B, S, l) \in \mathcal{D}_x$,

$$\mathcal{C}_x(B, S, l) = \begin{cases} \mathcal{C}^{\succ_x}(B, S, l) & \text{if } 1 \in B \cup S \\ \mathcal{C}^{\succ'_x}(B, S, l) & \text{otherwise.} \end{cases}$$

For each $(B, S, l) \in \mathcal{D}_y$,

$$\mathcal{C}_y(B, S, l) = \begin{cases} \mathcal{C}^{\succ_y}(B, S, l) & \text{if } 2 \in B \cup S \\ \mathcal{C}^{\succ'_y}(B, S, l) & \text{otherwise.} \end{cases}$$

Note that $\mathcal{C}_x$ and $\mathcal{C}_y$ satisfy *acceptance*, *monotonicity*, *sequence-substitutability*, and violate *substitutability* and *sequence-monotonicity*. The choice-based IA mechanism $IA^{\mathcal{C}}$ satisfies *weak non-wastefulness*, *favoring-higher-ranks*, and *RR-UTI*. It also satisfies *group non-bossiness* since the priority orderings the choice structure uses do not change when agents' allotments remain the same. To see that $IA^{\mathcal{C}}$ satisfies *resource-monotonicity*, let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Without loss of generality, assume that one more copy of real type $x$ is available, i.e., let $q' \in \mathcal{Q}$ such that $q'_x = q_x + 1$ and $q'_y = q_y$ (note that the argument for real type $y$ is symmetric). We prove *resource-monotonicity* by showing that for all $i \in N$, $\varphi_i(R, q') R_i \varphi_i(R, q)$ when going through the steps of the immediate $\mathcal{C}$-acceptance algorithm (which has at most three steps).

If $\varphi(R, q) = \varphi(R, q')$, then we are done. Let $t \in \{1, 2\}$ be the first step in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $(R, q')$ at which an agent receives different allotments. As in the proof of Theorem 2, some agent, say agent $j \in N$, can newly receive $x$ because one more copy is available and, using only *monotonicity*, all agents who have received their allotments until Step $t$ are weakly better off. After Step $t$ no copies of type $x$ are available anymore. If no further allotments change after Step $t$ we are done. Otherwise, $t = 1$ and now consider Step 2. Recall that priority structures $\succ_y$ and $\succ'_y$ only differ in the position of agent 1. Hence, for an agent to be worse off at $(R, q')$ than at $(R, q)$, agent 1 cannot have received his allotment at Step 1 and he has to apply for type $y$ at Step 2. But then, $R_1(x) = 1$ and by the definition of $\mathcal{C}_x$, $IA_1^{\mathcal{C}}(R, q) = IA_1^{\mathcal{C}}(R, q') = x$. Thus, also in Step 2, all agents receiving their allotments are weakly better off. □

The proof of Theorem 2 does not require more than one copy of any real type. Hence, Theorem 2 is also valid for the well-known *house allocation model*. Note that the choice function properties *monotonicity*, *sequence-monotonicity*, and *sequence-substitutability* are vacuously satisfied by any choice function in the house allocation model (since then the only possible capacities for any real type are 0 and 1). *Acceptance* and *substitutability* (which boils down to the *independence of rejected agents (IRA)* as defined in Remark 1) then define a priority structure for $N$ that determines choices.

The following corollary for house allocation is even valid when only one or two real object types are available. The reason for this "strengthening" of the result is that the two parts in the proof of Theorem 2 that require at least three real object types can now either be omitted (because *sequence-monotonicity* is vacuously satisfied by any choice function in the house allocation model) or adjusted to show a weaker property (since *substitutability* boils down to *independence of rejected agents (IRA)*).

**Corollary 2** (Theorem 2 for House Allocation but for all $|O|$).
*In the house allocation model, a mechanism satisfies weak non-wastefulness, resource-monotonicity, (group) non-bossiness, favoring-higher-ranks, and rank-respecting unavailable-type-invariance if and only if it is an IA mechanism.*

We would like to conclude this section with some remarks concerning Theorem 2 in relation to Theorem 1 and in comparison to Ehlers and Klaus (2016, Theorem 3).

**Remark 2** (Theorem 2 versus Theorem 1).
Lemma 5, which is essential for the proof of Theorem 1, shows that *weak non-wastefulness*, *resource-monotonicity*, *consistency*, and *RR-UTI* imply *uniform pairwise-tie-breaking*, which requires that for each type and each pair of agents, the same agent is favored over the other agent when they compete for a copy of a type, independent from the details of the choice environment such as who else is competing with the two agents or who received a copy before. Given that it may be desirable to make the choice depend on the details of the environment, *uniform pairwise-tie-breaking* may be a restrictive property. Since the only difference between the set of characterizing properties in Theorems 1 and 2 is that *consistency* is replaced with *non-bossiness*, our results shed some light on the implications of the *consistency* requirement, although, of course, *consistency* cannot be solely held responsible for *uniform pairwise-tie-breaking*. □

**Remark 3** (Theorem 2 versus Ehlers and Klaus (2016), Theorem 3).
Ehlers and Klaus (2016, Theorem 3) characterize choice-based deferred acceptance (DA) mechanisms (where the choice structure satisfies *acceptance*, *monotonicity*, and *substitu-*

*tability*) by *unavailable-type-invariance*[13], *weak non-wastefulness*, *resource-monotonicity*, *truncation-invariance*[14], and *strategy-proofness*.

Theorem 2 characterizes choice-based IA mechanisms (where the choice structure satisfies *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*) by *weak non-wastefulness*, *resource-monotonicity*, *(group) non-bossiness*, *favoring-higher-ranks*, and *RR-UTI*. Hence, both characterizations share the properties *weak non-wastefulness* and *resource-monotonicity* and in both characterizations, the properties imply the choice structure to be *acceptant*, *monotonic*, and *substitutable*.

Choice-based DA mechanisms satisfy *RR-UTI* but neither *favoring-higher-ranks* nor *non-bossiness*. Choice-based IA mechanisms satisfy *truncation-invariance* but neither *unavailable-type-invariance* nor *strategy-proofness*. □

# 6 Choice-Based Immediate-Acceptance Mechanisms to promote Affirmative Action

The class of mechanisms characterized in Theorem 2 is considerably larger than the class of IA mechanisms characterized in Theorem 1. One way to see that is the following observation, which was also noted in Section 5: take any auxiliary choice function that is not responsive but satisfies *acceptance*, *monotonicity*, and *substitutability*, and associate each $B \in 2^N$ with it; then, the IA mechanism based on the induced choice function is not an IA mechanism, but belongs to the class characterized in Theorem 2. Put differently, the IA mechanism based on the induced choice function satisfies *weak non-wastefulness*, *resource-monotonicity*, *(group) non-bossiness*, *favoring-higher-ranks*, and *RR-UTI*, but violates *consistency*.

Although the above observation reveals that a considerably larger class of mechanisms is characterized in Theorem 2, it does not highlight the richness of the class enough. The choice-based mechanisms in the above observation are based on a choice structures that do not depend on $B$, in other words, for any real type, the choice at a competition step does not depend on who received a copy of the type in an earlier (non-competition) step. Yet, there are interesting applications where it may be desirable to make the choice depend on who received what before, and such objectives can possibly be achieved with mechanisms in the

[13] *Unavailable-type-invariance*: the chosen allocation depends only on preferences over the set of available types.

[14] *Truncation-invariance*: if an agent truncates his preference relation in a way such that his allotment remains acceptable at the truncated preference relation, then the allocation is the same at both profiles.

class of choice based IA mechanisms. One example is the object allocation problem when there are affirmative action objectives. In this section, we provide a class of mechanisms which are designed to "promote" a group of "minority" agents. We show that each of these mechanisms is a choice-based IA mechanism based on a choice structure satisfying *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*.

There is a group of *minority agents* $N^m \subseteq N$. If an agent is a minority agent, we denote him with a superscript $m$, such as $i^m \in N^m$. Also, given any set of agents $S \subseteq N$, we denote the set of minority agents in $S$ by $S^m$. We introduce a class of choice-based IA mechanisms, called *IA mechanisms with affirmative-action-target*, that take the affirmative action objective into consideration.[15] Each mechanism in this class is parameterized with a priority profile $\succ$ and a *target profile* $t \in \mathbb{N}^{|O|}$; the target profile associates each real type $x$ with a target number of minority agents $t_x \in \mathbb{N}$ ($t_x$ is independent from the capacity of $x$, and possibly greater than the capacity). At each problem, the mechanism allocates objects based on the immediate $\succ$-acceptance algorithm, with the exception that, at a step at which there is competition for an object, the minority agents are given priority over the non-minority agents until the target is reached, taking the number of minority agents who were accepted before into account.

Formally, given $(\succ_x)_{x\in O}$ and $t = (t_x)_{x\in O}$, for each real type $x \in O$, let a choice function with an affirmative-action-target $\mathcal{C}^{\succ_x, t_x} : \mathcal{D}_x \to 2^N \setminus \emptyset$ be defined as follows. For each $(B, S, l) \in \mathcal{D}_x$,

- if $|B^m| < t_x$, $x$ first chooses the $(t_x - |B^m|)$ highest $\succ_x$-priority agents in $S^m$, and then it chooses the highest $\succ_x$-priority agents among the remaining agents in $S$ until either the set of agents in $S$ is exhausted or the capacity $l$ is reached; $\mathcal{C}^{\succ_x, t_x}(B, S, l)$ denotes the set of chosen agents,
- otherwise, $x$ chooses the highest $\succ_x$-priority agents in $S$ until either the set of agents in $S$ is exhausted or the capacity $l$ is reached; $\mathcal{C}^{\succ_x, t_x}(B, S, l)$ denotes the set of chosen agents.

An *IA mechanism with affirmative-action-target* is a choice-based IA mechanism associated with a choice structure consisting of choice functions with an affirmative-action-target.

[15] IA mechanisms with affirmative-action-target are the counterparts of deferred acceptance mechanisms that use "reserves" introduced by Hafalir et al. (2013). Achieving affirmative action by imposing reserves is also known as a reserve-based affirmative action policy.

**IA Mechanisms with Affirmative-Action-Target:** A mechanism $\varphi : \mathcal{R}^N \times \mathcal{Q} \to \mathcal{A}$ is an *IA mechanism with affirmative-action-target* if there is a priority structure $\succ$ and a target profile $t = (t_x)_{x \in O} \in \mathbb{N}^{|O|}$ such that for each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = IA^{\mathcal{C}}(R, q)$ where for each $x \in O$, $\mathcal{C}_x = \mathcal{C}^{\succ_x, t_x}$.

Next, we illustrate the working of an IA mechanism with affirmative-action-target with an example.

**Example 3.** Let $N \equiv \{1, 2, 3, 4, 5^m, 6^m\}$, $O \equiv \{x, y, z\}$, $q_x = 3$, $q_y = 1$, and $q_z = 2$. Let $t_x = 2$ and $t_y = t_z = 0$. Let the profile $R$ and the priority profile $\succ$ be as depicted below.

| $R_1$ | $R_2$ | $R_3$ | $R_4$ | $R_{5^m}$ | $R_{6^m}$ |
|---|---|---|---|---|---|
| $x$ | $y$ | $y$ | $y$ | $y$ | $y$ |
| $y$ | $x$ | $x$ | $x$ | $x$ | $x$ |
| $z$ | $z$ | $z$ | $z$ | $z$ | $z$ |

| $\succ_x$ | $\succ_y$ | $\succ_z$ |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |
| $5^m$ | $5^m$ | $5^m$ |
| $6^m$ | $6^m$ | $6^m$ |

The steps of the associated IA algorithm with affirmative-action-target are depicted below: the allotments are marked in boldface. Note that at the second step, $x$ rejects 3 and 4, and accepts $5^m$ and $6^m$ although they have lower priority.

| Step | $x$ | $y$ | $z$ |
|---|---|---|---|
| 1 | **1** | **2**, 3, 4, $5^m$, $6^m$ | |
| 2 | 3, 4, $\mathbf{5^m}$, $\mathbf{6^m}$ | | |
| 3 | | | **3**, **4** |

Now, everything else the same, if $R_{6^m}$ is such that $x$ is top-ranked, at the first step of the associated IA algorithm with affirmative-action-target agents 1 and $6^m$ are accepted by $x$ and at the second step agent $5^m$ is also accepted by $x$ while agents 3 and 4 are rejected (hence, the allocation did not change). However, if $R_{6^m}$ is such that $z$ is top-ranked, then at the first step of the associated IA algorithm with affirmative-action-target agent 1 is accepted by $x$ and agent $6^m$ is accepted by $z$ and at the second step agent $5^m$ is still accepted by $x$ while agent 3 is also accepted and agent 4 is still rejected. □

We conclude the section by showing that each IA mechanism with affirmative-action-target indeed belongs to the class characterized in Theorem 2.

**Proposition 2.** *Each choice function with an affirmative-action-target satisfies* acceptance, monotonicity, substitutability, sequence-monotonicity, *and* sequence-substitutability. *Consequently, each IA mechanism with affirmative-action-target satisfies* weak non-wastefulness, resource-monotonicity, *group* non-bossiness, favoring-higher-ranks, *and* rank-respecting unavailable-type-invariance.

The following example illustrates an IA mechanism with an affirmative-action-target that violates *consistency*.

**Example 4.** Let $N \equiv \{1, 2^m, 3^m\}$, $O \equiv \{x, y\}$, $q_x = 2$ and $q_y = 1$. Let $t_x = 1$ and $t_y = 0$. Let the profile $R$ and the priority profile $\succ$ be as depicted below.

| $R_1$ | $R_{2^m}$ | $R_{3^m}$ | $\succ_x$ | $\succ_y$ |
|---|---|---|---|---|
| $x$ | $x$ | $x$ | 1 | 1 |
| $y$ | $y$ | $y$ | 2 | 2 |
| | | | 3 | 3 |

Note that, for the problem $(R, q)$, the IA mechanism with the affirmative-action-target assigns $x$ to agents 1 and 2, and agent 3 receives $y$. Yet, when agent 2 leaves with his allotment, at the new problem the IA mechanism with the affirmative-action-target assigns $x$ to agent 3, and agent 1 receives $y$, which is a violation of *consistency*. □

**Remark 4** (Choice-Based IA Mechanisms with Affirmative-Action-Quota)**.**
Another way to promote minority agents is to impose quotas. Accordingly, we can introduce *choice-based IA mechanisms with affirmative-action-quota* that take the affirmative action objective into consideration.[16] Each mechanism in this class is parameterized with a priority profile $\succ$ and a *quota profile* $\kappa \in \mathbb{N}^{|O|}$; the quota profile associates each real type $x$ with a maximal number of nonminority agents $\kappa_x \in \mathbb{N}$ who can receive a copy of the object. At each problem, the mechanism allocates objects based on the immediate $\succ$-acceptance algorithm, with the exception that, at a step at which there is competition for an object, the number of accepted nonminority agents does not exceed the nonminority quota, taking the number of minority agents who were accepted before into account. It is straightforward to see that the associated choice functions satisfy *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*. Consequently, each IA mechanism

[16] IA mechanisms with affirmative-action-quota are the counterparts of deferred acceptance mechanisms that use quotas introduced by Abdulkadiroğlu and Sönmez (2003). Achieving affirmative action by imposing quotas and requiring that no object be assigned to more nonminority agents than its quota is also known as *quota-based affirmative action policy*.

with affirmative-action-quota satisfies *weak non-wastefulness*, *resource-monotonicity*, *group non-bossiness*, *favoring-higher-ranks*, and *rank-respecting unavailable-type-invariance*. □

# 7 Related Literature

Our study contributes to the literature on school choice initiated by Abdulkadiroğlu and Sönmez (2003). IA mechanisms have been widely studied, and besides our Theorem 1, Kojima and Ünver (2014) and Afacan (2013) also provide characterizations. Afacan (2013) takes a priority profile as given, whereas Kojima and Ünver (2014) and our paper derive a priority profile from the properties of a mechanism. In Section 4, we discuss in detail how our paper relates to Kojima and Ünver (2014).

Abdulkadiroğlu and Sönmez (2003) discuss three classes of well-known and "classic" mechanisms to assign types to agents: the deferred acceptance mechanisms (first introduced for marriage markets by Gale and Shapley, 1962), the immediate acceptance mechanisms that we have discussed already, and top trading cycles mechanisms (first introduced, but credited to David Gale, for object reallocation problems, also known as housing markets, by Shapley and Scarf, 1974). All these mechanisms have in common that they are based on linear orders of types over agents, interpreted either as priorities when tie-breaking between agents is necessary or as priorities of ownership for trade. Furthermore, Abdulkadiroğlu and Sönmez (2003) already pointed out some desirable properties of deferred acceptance (e.g., elimination of *justified envy* and *strategy-proofness*) as well as top trading cycles mechanisms (e.g., *Pareto efficiency* and *strategy-proofness*).

There are several studies in the literature that then started from sets of desirable properties that resulted in characterizations of larger classes of mechanisms. For the deferred acceptance mechanisms, Kojima and Manea (2010) and Ehlers and Klaus (2016) characterize choice-based deferred acceptance mechanisms and for the top trading cycles mechanisms, Pápai (2000) and Pycia and Ünver (2017) characterize classes of mechanisms that include the top trading cycles mechanisms. In all these studies, the structure the mechanism is based on (a choice structure in Kojima and Manea, 2010, and Ehlers and Klaus, 2016; an inheritance tree in Pápai, 2000; a control-rights structure in Pycia and Ünver, 2017) is derived from the properties of the mechanism, as in our study. Our paper is the first to provide such a result for immediate acceptance mechanisms by introducing and characterizing choice-based immediate acceptance mechanisms that are obtained by extending priority structures to choice structures.

In school choice, how to incorporate affirmative action has been receiving increasing attention in the literature. A version of the immediate acceptance mechanisms that uses quotas was in use in the City of Boston before 1999 (Abdulkadiroğlu and Sönmez, 2003), and versions of deferred acceptance and top trading cycles mechanisms that use quotas were proposed by Abdulkadiroğlu and Sönmez (2003). Recent findings show that affirmative action policies through quotas may have perverse welfare consequences (Kojima, 2012) and that a version of deferred acceptance mechanisms that uses "reserves" may improve upon the quota version (Hafalir et al., 2013). The mechanisms in Section 6 that we use to illustrate the richness of the class of IA mechanisms that we characterize in Theorem 2 are nothing but versions of immediate acceptance mechanisms that use reserves. In a recent paper, Afacan and Salman (2016) also study the reserve version of immediate acceptance mechanisms. Their objective is to understand whether such mechanisms satisfy a certain property, *responsiveness to affirmative action* (Kojima, 2012; Doğan, 2016). In contrast to their direct approach to affirmative action, neither does our model contain nor do our properties refer to any affirmative action concerns: we obtain (IA) mechanisms that can take affirmative action concerns into account as members of the general class of mechanisms that we characterize.

# 8 Conclusion

We have introduced a new property, *rank-respecting unavailable-type-invariance* (*RR-UTI*), which may be desirable for a social planner who deems the relative rank of a school in students' preferences crucial in determining who receives a seat at which school. We have characterized two classes of mechanisms, IA mechanisms and choice-based IA mechanisms, by *RR-UTI* and some already known properties. Our characterization results shed some light on the principles underlying IA mechanisms and also on what it takes, in terms of the underlying principles, to incorporate new design objectives, such as affirmative action. Note that our aim was not to tell a social planner which particular mechanism to use or which properties should be deemed desirable. In fact, there exist school districts that deem taking the relative rank of a school in students' preferences into consideration *undesirable*.[17] Our aim was to provide a detailed description of the implications of properties that may be of

[17] According to the School Admissions Code in England, a *first preference first* system is any "oversubscription criterion that gives priority to children according to the order of other schools named as a preference by their parents, or only considers applications stated as a first preference", and since 2007, such systems are banned in England, especially because they are vulnerable to strategic manipulation (Pathak and Sönmez, 2013).

interest to a social planner. In addition, by understanding which properties characterize IA mechanisms in comparison to other mechanisms (e.g., in Remark 3 we discuss the properties that IA and DA mechanisms have in common versus the properties that separate them), we help completing the picture of compatibility and incompatibility of desirable resource allocation properties and which set of mechanisms is characterized by which set of properties.

# References


Abdulkadiroğlu, A. and T. Sönmez (2003). School choice: A mechanism design approach. *American Economic Review 93*, 729–747.

Afacan, M. O. (2013). Alternative characterizations of Boston mechanism. *Mathematical Social Sciences 66*, 176–179.

Afacan, M. O. and U. Salman (2016). Affirmative actions: The Boston mechanism case. *Economics Letters 141*, 95–97.

Aizerman, M. A. and A. V. Malishevski (1981). General theory of best variants choice: Some aspects, automatic control. *IEEE Transactions on Automatic Control 26*, 1030–1040.

Aygün, O. and T. Sönmez (2013). Matching with contracts: Comment. *American Economic Review 103*, 2050–2051.

Balinski, M. and T. Sönmez (1999). A tale of two mechanisms: Student placement. *Journal of Economic Theory 84*, 73–94.

Barberà, S. and M. Jackson (1995). Strategy-proof exchange. *Econometrica 63*, 51–87.

Calsamiglia, C. and M. Güell (2017). Priorities in school choice: The case of the Boston mechanism in Barcelona. *Working paper*.

Chambers, C. P. and M. B. Yenmez (2017). Choice and matching. *American Economic Journal: Microeconomics 9*, 126–147.

Chun, Y. and W. Thomson (1988). Monotonicity properties of bargaining solutions when applied to economics. *Mathematical Social Sciences 15*, 11–27.

Doğan, B. (2016). Responsive affirmative action in school choice. *Journal of Economic Theory 165*, 69–105.

Ehlers, L. and B. Klaus (2014). Strategy-proofness makes the difference: Deferred-acceptance with responsive priorities. *Mathematics of Operations Research 39*, 949–966.

Ehlers, L. and B. Klaus (2016). Object allocation via deferred-acceptance: Strategy-proofness and comparative statics. *Games and Economic Behavior 97*, 128–146.

Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *American Mathematical Monthly 69*, 9–15.

Hafalir, I., B. Yenmez, and M. Yildirim (2013). Effective affirmative action in school choice. *Theoretical Economics 8*, 325–363.

Kelso, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica 50*, 1483–1504.

Kojima, F. (2012). School choice: Impossibilities for affirmative action. *Games and Economic Behavior 75*, 685–693.

Kojima, F. and M. Manea (2010). Axioms for deferred acceptance. *Econometrica 78*, 633–653.

Kojima, F. and U. Ünver (2014). The "Boston" school choice mechanism: An axiomatic approach. *Economic Theory 55*, 515–544.

Pápai, S. (2000). Strategyproof assignment by hierarchical exchange. *Econometrica 68*, 1403–1433.

Pathak, P. and T. Sönmez (2013). School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation. *American Economic Review 103*, 80–106.

Pycia, M. and U. Ünver (2017). Incentive compatible allocation and exchange of discrete resources. *Theoretical Economics 12*, 287–329.

Roth, A. E. (1984). Stability and polarization of interests in job matching. *Econometrica 52*, 47–57.

Satterthwaite, M. and H. Sonnenschein (1981). Strategy-proof allocation mechanisms at differentiable points. *Review of Economic Studies 48*, 587–597.

Shapley, L. and H. Scarf (1974). On cores and indivisibility. *Journal of Mathematical Economics 1*, 23–28.

Thomson, W. (2011). Strategy-proof allocation rules. *Working Paper*.

Thomson, W. (2015). *Consistent Allocation Rules*. Cambridge University Press. forthcoming.

Thomson, W. (2016). Non-bossiness. *Social Choice and Welfare 47*, 665–696.

# Appendix

# A Proofs

***Proof of Lemma 1***. Let $\varphi$ be a mechanism satisfying *resource-monotonicity*. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Let $q^0$ be such that for each $x \in O$, $q^0_x = 0$. Note that for each $i \in N$, $\varphi_i(R, q^0) = \emptyset$. Then, by *resource-monotonicity*, for each $i \in N$, $\varphi_i(R, q)\ R_i\ \emptyset$. □

***Proof of Lemma 2***. Let $\varphi$ be a mechanism satisfying *consistency*. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Let $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ be such that for each $i \in S$, $\varphi_i(R, q) = \varphi_i((R'_S, R_{-S}), q)$.

Let $R^*$ be a profile obtained from $R$ by changing the preference relation of each agent in $S$ to a preference relation for which the null object is top-ranked; and let $q^*$ be the capacity vector obtained by subtracting the aggregate allotments of $S$ at $\varphi(R, q)$. Formally, $R^*$ is such that for each $i \in S$, $A(R^*_i) = \emptyset$ and $R^*_{-S} = R_{-S}$; and for each $x \in O$, $q^*_x = q_x - |\{i \in S : \varphi_i(R, q) = x\}|$. By *consistency*, for each $j \in N \setminus S$, $\varphi_j(R, q) = \varphi_j(R^*, q^*)$.

Let $R^{**}$ be a profile obtained from $((R'_S, R_{-S}), q)$ by changing the preference relation of each agent in $S$ to a preference relation for which the null object is top-ranked such that $R^{**}_S = R^*_S$; and let $q^{**}$ be the capacity vector obtained by subtracting the aggregate allotments of $S$ at $\varphi((R'_S, R_{-S}), q)$. Formally, $R^{**}$ is such that $R^{**}_S = R^*_S$ and $R^{**}_{-S} = R_{-S}$, and for each $x \in O$, $q^{**}_x = q_x - |\{i \in S : \varphi_i((R'_S, R_{-S}), q) = x\}|$. By *consistency*, for each $j \in N \setminus S$, $\varphi_j((R'_S, R_{-S}), q) = \varphi_j(R^{**}, q^{**})$.

Note that $(R^*, q^*) = (R^{**}, q^{**})$. Hence, for each $j \in N \setminus S$, $\varphi_j(R, q) = \varphi_j(R^*, q^*) = \varphi_j(R^{**}, q^{**}) = \varphi_j((R'_S, R_{-S}), q)$. Then, since for each $i \in S$, $\varphi_i(R, q) = \varphi_i((R'_S, R_{-S}), q)$, $\varphi(R, q) = \varphi((R'_S, R_{-S}), q)$. □

***Proof of Lemma 3***. Let $\varphi$ be a mechanisms satisfying the three properties in the hypothesis of the lemma. By Lemma 1, it is *individually rational*. Suppose that $\varphi$ is not *non-wasteful*. Since $\varphi$ is *individually rational*, *weakly non-wasteful*, and *not non-wasteful*, there are $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $x \in O_+(q)$, and $i \in N$ such that $x\ P_i\ \varphi_i(R, q)\ P_i\ \emptyset$ and $|\varphi(R, q)(x)| < q_x$.

Let $R'$ be a profile obtained from $R$ by changing the preference relation of each agent other than $i$ to a preference relation for which the null object is top-ranked; and let $q'$ be the capacity vector obtained by subtracting the aggregate allotments of $N \setminus \{i\}$ at $\varphi(R, q)$. Formally, $R'$ is such that $R'_i = R_i$ and for each $j \in N \setminus \{i\}$, $A(R'_j) = \emptyset$; and for each $x \in O$, $q'_x = q_x - |\{k \in N \setminus \{i\} : \varphi_k(R, q) = x\}|$. By *consistency*, $\varphi_i(R', q') = \varphi_i(R, q)\ P_i\ \emptyset$. Also, $q'_x \geq 1$. Let $q''$ be such that there is only one copy of $x$ available and nothing else, i.e., $q''_x = 1$ and $q''_y = 0$ for each $y \in O \setminus \{x\}$. By *resource-monotonicity*, $x\ P_i\ \varphi_i(R', q')\ R_i\ \varphi_i(R', q'') = \emptyset$; which contradicts that $\varphi$ is *weakly non-wasteful*. □

***Proof of Lemma 4.*** Let $\varphi$ be a mechanisms satisfying the four properties in the hypothesis of the lemma. By Lemma 1, it is *individually rational*. Suppose that $\varphi$ is not *non-wasteful*. Since $\varphi$ is *individually rational*, *weakly non-wasteful*, and *not non-wasteful*, there are $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $x \in O_+(q)$, and $i \in N$ such that $x\, P_i\, \varphi_i(R, q)\, P_i\, \emptyset$ and $|\{j \in N : \varphi_j(R, q) = x\}| < q_x$.

Consider the following sets of agents, which form a partition of $N \setminus \{i\}$ when each of them is non-empty.

- Agents who receive the null object: $T_1 \equiv \{j \in N : \varphi_j(R, q) = \emptyset\}$
- Agents who do not receive the top-ranked type or $\emptyset$: $T_2 \equiv \{j \in N \setminus (T_1 \cup \{i\}) : R_j(\varphi_j(R, q)) \neq 1\}$
- Agents who receive the top-ranked type: $T_3 \equiv \{j \in N : R_j(\varphi_j(R, q)) = 1\}$

***Step 1:*** Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent in $T_1$ to a preference relation for which the null object is top-ranked. Formally, for each $j \in T_1$, $A(R^1_j) = \emptyset$ and for each $j \notin T_1$, $R^1_j = R_j$. By *individual rationality*, if $A(R^1_j) = \emptyset$, then $\varphi_j(R^1, q) = \emptyset$. By *group non-bossiness*, $\varphi(R^1, q) = \varphi(R, q)$. Alternatively, we can transform profile $R$ one preference relation at a time into $R^1$ and invoke *non-bossiness* at each step to show that $\varphi(R^1, q) = \varphi(R, q)$.

***Step 2:*** Let $R^2$ be a profile obtained from $R^1$ by changing the preference relation of each agent in $T_2$ to a preference relation for which his allotment is top-ranked and the null object is second-ranked. Formally, for each $j \in T_2$, $R^2_j(\varphi_j(R^1, q)) = 1$ and $R^2_j(\emptyset) = 2$, and for each $j \notin T_2$, $R^2_j = R^1_j$. We show that for each $j \in T_2$, $\varphi_j(R^2, q) = \varphi_j(R^1, q)$. Let $\varphi_j(R^1, q) = y$.

By *favoring-higher-ranks*, for each $k \in N$ such that $R^1_k(y) = 1$, we have $\varphi_k(R^1, q) = y$. Then, $|\{k \in N : R^2_k(y) = 1\}| \leq q_y$. Suppose by contradiction that $\varphi_j(R^2, q) \neq y$. By *individual rationality*, $\varphi_j(R^2, q) = \emptyset$. If $|\{k \in N : \varphi_k(R^2, q) = y\}| < q_y$, then we have a contradiction to *weak non-wastefulness*. If $|\{k \in N : \varphi_k(R^2, q) = y\}| = q_y$, then we have a contradiction to *favoring-higher-ranks*. Hence, for each $j \in T_2$, $\varphi_j(R^2, q) = \varphi_j(R^1, q)$. By *group non-bossiness*, $\varphi(R^2, q) = \varphi(R^1, q)$. Alternatively, we can transform profile $R^1$ one preference relation at a time into $R^2$ and invoke *non-bossiness* at each step to show that $\varphi(R^2, q) = \varphi(R^1, q)$.

***Step 3:*** Let $T_3' \subseteq T_3$ be the agents in $T_3$ who top-rank (and receive) a type in $L(R_i, \varphi_i(R, q))$. Since $x \; P_i \; \varphi_i(R, q)$, for any type $y \in L(R_i, \varphi_i(R, q))$, $R_i(y) \neq 1$. Let $R^3$ be a profile obtained from $R^2$ by changing the preference relation of each agent in $T_3'$ to a preference relation for which he stills top-ranks his allotment and second-ranks the null object. Formally, for each $j \in T_3'$, $R_j^3(\varphi_j(R, q)) = 1$ and $R_j^3(\emptyset) = 2$ and for each $j \notin T_3'$, $R_j^3 = R_j^2$. We show that for each $j \in T_3'$, $\varphi_j(R^3, q) = \varphi_j(R^2, q)$.

Let $j \in T_3'$ and let $\varphi_j(R^2, q) = y$. Note that $\{k \in N : R_k^2(y) = 1\} = \{k \in N : \varphi_k(R^2, q) = y\}$. Suppose by contradiction that $\varphi_j(R^3, q) \neq y$. By *individual rationality*, $\varphi_j(R^3, q) = \emptyset$. If $|\{k \in N : \varphi_k(R^3, q) = y\}| < q_y$, then we have a contradiction to *weak non-wastefulness*. If $|\{k \in N : \varphi_k(R^3, q) = y\}| = q_y$, then we have a contradiction to *favoring-higher-ranks*. Hence, for each $j \in T_3'$, $\varphi_j(R^3, q) = \varphi_j(R^2, q)$. By *group non-bossiness*, $\varphi(R^3, q) = \varphi(R^2, q)$. Alternatively, we can transform profile $R^2$ one preference relation at a time into $R^3$ and invoke *non-bossiness* at each step to show that $\varphi(R^3, q) = \varphi(R^2, q)$.

***Step 4:*** Note that $R_i^3 = R_i$ and $|\{j \in N : \varphi_j(R^3, q) = x\}| < q_x$ and therefore $|\{j \in N : \varphi_j(R^3, q) \notin L(R_i^3, \varphi_i(R^3, q))\}| < \sum_{y \notin L(R_i^3, \varphi_i(R, q))} q_y$. Now, let $q'$ be the capacity profile obtained from $q$ by removing all the real types in $L(R_i, \varphi_i(R, q))$. Formally, for each $y \in L(R_i, \varphi_i(R, q)) \setminus \{\emptyset\}$, $q_y' = 0$, and for each real type $z \notin L(R_i, \varphi_i(R, q))$, $q_z' = q_z$. By *resource-monotonicity* and *individual rationality*, for each $j \in N$ such that $\varphi_j(R^3, q) \in L(R_i^3, \varphi_i(R^3, q))$, $\varphi_j(R^3, q') = \emptyset$, and in particular $\varphi_i(R^3, q') = \emptyset$. Then, $|\{j \in N : \varphi_j(R^3, q') \notin L(R_i^3, \varphi_i(R^3, q'))\}| < \sum_{y \notin L(R_i^3, \varphi_i(R, q'))} q_y'$, which contradicts that $\varphi$ satisfies *weak non-wastefulness*. □

***Proof of Lemma 5***. Let $\varphi$ be a mechanisms satisfying the properties in the hypothesis of the lemma. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $x \in O$, and pair $i, j \in N$ be such that $R_i(x) = R_j(x)$, $\varphi_i(R, q) = x$, and $x \; P_j \; \varphi_j(R, q)$. Let $(R', q') \in \mathcal{R}^N \times \mathcal{Q}$ be such that $i$ and $j$ find only $x$ acceptable, every other agent top-ranks the null object, and there is only one copy of $x$ available and nothing else. Formally, $R'$ is such that $A(R_i') = A(R_j') = x$ and for each $k \in N \setminus \{i, j\}$, $A(R_k') = \emptyset$; and $q'$ is such that $q_x' = 1$ and $q_y = 0$ for each $y \in O \setminus \{x\}$. We show that $\varphi_i(R', q') = x$, which implies that $\varphi$ satisfies *uniform pairwise-tie-breaking*.

***Step 1:*** Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent other than $i$ and $j$ to a preference relation for which the null object is top-ranked and $x$ is bottom-ranked; and let $q^1$ be the capacity vector obtained by subtracting the aggregate allotments of $N \setminus \{i,j\}$ at $\varphi(R,q)$. Formally, $R^1$ is such that $R^1_i = R_i$, $R^1_j = R_j$, and for each $k \in N \setminus \{i,j\}$, $R^1_k$ is such that $A(R^1_k) = \emptyset$ and $y\ R^1_k\ x$ for each $y \in O$; and $q^1$ is such that for each $y \in O$, $q^1_y = q_y - |\{k \in N \setminus \{i,j\} : \varphi_k(R,q) = y\}|$. By *consistency*, $\varphi_i(R^1,q^1) = x$ and $x\ P^1_j\ \varphi_j(R^1,q^1) = \varphi_j(R,q)$.

***Step 2:*** Let $(R^2,q^2)$ be the problem obtained from $(R^1,q^1)$ by removing all the real types except for one copy of $x$. Formally, $R^2 = R^1$ and $q^2 = q'$. By *resource-monotonicity*, invoking Lemma 1, $\varphi$ is *individually rational*, and therefore $x\ P^1_i\ \emptyset$ and $x\ P^1_j\ \varphi_j(R^1,q^1)\ R^1_j\ \emptyset$. Then, by *resource-monotonicity*, $\varphi_j(R^2,q^2) = \emptyset$. By *weak non-wastefulness*, *resource-monotonicity*, and *consistency*, invoking Lemma 3, $\varphi$ is *non-wasteful*. Therefore $\varphi_i(R^2,q^2) = x$.

***Step 3:*** Let $(R^3,q^3)$ be the problem obtained from $(R^2,q^2)$ by changing $i$'s and $j$'s preferences to $R'_i$ and $R'_j$, respectively. Formally, $R^3_i = R'_i$, $R^3_j = R'_j$, $R^3_{-\{i,j\}} = R^2_{-\{i,j\}}$, and $q^3 = q^2 = q'$. By *weak uniform tie-breaking* (*RR-UTI*), $\varphi_i(R^3,q^3) = x$.

***Step 4:*** Let $(R^4,q^4)$ be the problem obtained from $(R^3,q^3)$ by changing $R^2_{-\{i,j\}}$ to $R'_{-\{i,j\}}$. Note that $(R^4,q^4) = (R',q')$. By *consistency*, $\varphi_i(R',q') = x$, which completes the proof. □

***Proof of Theorem 1. If part:*** That an IA mechanism $\varphi$ satisfies *(weak) non-wastefulness*, *resource-monotonicity*, *consistency*, and *favoring-higher-ranks* is clear; it also follows from Kojima and Ünver (2014). Let $\varphi$ be an IA mechanism with priority structure $\succ$. To see that $\varphi$ satisfies *rank-respecting unavailable-type-invariance* (and hence *weak uniform tie-breaking*), let $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$ and $R' \in \mathcal{R}^N$ be such that $R|_{O_+(q)\cup\{\emptyset\}} = R'|_{O_+(q)\cup\{\emptyset\}}$ and for all $x \in O_+(q)$ and for all $i,j \in N$, $R_i(x) \geq R_j(x)$ if and only if $R'_i(x) \geq R'_j(x)$.

Suppose that $\varphi(R,q) \neq \varphi(R',q)$. Let $t$ be the first step in the immediate $\succ$-acceptance algorithm at $(R',q)$ at which an agent's allotment differs at $\varphi(R,q)$ and $\varphi(R',q)$; let $i \in N$ be one of these agents and, without loss of generality, suppose that $\varphi_i(R',q)\ P_i\ \varphi_i(R,q)$. Hence, $\varphi_i(R',q) \in O$. Let $\varphi_i(R',q) \equiv x$. By *non-wastefulness*, there exists an agent $j$ who was assigned $x$ at $\varphi(R,q)$ and is assigned a different type at $\varphi(R',q)$. Since the relative rankings of $x$ are the same at $R$ and $R'$ and agent $j$ did obtain $x$ at $\varphi(R,q)$, agent $j$ had a higher priority for type $x$ than agent $i$ and thus prefers his allotment at $\varphi(R',q)$ to that at $\varphi(R,q)$; contradicting the assumption that $t$ is the first step in the immediate $\succ$-acceptance algorithm at $(R',q)$ at which an agent is assigned a type that he prefers to his allotment at $\varphi(R,q)$.

***Only if part:*** Let $\varphi$ be a mechanism satisfying the properties in the theorem. By Lemmas 1, 3, and 5, it also satisfies *individual rationality*, *non-wastefulness*, and *uniform pairwise-tie-breaking*.

For each $x \in O$, consider the following binary relation $\succ_x$ on $N$. For each distinct pair $i, j \in N$, $i \neq j$, let $i \succ_x j$ if and only if $i$ gets $x$ at every problem at which $i$ and $j$ find only $x$ acceptable, every other agent top-ranks the null object, and there is only one copy of $x$ available and nothing else. By *uniform pairwise-tie-breaking*, for each distinct pair $i, j \in N$, $i \neq j$, either $i \succ_x j$ or $j \succ_x i$. We claim that $\succ_x$ is a transitive relation. Suppose not, i.e., suppose that there are distinct $i, j, k \in N$ such that $i \succ_x j \succ_x k \succ_x i$. Consider a profile where $i, j, k$ find only $x$ acceptable, and every other agent top-ranks the null object, and there is one unit $x$ and nothing else. Since $k \succ_x i$, $i$ does not get $x$; since $i \succ_x j$, $j$ does not get $x$; since $j \succ_x k$, $k$ does not get $x$; and by *individual rationality*, no agent in $N \setminus \{i, j, k\}$ gets $x$. But then, no agent gets $x$, which contradicts *weak non-wastefulness*.

Now, $\varphi$ being an IA mechanism based on $(\succ_x)_{x \in O}$ is a straightforward consequence of *uniform pairwise-tie-breaking* together with *favoring-higher-ranks* and *non-wastefulness*. □

***Proof of Lemma 6.*** Let $\varphi$ be a mechanism satisfying *rank-respecting invariance*. Let $x \in O$, $i, j \in N$, and $q \in \mathcal{Q}$ such that $q_x = 1$ and $q_y = 0$ for each $y \in O \setminus \{x\}$, and preference profiles $R, R' \in \mathcal{R}^N$ such that $R_{-\{i,j\}} = R'_{-\{i,j\}}$, $R_i(x) = R_j(x)$, $x \in A(R_i) \cap A(R_j)$, $A(R'_i) = A(R'_j) = \{x\}$, and for each $k \in N \setminus \{i, j\}$, $R_k(x) = R'_k(x) = |O| + 1$.

Assume that $\varphi_i(R, q) = x$. Hence, for all $k \in N \setminus \{i\}$, $\varphi_k(R, q) = \emptyset$. Thus, $R'$ is a monotonic transformation of $R$ at $\varphi(R, q)$. Note that $U_i(R, \varphi(R, q)) = U_i(R', \varphi(R, q)) = \{i, j\}$ and $V_i(R, \varphi(R, q)) = V_i(R', \varphi(R, q)) = \emptyset$. Since agent $i$ is the only agent such that $\varphi_i(R, q) \in O$, $R'$ is a rank-respecting monotonic transformation of $R$ at $\varphi(R, q)$. Then, *rank-respecting invariance* implies $\varphi(R', q) = \varphi(R, q)$. In particular, $\varphi_i(R', q) = \varphi_i(R, q)$, which proves *weak uniform tie-breaking*. □

***Proof of Proposition 1.*** Let $\varphi$ be a mechanism and $\mathcal{C}$ be a choice structure satisfying the properties in the hypothesis of the proposition.

**(*i.*)** ***If part:*** Suppose that $\mathcal{C}$ is *sequence-monotonic*. Consider a problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, a real type $x \in O$ and a capacity vector $q' \in \mathcal{Q}$ such that $q'_x = q_x + 1$ and $q'_y = q_y$ for each $y \in O \setminus \{x\}$. We prove *resource-monotonicity* by showing that for all $i \in N$, $\varphi_i(R, q') R_i \varphi_i(R, q)$ when going through the steps of the immediate $\mathcal{C}$-acceptance algorithm.

If $\varphi(R, q) = \varphi(R, q')$, then we are done. Let $t$ be the first step in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $(R, q')$ at which an agent receives different allotments. All agents who applied to types before Step $t$ and agents who apply to types $y \in (O \setminus \{x\}) \cup \{\emptyset\}$ at Step $t$ receive the same allotments. Note that at Step $t$, some agent can newly receive $x$ because one more copy is available. At problem $(R, q)$, let $B$ be the set of agents who have received a copy of type $x$ before Step $t$ and let $S$ be the set of agents applying to type $x$ at Step $t$. Note that $q'_x - |B| = q_x - |B| + 1$. By *monotonicity*, $\mathcal{C}_x(B, S, q_x - |B|) \subseteq \mathcal{C}_x(B, S, q'_x - |B|)$. Hence, for any agent $i \in N$ who received type $x$ at problem $(R, q)$ at Step $t$, it follows that $\varphi_i(R, q) = \varphi_i(R, q') = x$. For the agent who newly receives type $x$ at Step $t$, say agent $j \in N$, we have $\varphi_j(R, q')\ P_j\ \varphi_j(R, q)$. Hence, no agent who has received his allotment until Step $t$ is worse off.

By *substitutability*, in all steps after Step $t$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ at which agent $j$ applied to a type $y \in (O \setminus \{x\}) \cup \{\emptyset\}$ (possibly receiving it), the allotments made to other agents at $(R, q')$ do not change now that agent $j$ does not apply anymore. Let $t'$ be the next step at which an agent receives different allotments in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $(R, q')$. Since in the previous steps only one copy of type $x$ was assigned differently, at Step $t'$ exactly one real type $y \in O \setminus \{x\}$ is affected. Furthermore, by *substitutability*, a change in allotment at Step $t'$ happens because agent $j$ did receive type $y$ at $(R, q)$ and another agent $k \in N \setminus \{j\}$ can now receive this copy of type $y$ at Step $t'$. At problem $(R, q)$, let $B'$ be the set of agents who have received a copy of type $y$ before Step $t'$ and let $S'$ be the set of agents applying to type $y$ at Step $t'$. There are two cases:

***Case (a)*** At $(R, q)$, $j \in S'$, i.e., agent $j$ applied to type $y$ in Step $t'$ (and received it). Then, at $(R, q')$, $B'$ is the set of agents who have received a copy of type $y$ before Step $t'$ and $S' \setminus \{j\}$ is the set of agents applying to type $y$ at Step $t'$. By *substitutability*, for each $i \in N \setminus \{j\}$ such that $i \in \mathcal{C}_y(B', S', q_y - |B'|)$ we have $i \in \mathcal{C}_y(B', S' \setminus \{j\}, q_y - |B'|)$. Hence, for any agent $i \in N \setminus \{j\}$ who received type $y$ at $(R, q)$ at Step $t'$, it follows that $\varphi_i(R, q) = \varphi_i(R, q') = y$. For agent $k$ who newly receives type $y$ at Step $t$, we have $\varphi_k(R, q')\ P_k\ \varphi_k(R, q)$. All agents who apply to types $z \in (O \setminus \{y\}) \cup \{\emptyset\}$ at Step $t'$ receive the same allotments.

***Case (b)*** At problem $(R, q)$, $j \in B'$, i.e., agent $j$ applied to type $y$ before Step $t'$ (and received it at $(R, q)$). Then, at problem $(R, q')$, $B' \setminus \{j\}$ is the set of agents who have received a copy of type $y$ before Step $t'$ and $S'$ is the set of agents applying to type $y$ at Step $t'$. Note that $q'_y - |B' \setminus \{j\}| = q_y - |B'| + 1$, i.e., at problem $(R, q')$ at Step $t'$ an additional copy of type $y$ is available. By *sequence-monotonicity*, $\mathcal{C}_y(B', S', q_y - |B'|) \subseteq \mathcal{C}_y(B' \setminus \{j\}, S', q'_y - |B' \setminus \{j\}|)$.

Hence, for any agent $i \in N \setminus \{j\}$ who received type $y$ at $(R, q)$ at Step $t'$ it follows that $\varphi_i(R, q) = \varphi_i(R, q') = y$. For agent $k$ who newly receives type $y$ at Step $t'$, we have $\varphi_k(R, q') \; P_k \; \varphi_k(R, q)$. All agents who apply to types $z \in (O \setminus \{y\}) \cup \{\emptyset\}$ at Step $t'$ receive the same allotments.

Hence, no agent who has received his allotment until Step $t'$ is worse off. We now repeat the previous arguments at each step of the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $(R, q')$ at which an agent receives different allotments. Finally, *resource-monotonicity* for an arbitrary capacity vector $q' \in \mathcal{Q}$ with $q'_x \geq q_x$ for each $x \in O$ follows by applying the above argument repeatedly to increase capacities from $q$ to $q'$ one additional object at a time.

***(i.)* *Only if part:*** Suppose that $|O| \geq 3$ and that $\mathcal{C}$ is not *sequence-monotonic*. Then, there are $x \in O$, $(B, S, l) \in \mathcal{D}_x$, $i \in \mathcal{C}_x(B, S, l)$, and $j \in B$ such that $i \notin \mathcal{C}_x(B \setminus \{j\}, S, l+1)$. Let $y, z \in O \setminus \{x\}$ such that $y \neq z$.

Let $q \in \mathcal{Q}$ be such that $q_x = |B| + l$ and $q_{x'} = 0$ for each $x' \in O \setminus \{x\}$. Let $R \in \mathcal{R}^N$ such that agent $j$ top-ranks $y$, second-ranks $x$, and third-ranks the null object; each agent in $B \setminus \{j\}$ top-ranks $x$ and second-ranks the null object; each agent in $S$ top-ranks $z$, second-ranks $y$, third-ranks $x$, and fourth-ranks the null object; and each agent in $N \setminus (B \cup S)$ top-ranks the null object. Formally, $R$ is such that $R_j(y) = 1$, $R_j(x) = 2$, and $R_j(\emptyset) = 3$; for each $k \in B \setminus \{j\}$, $R_k(x) = 1$ and $R_j(\emptyset) = 2$; for each $k \in S$, $R_k(z) = 1$, $R_k(y) = 2$, $R_k(x) = 3$, and $R_k(\emptyset) = 4$; and for each $k \in N \setminus (B \cup S)$, $A(R_k) = \emptyset$. Since $\varphi$ is a choice-based IA mechanism, for each agent $k \in B$ we have that $\varphi_k(R, q) = x$. Next, since $i \in \mathcal{C}_x(B, S, l)$, we have $\varphi_i(R, q) = x$.

Let $q'$ be such that $q'_y = 1$ and $q'_{y'} = q_{y'}$ for each $y' \in O \setminus \{y\}$. Since $\varphi$ is a choice-based IA mechanism, $\varphi_j(R, q) = y$ and for each agent $k \in B \setminus \{j\}$ we have that $\varphi_k(R, q) = x$. Next, since $i \notin \mathcal{C}_x(B \setminus \{j\}, S, l+1)$, $\varphi_i(R, q') = \emptyset$, implying that $\varphi$ violates *resource-monotonicity*.

***(ii.)* *If part:*** Suppose that $\mathcal{C}$ is *sequence-substitutable*. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Let $S \subseteq N$ and $R'_S \in \mathcal{R}^S$ such that for each $i \in S$, $\varphi_i(R, q) = \varphi_i((R'_S, R_{-S}), q)$. We prove $\varphi(R, q) = \varphi((R'_S, R_S), q)$ in two parts.

***Part 1:*** Let $i \in N$ and denote $\varphi_i(R, q) = x$. We show that if we replace $R_i$ with another preference relation $\bar{R}_i$ where $i$ top-ranks $x$, the allocation does not change, that is, $\varphi(R, q) = \varphi((\bar{R}_i, R_{-i}), q)$.

Let $t$ be the step in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ at which agent $i$ receives $x$. Note that $R_i(x) = t$. Change agent $i$'s preference relation $R_i$ into $\bar{R}_i$ where type $x$ is top-ranked, i.e., $\bar{R}_i(x) = 1$. If $t = 1$, then clearly $\varphi(R, q) = \varphi((\bar{R}_i, R_{-i}), q)$. Hence, assume $1 = \bar{R}_i(x) < R_i(x) = t$. Let $\{y_1, \ldots, y_{t-1}\}$ be the set of real types that are ranked higher than $x$ at $R_i$ such that $R_i(y_1) = 1, \ldots, R_i(y_{t-1}) = t - 1$.

At *Step 1* in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $((\bar{R}_i, R_{-i}), q)$, two differences occur. First, agent $i$ applies to $x$ at $((\bar{R}_i, R_{-i}), q)$ but not at $(R, q)$. At problem $(R, q)$, let $S^1$ be the set of agents applying to type $x$ at Step 1. Then, at problem $((\bar{R}_i, R_{-i}), q)$ set $S^1 \cup \{i\}$ is the set of agents applying to type $x$ at Step 1. By *acceptance*, $\mathcal{C}_x(\emptyset, S^1, q_x) \cup \{i\} = \mathcal{C}_x(\emptyset, S^1 \cup \{i\}, q_x) = S^1 \cup \{i\}$. Hence, for all $j \in S^1 \cup \{i\}$, $\varphi_j(R, q) = \varphi_j((\bar{R}_i, R_{-i}), q) = x$. Thus, all agents who are assigned $x$ at Step 1 in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, are assigned $x$ at $((\bar{R}_i, R_{-i}), q)$.

Second, agent $i$ applies to $y_1$ at $(R, q)$ but not at $((\bar{R}_i, R_{-i}), q)$. At problem $(R, q)$, let $\bar{S}^1$ be the set of agents applying to type $y_1$ at Step 1. Then, at $((\bar{R}_i, R_{-i}), q)$ set $\bar{S}^1 \setminus \{i\}$ is the set of agents applying to type $y_1$ at Step 1. By *substitutability*, for each $j \in N \setminus \{i\}$ such that $j \in \mathcal{C}_{y_1}(\emptyset, \bar{S}^1, q_{y_1})$ we have $j \in \mathcal{C}_{y_1}(\emptyset, \bar{S}^1 \setminus \{i\}, q_{y_1})$. Thus, all agents who are assigned $y_1$ at Step 1 in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, are assigned $y_1$ at $((\bar{R}_i, R_{-i}), q)$.

Since all other objects in $(O \setminus \{x, y_1\}) \cup \{\emptyset\}$ receive the same proposals in Step 1, all agents who obtain their allotments at Step 1 in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain the same allotments at $((\bar{R}_i, R_{-i}), q)$.

At *Step k* ($k \in \{2, \ldots, t-1\}$) in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $((\bar{R}_i, R_{-i}), q)$, one difference occurs. Agent $i$ applies to $y_k$ at $(R, q)$ but not at $((\bar{R}_i, R_{-i}), q)$. Assume that at problem $(R, q)$ set $B^k$ is the set of agents who have received a copy of type $y_k$ before Step $k$ and set $S^k$ is the set of agents applying to type $y_k$ at Step $k$. Then, at problem $((\bar{R}_i, R_{-i}), q)$ set $B^k$ is the set of agents who have received a copy of type $y_k$ before Step $k$ and set $S^k \setminus \{i\}$ is the set of agents applying to type $y_k$ at Step $k$. By *substitutability*, for each $j \in N \setminus \{i\}$ such that $j \in \mathcal{C}_{y_k}(B^k, S^k, q_{y_k} - |B^k|)$ we have $j \in \mathcal{C}_{y_k}(B^k, S^k \setminus \{i\}, q_{y_k} - |B^k|)$. Thus, all agents who obtain allotment $y_k$ at Step $k$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain $y_k$ as allotment at $((\bar{R}_i, R_{-i}), q)$.

By *acceptance*, agents who apply to type $x$ at Step $k$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain $x$ as allotment at $((\bar{R}_i, R_{-i}), q)$. Since all other objects in $(O \setminus \{x, y_k\}) \cup \{\emptyset\}$ receive the same proposals in Step $k$, all agents who obtain their allotments at Step $k$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain the same allotments at $((\bar{R}_i, R_{-i}), q)$.

At *Step t* in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $((\bar{R}_i, R_{-i}), q)$, one difference occurs. Agent $i$ applies to $x$ at $(R, q)$ but not at $((\bar{R}_i, R_{-i}), q)$. At problem $(R, q)$, let $B^t$ be the set of agents who have received a copy of type $x$ before Step $t$ and let $S^t$ be the set of agents applying to type $x$ at Step $t$. Then, at $((\bar{R}_i, R_{-i}), q)$, $B^t \cup \{i\}$ is the set of agents who have received a copy of type $x$ before Step $t$ and $S^t \setminus \{i\}$ is the set of agents applying to type $x$ at Step $t$. Note that $q_x - |B^t \cup \{i\}| = (q_x - |B^t|) - 1$, i.e., at problem $((\bar{R}_i, R_{-i}), q)$ at Step $t$ one less copy of type $x$ is available. By *sequence-substitutability*, $\mathcal{C}_x(B^t, S^t, q_x - |B^t|) = \mathcal{C}_x(B^t \cup \{i\}, S^t \setminus \{i\}, q_x - |B^t \cup \{i\}|) \cup \{i\}$. Thus, all agents in $N \setminus \{i\}$ who obtain allotment $x$ at Step $t$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain $x$ as allotment at $((\bar{R}_i, R_{-i}), q)$.

Since all other objects in $(O \setminus \{x\}) \cup \{\emptyset\}$ receive the same proposals in Step $t$, all agents who obtain their allotments at Step $t$ in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$, obtain the same allotments at $((\bar{R}_i, R_{-i}), q)$. After Step $t$, there are no further differences between the immediate $\mathcal{C}$-acceptance algorithms at $(R, q)$ and $((\bar{R}_i, R_{-i}), q)$ and it follows that $\varphi(R, q) = \varphi((\bar{R}_i, R_{-i}), q)$.

***Part 2:*** Starting from $(R, q)$, for each $i \in S$ replace $R_i$ with another preference relation $\bar{R}_i$ where $i$ top-ranks $\varphi_i(R, q)$, i.e., $\bar{R}_i(\varphi_i(R, q)) = 1$. Denote the resulting problem by $((\bar{R}_S, R_S), q)$. By stepwise application of the result of *Part 1*, we obtain that $\varphi(R, q) = \varphi((\bar{R}_S, R_{-S}), q)$.

Next, starting from $((R'_S, R_{-S}), q)$, for each $i \in S$ replace $R'_i$ with $\bar{R}_i$; note that $i$ top-ranks $\varphi_i((R'_S, R_{-S}), q) = \varphi_i(R, q)$. The resulting problem is $((\bar{R}_S, R_S), q)$. By stepwise application of the result of *Part 1*, we obtain that $\varphi((R'_S, R_{-S}), q) = \varphi((\bar{R}_S, R_{-S}), q)$. Hence, $\varphi(R, q) = \varphi((R'_S, R_{-S}), q)$.

***(ii.) Only if part:*** Suppose that $\mathcal{C}$ is not *sequence-substitutable*. Then, there are $x \in O$, $(B, S, l) \in \mathcal{D}_x$ such that $l \geq 2$, and $i, j \in \mathcal{C}_x(B, S, l)$, $i \neq j$, and $i \notin \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l - 1)$. Let $y \in O \setminus \{x\}$.

Let $q \in \mathcal{Q}$ be such that $q_x = |B| + l$ and $q_{x'} = 0$ for each $x' \in O \setminus \{x\}$. Let $R \in \mathcal{R}^N$ such that each agent in $B$ top-ranks $x$ and second-ranks the null object; each agent in $S$ top-ranks $y$, second-ranks $x$, and third-ranks the null object; and each agent in $N \setminus (B \cup S)$ top-ranks the null object. Formally, $R$ is such that for each $k \in B$, $R_k(x) = 1$ and $R_k(\emptyset) = 2$; for each $k \in S$, $R_k(y) = 1$, $R_k(x) = 2$, and $R_k(\emptyset) = 3$; and for each $k \in N \setminus (B \cup S)$, $A(R_k) = \emptyset$. Since $i, j \in \mathcal{C}_x(B, S, l)$, we have $\varphi_i(R, q) = \varphi_j(R, q) = x$. Let $R'_j$ be such that $j$ top-ranks $x$. Note that $\varphi_j((R'_j, R_{-j}), q) = x$ and since $i \notin \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l - 1)$, $\varphi_i((R'_j, R_{-j}), q) = \emptyset$.

Hence, $\varphi_j((R'_j, R_{-j}), q) = \varphi_j(R, q)$ but $\varphi_i((R'_j, R_{-j}), q) \neq \varphi_i(R, q)$, implying that $\varphi$ violates *non-bossiness*. □

***Proof of Lemma 7.*** Let $\varphi$ be a mechanisms satisfying the properties in the hypothesis of the lemma. By Lemmas 1 and 4, it also satisfies *individual rationality* and *non-wastefulness*. Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $x \in O$, $(B, S, l) \in \mathcal{D}_x$, $S \neq \emptyset$, and $r \in \mathbb{N}$ such that $S = \{i \in N : R_i(x) = r \text{ and } x \ R_i \ \varphi_i(R, q)\}$ and $B = \{i \in N : R_i(x) < r \text{ and } \varphi_i(R, q) = x\}$; $B = \emptyset$ is possible.

Consider the following sets of agents, which form a partition of $N \setminus (B \cup S)$ when each of them is non-empty.

- Agents who receive the null object: $T_1 \equiv \{j \in N \setminus (B \cup S) : \varphi_j(R, q) = \emptyset\}$
- Agents who do not receive the top-ranked type, type $x$, or $\emptyset$: $T_2 \equiv \{j \in N \setminus (T_1 \cup B \cup S) : \varphi_j(R, q) \neq x \text{ and } R_j(\varphi_j(R, q)) \neq 1\}$
- Agents who receive their top-ranked real type, which is different from $x$: $T_3 \equiv \{j \in N \setminus (T_1 \cup T_2 \cup B \cup S) : \varphi_j(R, q) \neq x \text{ and } R_j(\varphi_j(R, q)) = 1\}$

***Step 1:*** Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent in $T_1$ to a preference relation for which the null object is top-ranked and $x$ is bottom-ranked. Formally, for each $j \in T_1$, $A(R^1_j) = \emptyset$, $R^1_j(x) = |O| + 1$, and for each $j \notin T_1$, $R^1_j = R_j$. By *individual rationality*, if $A(R^1_j) = \emptyset$, then $\varphi_j(R^1, q) = \emptyset$. By *group non-bossiness*, $\varphi(R^1, q) = \varphi(R, q)$. Alternatively, we can transform profile $R$ one preference relation at a time into $R^1$ and invoke *non-bossiness* at each step to show that $\varphi(R^1, q) = \varphi(R, q)$.

***Step 2:*** Let $R^2$ be a profile obtained from $R^1$ by changing the preference relation of each agent in $T_2$ to a preference relation for which his allotment is the only acceptable type and $x$ is bottom-ranked. Formally, for each $j \in T_2$, $A(R^2_j) = \{\varphi_j(R^1, q)\}$, $R^2_j(x) = |O| + 1$, and for each $j \notin T_2$, $R^2_j = R^1_j$. We show that for each $j \in T_2$, $\varphi_j(R^2, q) = \varphi_j(R^1, q)$.

Let $\varphi_j(R^1, q) = y$. By *favoring-higher-ranks*, for each $k \in N$ such that $R^1_k(y) = 1$, we have $\varphi_k(R^1, q) = y$. Then, $|\{k \in N : R^2_k(y) = 1\}| \leq q_y$. Suppose by contradiction that $\varphi_j(R^2, q) \neq y$. By *individual rationality*, $\varphi_j(R^2, q) = \emptyset$. If $|\{k \in N : \varphi_k(R^2, q) = y\}| < q_y$, then we have a contradiction to *weak non-wastefulness*. If $|\{k \in N : \varphi_k(R^2, q) = y\}| = q_y$, then we have a contradiction to *favoring-higher-ranks*. Hence, for each $j \in T_2$, $\varphi_j(R^2, q) = \varphi_j(R^1, q)$. By *group non-bossiness*, $\varphi(R^2, q) = \varphi(R^1, q)$. Alternatively, we can transform profile $R^1$ one preference relation at a time into $R^2$ and invoke *non-bossiness* at each step to show that

$\varphi(R^2, q) = \varphi(R^1, q)$.

***Step 3:*** Let $R^3$ be obtained from $R^2$ by changing the preference relation of each agent in $T_3$ to a preference relation for which his allotment is the only acceptable type and $x$ is bottom-ranked. Formally, for each $j \in T_3$, $A(R^3_j) = \{\varphi_j(R^2, q)\}$, $R^3_j(x) = |O| + 1$, and for each $j \notin T_3$, $R^3_j = R^2_j$. We show that for each $j \in T_3$, $\varphi_j(R^3, q) = \varphi_j(R^2, q)$.

Let $\varphi_j(R^2, q) = y$. Suppose by contradiction that $\varphi_j(R^3, q) \neq y$. By *individual rationality*, $\varphi_j(R^3, q) = \emptyset$. If $|\{k \in N : \varphi_k(R^3, q) = y\}| < q_y$, then we have a contradiction to *weak non-wastefulness*. Hence, $|\{k \in N : \varphi_k(R^3, q) = y\}| = q_y$. Let $q^*$ be the capacity profile obtained from $q$ by removing all the real types except for $y$. Formally, $q^*_y = q_y$ and $q^*_z = 0$ for each $z \in O \setminus \{y\}$. By *resource-monotonicity*, $\varphi_j(R^3, q^*) = \emptyset$. Now, by *RR-UTI* $\varphi_j(R^2, q^*) = \emptyset$. But then, by *weak non-wastefulness* and *favoring-higher-ranks*, there exists $l \in B \cup S$ such that $\varphi_l(R^2, q) \neq y$, $\varphi_l(R^2, q^*) = y$, and $R^2_l(y) = 1$. However, now $\varphi_l(R^2, q^*)\ P_l\ \varphi_l(R^2, q)$, a contradiction to *resource-monotonicity*. Hence, for each $j \in T_3$, $\varphi_j(R^3, q) = \varphi_j(R^2, q)$. By *group non-bossiness*, $\varphi(R^3, q) = \varphi(R^2, q)$. Alternatively, we can transform profile $R^2$ one preference relation at a time into $R^3$ and invoke *non-bossiness* at each step to show that $\varphi(R^3, q) = \varphi(R^2, q)$.

***Step 4:*** If $B = \emptyset$, set $R^4 = R^3$; otherwise let $R^4$ be a profile obtained from $R^3$ by changing the preference relation of each agent in $B$ to a preference relation for which $x$ is the only acceptable type. Formally, for each $j \in B$, $A(R^4_j) = \{x\}$ and for each $j \notin B$, $R^4_j = R^3_j$. We show that for each $j \in B$, $\varphi_j(R^4, q) = \varphi_j(R^3, q)$.

By *favoring-higher-ranks*, for each $k \in N$ such that $R^3_k(x) = 1$, we have $\varphi_k(R^3, q) = x$. Then, $|\{k \in N : R^3_k(x) = 1\}| \leq q_x$. Suppose by contradiction that $\varphi_j(R^4, q) \neq x$. By *individual rationality*, $\varphi_j(R^4, q) = \emptyset$. If $|\{k \in N : \varphi_k(R^4, q) = x\}| < q_x$, then we have a contradiction to *weak-non-wastefulness*. If $|\{k \in N : \varphi_k(R^4, q) = x\}| = q_x$, then we have a contradiction to *favoring-higher-ranks*. Hence, for each $j \in B$, $\varphi_j(R^4, q) = \varphi_j(R^3, q)$. By *group non-bossiness*, $\varphi(R^4, q) = \varphi(R^3, q)$. Alternatively, we can transform profile $R^3$ one preference relation at a time into $R^4$ and invoke *non-bossiness* at each step to show that $\varphi(R^4, q) = \varphi(R^3, q)$.

To summarize, at problem $(R^4, q)$, we have $\varphi(R^4, q) = \varphi(R, q)$ and all agents in $N \setminus (B \cup S)$ receive either $\emptyset$ or their only acceptable real type and they all bottom-rank type $x$. Furthermore, if $B \neq \emptyset$, then for all agents in $B$ type $x$ is the only acceptable real type. Let

$$T \equiv \{i \in S : \varphi_i(R, q) = x\}.$$

***Step 5:*** Let $\bar{q}$ be the capacity profile obtained from $q$ by removing all the real types except for $x$. Formally, $\bar{q}_x = q_x$ and $\bar{q}_y = 0$ for each $y \in O \setminus \{x\}$. By *resource-monotonicity* and *individual rationality*, for all $i \in N$ such that $\varphi_i(R^4, q) \neq x$, $\varphi_i(R^4, \bar{q}) = \emptyset$. By *weak non-wastefulness*, for all $i \in N$ such that $\varphi_i(R^4, q) = x$, $\varphi_i(R^4, \bar{q}) = x$.

***Step 6:*** Let $R^5$ be a profile obtained from $R^4$ by changing the preference relation of each agent in $N \setminus (B \cup S)$ who newly obtains the null object to a preference relation for which the null object is top-ranked while keeping $x$ bottom-ranked; this transformation only affects agents (not belonging to $B \cup S$) who have "lost" a real type in the last step. Formally, if $j \in N \setminus (B \cup S)$, $\varphi_j(R^4, q) \neq \emptyset$, and $\varphi_j(R^4, \bar{q}) = \emptyset$, then $A(R^5_j) = \emptyset$ and $R^5_j(x) = |O| + 1$; otherwise, $R^5_j = R^4_j$. By *individual rationality*, if $A(R^5_j) = \emptyset$, then $\varphi_j(R^5, \bar{q}) = \emptyset$. By *group non-bossiness*, $\varphi(R^5, \bar{q}) = \varphi(R^4, \bar{q})$. Alternatively, we can transform profile $R^4$ one preference relation at a time into $R^5$ and invoke *non-bossiness* at each step to show that $\varphi(R^5, \bar{q}) = \varphi(R^4, \bar{q})$.

***Case 1:*** $\boldsymbol{B = \emptyset}$.

***Step 1.7:*** Note that for all $i \in N \setminus S$, $R^5_i(\emptyset) = 1$ and $R^5_i(x) = |O| + 1$. Let $R^6$ be a profile obtained from $R^5$ by changing the preference relations of each agent in $S$ to a preference relation for which $x$ is the only acceptable type. Formally, for each $j \in S$, $A(R^6_j) = \{x\}$ and for each $j \notin S$, $R^6_j = R^5_j$. By *RR-UTI*, $\varphi(R^6, \bar{q}) = \varphi(R^5, \bar{q})$.

***Step 1.8:*** Finally, by *RR-UTI*, at each problem $(\bar{R}, \bar{q}) \in \mathcal{R}^N \times \mathcal{Q}$ such that $q_x = \bar{q}_x$, each agent in $S$ top-ranks $x$ and second-ranks the null object and each agent in $N \setminus S$ top-ranks the null object and bottom-ranks $x$, we have $\varphi(\bar{R}, \bar{q}) = \varphi(R^6, \bar{q})$ and thus $T = \{i \in S : \varphi_i(\bar{R}, \bar{q}) = x\}$.

***Case 2:*** $\boldsymbol{B \neq \emptyset}$. Note that $r > 1$.

***Step 2.7:*** Note that for all $i \in N \setminus S$, either $R^5_i(x) = 1$ and $R^5_i(\emptyset) = 2$ (for $i \in B$) or $R^5_i(\emptyset) = 1$ and $R^5_i(x) = |O| + 1$ (for $i \in N \setminus (B \cup S)$). Let $R^6$ be a profile obtained from $R^5$ by changing the preference relations of each agent in $S$ to a preference relation for which $x$ is second-ranked and the null object is third-ranked. Formally, for each $j \in S$, $A(R^6_j) = \{x\}$ and for each $j \notin S$, $R^6_j = R^5_j$. By *RR-UTI*, $\varphi(R^6, \bar{q}) = \varphi(R^5, \bar{q})$.

***Step 2.8:*** Finally, by *RR-UTI*, at each problem $(\bar{R}, \bar{q}) \in \mathcal{R}^N \times \mathcal{Q}$ such that $q_x = \bar{q}_x$, each agent in $S$ second-ranks $x$ and third-ranks the null object, each agent in $B$ top-ranks $x$ and second-ranks the null object, and each agent in $N \setminus (B \cup S)$ top-ranks the null object and bottom-ranks $x$, we have $\varphi(\bar{R}, \bar{q}) = \varphi(R^6, \bar{q})$ and thus $T = \{i \in S : \varphi_i(\bar{R}, \bar{q}) = x\}$.

With the derivation of the reduced problem $(\bar{R}, \bar{q})$ in either Step 1.8 or 2.8 such that $\varphi(\bar{R}, \bar{q}) = \varphi(R^6, \bar{q})$ and $T = \{i \in S : \varphi_i(\bar{R}, \bar{q}) = x\}$, *uniform choice* follows (because for any problem $(R', q')$ that relates to problem $(R, q)$ as specified in the definition of *uniform choice*, we can perform the same transformation into problem $(\bar{R}, \bar{q})$). □

***Proof of Theorem 2. If part:*** Let $\varphi$ be a choice-based IA mechanism with a choice structure $\mathcal{C}$ that satisfies *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity* and *sequence-substitutability*. Mechanism $\varphi$ being an IA mechanism with an *acceptant* choice structure implies *weak non-wastefulness* and *favoring-higher-ranks*. *Resource-monotonicity* follows from Proposition 1 $(i)$ and *group non-bossiness* follows from Proposition 1 $(ii)$.

*RR-UTI*: Let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and $R' \in \mathcal{R}^N$ be such that $R|_{O_+(q)\cup\{\emptyset\}} = R'|_{O_+(q)\cup\{\emptyset\}}$ and for all $x \in O_+(q)$ and for all $i, j \in N$, $R_i(x) \geq R_j(x)$ if and only if $R'_i(x) \geq R'_j(x)$. Suppose that $\varphi(R, q) \neq \varphi(R', q)$.

Let $t$ be the first step in the immediate $\mathcal{C}$-acceptance algorithm at $(R, q)$ and $(R', q)$ at which an agent receives different allotments, say agent $i \in N$ is one of these agents. Without loss of generality, suppose that $\varphi_i(R, q)\, P_i\, \varphi_i(R', q)$. Hence, $\varphi_i(R, q) \in O$ and assume $\varphi_i(R, q) = x$. Note that $R_i(x) = t$. At problem $(R, q)$, let $B$ be the set of agents who have received a copy of type $x$ before Step $t$ and let $S$ be the set of agents applying to type $x$ at Step $t$. By the definition of Step $t$, at problem $(R', q)$ set $B$ also is the set of agents who have received a copy of type $x$ before Step $t$. Furthermore, at problem $(R', q)$ agent $i \in S$ applies to type $x$ at some Step $t'$ such that $t' \geq t$. Now, by the assumptions on preference profiles $R$ and $R'$, at problem $(R', q)$ the set of agents applying to type $x$ at Step $t'$ equals $S$. Hence, at both problems $(R, q)$ and $(R', q)$, agents in $\mathcal{C}_x(B, S, q_x - |B|)$ receive type $x$; contradicting that $x \neq \varphi_i(R', q)$.

***Only if part:*** Suppose that $|O| \geq 3$ and that $\varphi$ satisfies *resource-monotonicity*, *weak non-wastefulness*, *(group) non-bossiness*, *favoring-higher-ranks*, and *RR-UTI*. By Lemmas 1, 4, and 7, it also satisfies *individual rationality*, *non-wastefulness*, and *uniform choice*.

Consider the following choice structure $\mathcal{C} = (\mathcal{C}_x)_{x\in O}$. Given $x \in O$ and $(B, S, l) \in \mathcal{D}_x$, define a profile $R$ as follows.

***Case 1: $B = \emptyset$.*** Let $R$ be a profile such that each agent in $S$ top-ranks $x$ and second-ranks the null object and each agent in $N \setminus S$ top-ranks the null object and bottom-ranks $x$. Formally, $R$ is such that for each $i \in S$, $R_i(x) = 1$ and $R_i(\emptyset) = 2$, and for each $i \in N \setminus S$, $A(R_i) = \emptyset$ and $R_i(x) = |O| + 1$.

***Case 2:* $B \neq \emptyset$.** Let $R$ be a profile such that each agent in $B$ top-ranks $x$ and second-ranks the null object; each agent in $S$ second-ranks $x$ and third-ranks the null object; and each agent in $N \setminus (B \cup S)$ top-ranks the null object and bottom-ranks $x$. Formally, $R$ is such that for each $i \in B$, $R_i(x) = 1$ and $R_i(\emptyset) = 2$, and for each $i \in S$, $R_i(x) = 2$ and $R_i(\emptyset) = 3$, and for each $i \in N \setminus S$, $A(R_i) = \emptyset$ and $R_i(x) = |O| + 1$.

Let $q_x = |B| + l$ and $q_y = 0$ for each $y \neq x$. By *uniform choice*, at each problem $(R, q)$ as defined in either Case 1 or Case 2, the same subset of $S$, say $S' \subseteq S$, receives $x$. Let $\mathcal{C}_x(B, S, l) \equiv S'$.

***$\mathcal{C}_x$ satisfies acceptance and monotonicity:*** By *weak non-wastefulness*, it follows easily that $\mathcal{C}_x$ is *acceptant*. By *resource-monotonicity*, it follows easily that $\mathcal{C}_x$ is *monotonic*.

***$\mathcal{C}_x$ satisfies substitutability:*** Let $(B, S, l) \in \mathcal{D}_x$, $i, j \in S$, $i \neq j$, and $i \in \mathcal{C}_x(B, S, l)$. Let $(R, q)$ be a problem defining $\mathcal{C}_x(B, S, l)$. Hence, $\varphi_i(R, q) = x$. Let $y, z \in O \setminus \{x\}$ and $y \neq z$ ($|O| \geq 3$).[18]

Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent in $S$ to a preference relation for which agent $j \in S$ top-ranks $y$, second-ranks $x$, and third-ranks the null object and each agent in $S \setminus \{j\}$ top-ranks $z$, second-ranks $x$, and third-ranks the null object. Formally, $R^1_j(y) = 1$, $R^1_j(x) = 2$, and $R^1_j(\emptyset) = 3$; for each $k \in S \setminus \{j\}$, $R^1_k(z) = 1$, $R^1_k(x) = 2$, and $R^1_k(\emptyset) = 3$; and for each $k \notin S$, $R^1_k = R_k$. By *RR-UTI*, $\varphi(R^1, q) = \varphi(R, q)$, particularly $\varphi_i(R^1, q) = x$. Let $q'$ be such that $q_y = 1$ and $q'_t = q_t$ for each $t \in O \setminus \{y\}$ (i.e., $q'_x = l$ and for each $w \in O \setminus \{x, y\}$, $q'_w = 0$). By *non-wastefulness* and *resource-monotonicity*, $\varphi_j(R^1, q') = y$ and $\varphi_i(R^1, q') = x$.

Let $R^2$ be a profile obtained from $R^1$ by changing the preference relation of agent $j$ to a preference relation for which $y$ is the only acceptable type and type $x$ is bottom-ranked. Formally, $A(R^2_j) = y$, $R^2_j(x) = |O| + 1$, and for each $k \in N \setminus \{j\}$, $R^2_k = R^1_k$. By *weak non-wastefulness* and *individual rationality*, $\varphi_j(R^2, q') = y$. By *non-bossiness*, $\varphi(R^2, q') = \varphi(R^1, q')$, particularly $\varphi_i(R^2, q') = x$. Next, we decrease the capacity of $y$ back to zero, i.e., we switch from $q'$ to $q$. By *individual rationality*, $\varphi_j(R^2, q) = \emptyset$. By *weak non-wastefulness* and *resource-monotonicity*, agent $i$ still receives a copy of $x$, that is, $\varphi_i(R^2, q) = x$.

Finally, let $R^3$ be a profile obtained from $R^2$ by changing the preference relation of agent $j$ to a preference relation for which the null object is top-ranked and type $x$ is bottom-

[18] Note that this proof step requires at least three real object types, i.e., we need $|O| \geq 3$. Our counter Example 2 uses a choice structure that violates *substitutability*.

ranked. Formally, $A(R^3_j) = \emptyset$, $R^3_j(x) = |O| + 1$, and for each $k \in N \setminus \{j\}$, $R^3_k = R^2_k$. By *individual rationality*, $\varphi_j(R^3, q) = \emptyset$. By *non-bossiness*, $\varphi(R^3, q) = \varphi(R^2, q)$, particularly $\varphi_i(R^3, q) = x$. Since $(R^3, q)$ is a problem that defines $\mathcal{C}_x(B, S \setminus \{j\}, l)$ (by *RR-UTI*), it follows that $i \in \mathcal{C}_x(B, S \setminus \{j\}, l)$.

***$\mathcal{C}_x$ satisfies sequence-monotonicity:*** Let $(B, S, l) \in \mathcal{D}_x$ and $j \in B$. Let $(R, q)$ be a problem defining $\mathcal{C}_x(B, S, l) \equiv S'$. Hence, $\varphi_j(R, q) = x$. Let $y, z \in O \setminus \{x\}$ and $y \neq z$.[19]

Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent in $B \cup S$ to a preference relation for which $j$ top-ranks $y$, second-ranks $x$, and third-ranks the null object; each agent in $B \setminus \{j\}$ top-ranks $z$, second-ranks $x$, and third-ranks the null object; and each agent in $S$ top-ranks $z$, second-ranks $y$, third-ranks $x$, and fourth-ranks the null object. Formally, $R^1_j(y) = 1$, $R^1_j(x) = 2$, and $R^1_j(\emptyset) = 3$; for each $k \in B \setminus \{j\}$, $R^1_k(z) = 1$, $R^1_k(x) = 2$, and $R^1_k(\emptyset) = 3$; each $k \in S$, $R^1_k(z) = 1$, $R^1_k(y) = 2$, $R^1_k(x) = 3$, and $R^1_k(\emptyset) = 4$; and for each $k \in N \setminus (B \cup S)$, $R^1_k = R_k$. By *RR-UTI*, $\varphi(R^1, q) = \varphi(R, q)$. Let $q'$ be such that $q_y = 1$ and $q'_t = q_t$ for each $t \in O \setminus \{y\}$. By *favoring-higher-ranks*, *non-wastefulness*, and *resource-monotonicity*, $\varphi_j(R^1, q') = y$ and for each $k \in S'$, $\varphi_k(R^1, q') = x$.

Let $R^2$ be a profile obtained from $R^1$ by changing the preference relation of agent $j$ to a preference relation for which $y$ is the only acceptable type and type $x$ is bottom-ranked. Formally, $A(R^2_j) = y$, $R^2_j(x) = |O| + 1$, and for each $k \in N \setminus \{j\}$, $R^2_k = R^1_k$. By *weak non-wastefulness* and *individual rationality*, $\varphi_j(R^2, q') = y$. By *non-bossiness*, $\varphi(R^2, q') = \varphi(R^1, q')$, particularly, for each $k \in S'$, $\varphi_k(R^2, q') = x$. Next, we decrease the capacity of $y$ back to zero, i.e., we switch from $q'$ to $q$. By *individual rationality*, $\varphi_j(R^2, q) = \emptyset$. By *weak non-wastefulness* and *resource-monotonicity*, agents in $S'$ still receives a copy of $x$, that is, for each $k \in S'$, $\varphi_k(R^2, q) = x$. Recall that $q_x = |B| + l$ and that at problem $(R^2, q)$ agent $j \in B$ does not receive $x$ anymore and that hence $l + 1$ copies of type $x$ are available to agents in $S$. Hence, $(R^2, q)$ is a problem that defines $\mathcal{C}_x(B \setminus \{j\}, S, l + 1)$. Then, $S' = \mathcal{C}_x(B, S, l) \subseteq \mathcal{C}_x(B \setminus \{j\}, S, l + 1)$. Thus, if $i \in \mathcal{C}_x(B, S, l)$, then $i \in \mathcal{C}_x(B \setminus \{j\}, S, l + 1)$.

***$\mathcal{C}_x$ satisfies sequence-substitutability:*** Let $(B, S, l) \in \mathcal{D}_x$ such that $l \geq 2$ and $j \in \mathcal{C}_x(B, S, l)$. Let $(R, q)$ be a problem defining $\mathcal{C}_x(B, S, l)$. Hence, $\varphi_j(R, q) = x$.

Let $R^1$ be a profile obtained from $R$ by changing the preference relation of each agent in $S$ to a preference relation for which each agent $i \in S$ second-ranks $x$ and third-ranks the null object. Formally, for each $i \in S$, $R^1_i(x) = 2$ and $R^1_i(\emptyset) = 3$ and for each $k \notin S$, $R^1_k = R_k$. By

[19]Note that this proof step requires at least three real object types, i.e., we need $|O| \geq 3$. Our counter Example 2 uses a choice structure that violates *sequence-monotonicity*.

*RR-UTI*, $\varphi(R^1, q) = \varphi(R, q)$, particularly $\varphi_j(R^1, q) = x$.

Let $R^2$ be a profile obtained from $R^1$ by changing the preference relation of agent $j \in S$ to a preference relation for which $x$ is the only acceptable type. Formally, $A(R^2_j) = \{x\}$ and for each $k \in N \setminus \{j\}$, $R^2_k = R^1_k$. By *favoring-higher-ranks*, $\varphi_j(R^2, q) = x$. By *non-bossiness*, $\varphi(R^2, q) = \varphi(R^1, q)$. Since $(R^2, q)$ is a problem that defines $\mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1)$ (by *RR-UTI*), it follows that $\mathcal{C}_x(B, S, l) = \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1) \cup \{j\}$. In particular, $i \in \mathcal{C}_x(B, S, l)$ and $i \neq j$ imply $i \in \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1)$.

Now, the mechanism $\varphi$ being a choice-based IA mechanism based on $\mathcal{C}$ is a straightforward consequence of *uniform choice* together with *favoring-higher-ranks* and *non-wastefulness*. □

***Proof of Corollary 2.*** As already explained before stating the corollary, the proof of Theorem 2 remains valid for the house allocation model. Hence, we only need to show that it also remains valid if fewer than three real types are available, i.e., $|O| \leq 2$. Note the two steps that require $|O| > 3$ in the proof of Theorem 2 are in the *only if part* of the proof and concern the *substitutability* of $\mathcal{C}_x$ (see Footnote 18) and the *sequence-monotonicity* of $\mathcal{C}_x$ (see Footnote 19). Since any choice function $\mathcal{C}_x$ vacuously satisfied *sequence-monotonicity* in the house allocation model, we only need to prove the *substitutability* of $\mathcal{C}_x$.

***$\mathcal{C}_x$ satisfies substitutability:*** Let $(\emptyset, S, l) \in \mathcal{D}_x$, $i, j \in S$, $i \neq j$, and $i \in \mathcal{C}_x(B, S, 1)$. Hence, $\varphi_i(R, q) = x$ and $\varphi_j(R, q) = \emptyset$. Change agent $j$'s preference relation $R_j$ into $R'_j$ where the null object is top-ranked and $x$ is bottom-ranked. Formally, $A(R'_j) = \emptyset$ and $R'_j(x) = |O| + 1$. By *individual rationality* and *non-bossiness*, $\varphi(R, q) = \varphi((R'_j, R_{-j}), q)$. Hence, $i \in \mathcal{C}_x(\emptyset, S \setminus \{j\}, 1)$. □

***Proof of Proposition 2.*** Let $x \in O$ and let $\mathcal{C}^{\succ_x, t_x}$ be a choice function with an affirmative-action-target.

***$\mathcal{C}^{\succ_x, t_x}$ satisfies acceptance and monotonicity:*** by definition (the latter is due to the fact that affirmative-action-targets are fixed while the capacities of real types can vary).

***$\mathcal{C}^{\succ_x, t_x}$ satisfies substitutability:*** Let $(B, S, l) \in \mathcal{D}_x$, $i, j \in S$, $i \neq j$, and $i \in \mathcal{C}_x(B, S, l)$. We want to show that if $j$ is removed from $S$, agent $i$ should still be chosen, that is, $i \in \mathcal{C}_x(B, S \setminus \{j\}, l)$.

Assume that agent $i$ is only chosen to meet the minority target (thus, $i \in S^m$). Then, that is still the case after the removal of agent $j$ and $i \in \mathcal{C}_x(B, S \setminus \{j\}, l)$. Alternatively, assume that agent $i$ is chosen solely on account of a high priority (being a minority agent or not). Then, after the removal of agent $j$, agent $i$ can still be chosen on account of his high priority (he might also be chosen as a minority agent to meet the target if agent $j$ was a minority agent that helped meet the target before leaving). Hence, $i \in \mathcal{C}_x(B, S \setminus \{j\}, l)$.

***$\mathcal{C}^{\succ_x, t_x}$ satisfies sequence-monotonicity:*** Let $(B, S, l) \in \mathcal{D}_x$, $j \in B$, and $i \in \mathcal{C}^{\succ_x, t_x}(B, S, l)$. We want to show that if agent $j$ is removed from $B$, agent $i$ should still be chosen, that is, $i \in \mathcal{C}^{\succ_x, t_x}(B \setminus \{j\}, S, l+1)$.

***Case 1: $j \notin B^m$***. In this case, if agent $j$ is removed from $B$, the number of minority agents who are given priority, which is $\max\{0, t_x - |B^m|\}$, stays the same. Moreover, there is one more copy to be allocated, therefore clearly $i \in \mathcal{C}^{\succ_x, t_x}(B \setminus \{j\}, S, l+1)$.

***Case 2: $j \in B^m$* and *$i \in S^m$*.** In this case, if agent $j$ is removed from $B$, the number of minority agents who are given priority is $\max\{0, t_x - |B^m| + 1\}$. Moreover, there is one more copy to be allocated, and since $i \in S^m$, clearly $i \in \mathcal{C}^{\succ_x, t_x}(B \setminus \{j\}, S, l+1)$.

***Case 3: $j \in B^m$* and *$i \notin S^m$*.** In this case, if agent $j$ is removed from $B$, the number of minority agents who are given priority is $\max\{0, t_x - |B^m| + 1\}$ as in the previous case. Moreover, there is one more copy to be allocated. Now, even if $\max\{0, t_x - |B^m| + 1\} > \max\{0, t_x - |B^m|\}$, in which case one more minority agent is given priority, since there is one more copy we have $i \in \mathcal{C}^{\succ_x, t_x}(B \setminus \{j\}, S, l+1)$.

***$\mathcal{C}^{\succ_x, t_x}$ satisfies sequence-substitutability:*** Let $(B, S, l) \in \mathcal{D}_x$ such that $l \geq 2$, $i, j \in \mathcal{C}^{\succ_x, t_x}(B, S, l)$, $i \neq j$. We want to show that if agent $j$ is moved to $B$ and one less agent is to be chosen, agent $i$ should still be chosen, that is, $i \in \mathcal{C}^{\succ_x, t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$.

***Case 1: $j \notin S^m$***. In this case, if agent $j$ is moved to $B$, the number of minority agents who are given priority, which is $\max\{0, t_x - |B^m|\}$, stays the same. Although there is one fewer copy to be allocated, since one of the agents who received a copy before is removed, $i \in \mathcal{C}^{\succ_x, t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$.

***Case 2: $j \in S^m$* and *$i \notin S^m$*.** In this case, if agent $j$ is removed from $B$, the number of minority agents who are given priority is $\max\{0, t_x - |B^m| - 1\}$, which is less than or equal to before and which is an advantage for agent $i$ since $i \notin S^m$. Now, although there is one less copy to be allocated, since one of the agents who received a copy before is removed, $i \in \mathcal{C}^{\succ_x, t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$.

***Case 3: $j \in S^m$* and *$i \in S^m$*.** In this case, if agent $j$ is moved to $B$, the number of minority agents who are given priority is $\max\{0, t_x - |B^m| - 1\}$.

Suppose that before moving agent $j$ to $B$, agent $i$ was not receiving one of the copies intended for minorities, that is, agent $i$ is not one of the top $\max\{0, t_x - |B^m|\}$ $\succ_x$-priority agents in $S^m$. Clearly, after moving agent $j$ to $B$, although there is one less copy to be allocated, since one of the agents who received a copy before is removed, $i \in \mathcal{C}^{\succ_x,t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$.

Suppose that before moving agent $j$ to $B$, agent $i$ was receiving one of the copies intended for minorities, that is, agent $i$ is one of the top $\max\{0, t_x - |B^m|\}$ $\succ_x$-priority agents in $S^m$. If agent $j$ was also receiving one of the copies intended for minorities, then agent $i$ is one of the top $\max\{0, t_x - |B^m| - 1\}$ $\succ_x$-priority agents in $S^m \setminus \{j\}$ and therefore $i \in \mathcal{C}^{\succ_x,t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$. If agent $j$ was not receiving one of the copies intended for minorities, then $i \succ_x j$ and also $j$ was chosen after the copies intended for the minorities were exhausted. Then either agent $i$ is one of the top $\max\{0, t_x - |B^m| - 1\}$ $\succ_x$-priority agents in $S^m \setminus \{j\}$ or agent $i$ is chosen after the copies intended for the minorities are exhausted, which implies that $i \in \mathcal{C}^{\succ_x,t_x}(B \cup \{j\}, S \setminus \{j\}, l-1)$. □

# B Independence of Characterizing Properties

The following examples establish the independence of the properties in Theorems 1 and 2. Throughout the examples, let $N = \{1, \ldots, n\}$. Recall that *non-wastefulness* implies *weak non-wastefulness* and that *consistency* implies *group non-bossiness* and the latter implies *non-bossiness*.

**Example 5** (A mechanism satisfying *resource-monotonicity*, *consistency*, *favoring-higher–ranks*, *RR-UTI*, and ***violating weak non-wastefulness*****).** Let $\varphi$ be the mechanism that assigns the null object to any agent at any problem, that is, for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ and each agent $i \in N$, $\varphi_i(R, q) = \emptyset$. □

**Example 6** (A mechanism satisfying *non-wastefulness*, *consistency*, *favoring-higher-ranks*, *RR-UTI*, and ***violating resource-monotonicity*****).** Let $\succ$ and $\succ'$ be priority structures such that for each real type $x \in O$, $2 \succ_x \cdots \succ_x n \succ_x 1$ and $1 \succ'_x 2 \succ'_x \cdots \succ'_x (n-1) \succ'_x n$. Let $\varphi$ be the mechanism such that for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$,

$$\varphi(R, q) = \begin{cases} IA^{\succ}(R, q) & \text{if } IA_1^{\succ}(R, q) \neq \emptyset \\ IA^{\succ'}(R, q) & \text{otherwise.} \end{cases}$$

Mechanism $\varphi$ clearly satisfies *weak non-wastefulness*, *favoring-higher-ranks*, and *RR-UTI*. To see that it satisfies *consistency*, let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$. Suppose that $IA_1^{\succ}(R, q) = \emptyset$, and therefore $\varphi(R, q) = IA^{\succ'}(R, q)$. Now, suppose that some agents (possibly including agent 1) leave with their $IA^{\succ'}(R,q)$ allotments, resulting in problem $(R',q')$. If 1 is a member of the leaving group, $IA_1^{\succ}(R',q') = \emptyset$ and $\varphi(R',q') = IA^{\succ'}(R',q')$. Since $IA^{\succ'}$ is *consistent*, the allotments of the remaining agents don't change. If 1 is not a member of the leaving group, by *consistency* of $IA^{\succ}$, $IA_1^{\succ}(R',q') = \emptyset$ and $\varphi(R',q') = IA^{\succ'}(R',q')$. Since $IA^{\succ'}$ is *consistent*, the allotments of the remaining agents, including agent 1, don't change.

Suppose that $IA_1^{\succ}(R,q) \neq \emptyset$, and therefore $\varphi(R,q) = IA^{\succ}(R,q)$. Suppose that some agents (possibly including agent 1) leave with their $IA^{\succ}(R,q)$ allotments, resulting in problem $(R',q')$. If 1 is a member of the leaving group, $IA_1^{\succ}(R',q') = \emptyset$ and $\varphi(R',q') = IA^{\succ'}(R',q')$. Since agent 1 top-ranks the null object at $R'$ and the restrictions of $\succ$ and $\succ'$ to $N \setminus \{1\}$ are the same, $IA^{\succ}(R',q') = IA^{\succ'}(R',q')$. Since $IA^{\succ}$ is *consistent*, the allotments of the remaining agents don't change. If 1 is not a member of the leaving group, by *consistency* of $IA^{\succ}$, $IA_1^{\succ}(R',q') \neq \emptyset$ and $\varphi(R',q') = IA^{\succ}(R',q')$. Since $IA^{\succ}$ is *consistent*, the allotments of the remaining agents, including agent 1, don't change.

To see that $\varphi$ violates *resource-monotonicity*, let types $x, y \in O$, $x \neq y$, and let problem $(R,q) \in \mathcal{R}^N \times \mathcal{Q}$ be such that agents 1, 2, and 3 top-rank $x$, second-rank $y$, and third-rank the null object, every other agent top-ranks the null object, $q_x = 2$, and $q_z = 0$ for each $z \in O \setminus \{x\}$. Let $q'$ be such that $q'_y = 1$ and $q'_z = q_z$ for each $z \in O \setminus \{y\}$, i.e., $q'_x = 2$. Note that $\varphi_1(R,q) = x$ and $\varphi_1(R,q') = y$, meaning that the increase in the capacity of $y$ makes agent 1 worse-off. □

The above example shows the independence of *resource-monotonicity* from the other characterizing properties in Theorem 2, as well as in Theorem 1, since *consistency* implies *group non-bossiness*, which in turn implies *non-bossiness*. Since Proposition 1 reveals that *resource-monotonicity* of a choice-based IA mechanism is intimately related to *sequence-monotonicity* of the associated choice structure, such a relation can possibly be further highlighted with an example of a choice-based IA mechanism which satisfies all the properties in Theorem 2 but not *resource-monotonicity*, and the choice structure of which violates *sequence-monotonicity*. The above example cannot serve for that purpose since the mechanism in the example is not a choice-based IA mechanism. The following example, which shows the independence of *resource-monotonicity* from the other characterizing properties in Theorem 2, but not in Theorem 1 since *consistency* is violated, serves that purpose.

**Example 7** (A mechanism satisfying *non-wastefulness*, *group non-bossiness*, *favoring-higher-ranks*, *RR-UTI*, and ***violating resource-monotonicity*****).** This example requires that $n \geq 4$. Let $x \in O$. Let $\succ_x$ and $\succ'_x$ be the priority orderings defined as $1 \succ_x 2 \succ_x 3 \succ_x \cdots \succ_x n$

and $3 \succ'_x 2 \succ'_x 1 \succ'_x \cdots \succ'_x n$. Let the choice function $\mathcal{C}_x : \mathcal{D}_x \to 2^N \setminus \emptyset$ be defined as follows: for each $(B, S, l) \in \mathcal{D}_x$, if $n \in B$, then $\mathcal{C}_x(B, S, l)$ coincides with the responsive choice function based on $\succ$; if $n \notin B$, then $\mathcal{C}_x(B, S, l)$ coincides with the responsive choice function based on $\succ'$. Formally, for each $(B, S, l) \in \mathcal{D}_x$, if $n \in B$, then $\mathcal{C}_x(B, S, l)$ chooses the $\min\{|S|, l\}$ highest $\succ_x$-priority agents in $S$; otherwise, $\mathcal{C}_x(B, S, l)$ chooses the $\min\{|S|, l\}$ highest $\succ'_x$-priority agents in $S$. Let $\mathcal{C}$ be a choice structure such that the choice function of $x$, $\mathcal{C}_x$, is the one defined above, and every other type has an arbitrary choice function satisfying *acceptance*, *monotonicity*, *substitutability*, *sequence-monotonicity*, and *sequence-substitutability*. Let $\varphi$ be the choice-based IA mechanism based on $\mathcal{C}$, that is for each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = IA^{\mathcal{C}}(R, q)$.

Mechanism $\varphi$ clearly satisfies *weak non-wastefulness*, *favoring-higher-ranks*, and *RR-UTI*. Note that $\mathcal{C}$ clearly satisfies *acceptance*, *monotonicity*, and *substitutability*. Therefore, to see that it satisfies *group non-bossiness*, by Proposition 1 it is sufficient to show that $\mathcal{C}$ satisfies *sequence-substitutability*. Let $(B, S, l) \in \mathcal{D}_x$, $l \geq 2$. Let $i, j \in \mathcal{C}_x(B, S, l)$ and $i \neq j$. If $j \neq n$, then clearly $i \in \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1)$. If $j = n$, then since $n$ has the least $\succ_x$ priority, $n \in \mathcal{C}_x(B, S, l)$ implies that $\mathcal{C}_x(B, S, l) = S$. Then, $\mathcal{C}_x(B \cup \{n\}, S \setminus \{n\}, l-1) = S \setminus \{n\}$ and therefore $i \in \mathcal{C}_x(B \cup \{j\}, S \setminus \{j\}, l-1)$.

To see that $\varphi$ violates *resource-monotonicity*, let $y, z \in O \setminus \{x\}$ and let $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$ be such that agents 1, 2, and 3 top-rank $z$, second-rank $y$, and third-rank $x$, agent $n$ top-ranks $y$ and second-ranks $x$, and every other agent top-ranks the null object, $q_x = 2$ and $q_t = 0$ for each $t \in O \setminus \{x\}$. Let $q'$ be such that $q'_y = 1$ and $q'_t = q_t$ for each $t \in O \setminus \{y\}$. Note that $\varphi_1(R, q) = x$ and $\varphi_1(R, q') = \emptyset$, meaning that the increase in the capacity of $y$ makes agent 1 worse-off. □

**Example 8** (A mechanism satisfying *non-wastefulness*, *resource-monotonicity*, *favoring-higher-ranks*, *RR-UTI*, and ***violating non-bossiness***)**.** Let $\succ$ and $\succ'$ be priority structures such that for each real type $x \in O$, $1 \succ_x 2 \succ_x \cdots \succ_x (n-1) \succ_x n$ and $n \succ'_x (n-1) \succ'_x \cdots \succ'_x 2 \succ'_x 1$. For each real type $x \in O$, let the choice function $\mathcal{C}_x : \mathcal{D}_x \to 2^N \setminus \emptyset$ be defined as follows. For each $(B, S, l) \in \mathcal{D}_x$, first $x$ chooses the highest $\succ_x$-priority agent in $S$ (only one agent), and then chooses the highest $\succ'_x$-priority agents until capacity is full, and $\mathcal{C}_x(B, S, l)$ is defined to be the set of chosen agents. Let $\varphi$ be the choice-based IA mechanism based on $\mathcal{C} = (\mathcal{C}_x)_{x \in O}$, that is for each $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, $\varphi(R, q) = IA^{\mathcal{C}}(R, q)$.

To see that $\varphi$ violates *non-bossiness*, observe that we can construct a problem such that agents 1, 2, and 3 second-rank type $x \in O$ and compete for two copies of it, in which case agents 1 and 3 receive the copies, however when agent 1 changes his preference relation to a preference relation for which he now top-ranks $x$, agents 2 and 3 compete for one copy of $x$ and agent 2 receives the copy. Note that the choice structure $\mathcal{C}$ violates *sequence-substitutability*. $\square$

**Example 9** (A mechanism satisfying *non-wastefulness*, *resource-monotonicity*, *consistency*, *RR-UTI*, and ***violating favoring-higher-ranks***)**.** Let $\succ$ be a priority structure. Let $\varphi$ be the mechanism such that for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, first agent 1 is assigned a copy of his most preferred available type, and then the remaining resources are allocated among the remaining agents via IA mechanism $IA^{\succ}$. $\square$

**Example 10** (A mechanism satisfying *non-wastefulness*, *resource-monotonicity*, *consistency*, *favoring-higher-ranks*, and ***violating RR-UTI***)**.** Let $\succ$ and $\succ'$ be priority structures such that $\succ \neq \succ'$. Let mechanism $\varphi$ be the following modification of an IA mechanism: for each problem $(R, q) \in \mathcal{R}^N \times \mathcal{Q}$, priority structure $\succ$ is used in Step 1 of the IA algorithm and thereafter priority structure $\succ'$. $\square$