Preference revelation games and strict cores of multiple-type housing market problems*

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Abstract

We consider multiple-type housing market problems as introduced by Moulin (1995) and study the relationship between strict strong Nash equilibria and the strict core (two solution concepts that are defined in terms of the absence of weak blocking coalitions). We prove that for lexicographically separable preferences, the set of all strict strong Nash equilibrium outcomes of each preference revelation game that is induced by a strictly core-stable mechanism is a subset of the strict core, but not vice versa, i.e., there are strict core allocations that cannot be implemented in strict strong Nash equilibrium (Theorem 1). This result is extended to a more general set of preference domains that satisfy strict core non-emptiness and a minimal preference domain richness assumption (Theorem 2).

Keywords: multiple-type housing market problems; strict core; strict strong Nash equilibria.

JEL codes: C71, C72, C78.

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1 Introduction

In a classical Shapley-Scarf housing market (Shapley and Scarf, 1974), each agent is endowed with an indivisible object, e.g., a house, wishes to consume exactly one house, and ranks all houses in the market. The problem then is to (re)allocate houses among the agents without using monetary transfers and by taking into account agents' preferences and endowments.

A common solution concept for Shapley-Scarf housing markets is the strict core solution, which assigns the set of allocations where no group of agents has an incentive (via weak blocking) to deviate by exchanging their endowments within the group. When agents' preferences are strict, the strict core solution exhibits a remarkable number of positive features: it is non-empty (Shapley and Scarf, 1974), always a singleton, and coincides with the unique competitive allocation (Roth and Postlewaite, 1977). In addition, it can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Furthermore, the TTC mechanism that assigns the unique strict core allocation for any housing market is strategy-proof (Roth, 1982), and it is the unique mechanism satisfying individual rationality, Pareto efficiency, and strategy-proofness (Ma, 1994).

Multiple-type housing markets are an extension of Shapley-Scarf housing markets, which were first introduced by Moulin (1995).[1] In multiple-type housing markets, there are multiple types of indivisible objects, each agent is endowed with one object of each type and wishes to consume exactly one object of each type. Multiple-type housing markets are often described with houses and cars as metaphors for indivisible object types. While these and related housing market models appear to be rather stylized, they give valuable insights into many real-world applications such as dynamic resource allocation problems (Monte and Tumennasan, 2015), the assignment of student-presentations (Mackin and Xia, 2016), cloud computing (Ghodsi et al., 2011, 2012), the assignment of medical resources (Huh et al., 2013), and

[1] There are many other extensions, such as the multi-demand models of Pápai (2001, 2007), Ehlers and Klaus (2003), and Manjunath and Westkamp (2021).

5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al., 2019). A more familiar example for most readers would be the situation of students' enrollment at many universities where courses are taught in parallel sessions (Klaus, 2008).

Konishi et al. (2001) were the first to analyze multiple-type housing markets. They demonstrated that when increasing the dimension of the classical Shapley-Scarf housing market model by adding other types of indivisible objects, most of the positive results obtained for the one-dimensional single-type case disappear: even for additively separable preferences, the strict core may be empty and no individually rational, Pareto efficient, and strategy-proof mechanism exists. One of the reasons for this is that, in contrast to single-type housing market problems, multiple-type housing market problems cannot be transformed into well-behaved coalition formation games (Banerjee et al., 2001; Bogomolnaia and Jackson, 2002; Quint and Wako, 2004); e.g., an agent may exchange his house within a trading coalition S_1 but exchange his car with a different trading coalition S_2.

There has been very little work on multiple-type housing market problems after Konishi et al. (2001)'s negative results. The following papers considered different solutions for different sub-domains of preferences.

For separable preferences, Konishi et al. (2001) and Wako (2005) suggested an alternative solution to the strict core solution by first using separability to decompose a multiple-type housing market into "coordinate-wise submarkets" and second, determining the strict core in each submarket. Wako (2005) called the resulting outcome the commodity-wise competitive allocation and showed that it is implementable in strong Nash equilibria. Klaus (2008) called the mechanism that always assigns this unique allocation the coordinate-wise core rule, and showed that it satisfies individual rationality, constrained efficiency,[2] and strategy-proofness.

For a very general domain of lexicographic preferences, Sikdar et al. (2017, 2019) extended the TTC algorithm and defined a new mechanism: the multiple-type top-

[2]There exists no other strategy-proof mechanism that Pareto dominates the coordinate-wise core rule.

trading-cycles (MTTC) mechanism, and they showed that the MTTC mechanism determines a strict core allocation; hence, the strict core for general lexicographic preferences is non-empty. Strict-core stability implies individual rationality and Pareto efficiency of the MTTC mechanism. However, they demonstrated that the MTTC mechanism is not strategy-proof and that the strict core may be multi-valued.

Our contributions

Takamiya (2009) considered the more generalized model of indivisible goods allocation introduced by Sönmez (1999) that contains Shapley-Scarf housing market problems as special case. In particular, Takamiya's results imply that for Shapley-Scarf housing market problems and for individually rational and Pareto efficient mechanisms, the set of strict strong Nash equilibrium outcomes of the preference revelation game equals the strict core (we state this result as Corollary 1).

Similarly, we examine the relationship between strict strong Nash equilibrium outcomes of the preference revelation games and strict core allocations for multiple-type housing markets. Takamiya's (2009) results do not translate into our higher dimensional model. First, multiple-type housing market problems may have an empty strict core, even if preferences are separable. Then, a promising subdomain that guarantees the non-emptiness of the strict core is the domain of lexicographically separable preferences. However, lexicographically separable preferences do not satisfy the domain richness condition Takamiya (2009) needs for his main result.

We prove that for lexicographically separable preferences, the set of all strict strong Nash equilibrium outcomes of each preference revelation game that is induced by a strictly core-stable mechanism is a subset of the strict core, but not vice versa, i.e., there are strict core allocations that cannot be implemented in strict strong Nash equilibrium (Theorem 1). This result is extended to a more general set of preference domains that satisfy strict core non-emptiness and a minimal preference domain richness assumption (Theorem 2). Throughout the paper, we motivate our approach and discuss some comparative statics aspects of our results via various examples.

Our paper is organized as follows. In the following section we introduce multiple-type housing market problems, solutions / mechanisms and their properties, and preference revelation games. We develop and prove our results in Section 3 and conclude in Section 4. Appendix A describes the generalized indivisible goods allocation model of Sönmez (1999) and Takamiya (2009) and compares their preference domain richness conditions with the domain richness condition we use here.

2 The model

Multiple-type housing market problems

Let $N = \{1, \ldots, n\}$ be a finite *set of agents*. A nonempty subset of agents $S \subseteq N$ is a *coalition*. We assume that there exist $m \geq 1$ *(distinct) types of indivisible objects* and n *(distinct) indivisible objects of each type*. We denote the *set of types* by $T = \{1, ..., m\}$. Note that for $m = 1$, our model equals the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974). Throughout this paper, we focus on the multiple-type extension of the Shapley-Scarf housing market model as introduced by Moulin (1995), where $|N| = n \geq 3$ and $|T| = m \geq 2$.[3]

Each agent $i \in N$ is endowed with exactly one object of each type $t \in T$, denoted by o_i^t. Hence, each *agent i's endowment* is a list $o_i = (o_i^1, \ldots, o_i^m)$. The *set of type-t objects* is $O^t = \{o_1^t, ..., o_n^t\}$, and the *set of all objects* is $O = \{o_1^1, o_1^2, \ldots, o_n^1, o_n^2, \ldots, o_n^m\}$. In particular, $|O| = n \times m$.

For each agent i, an *allotment* x_i assigns one object of each type to agent i, i.e., x_i is a list $x_i = (x_i^1, \ldots, x_i^m) \in \Pi_{t \in T} O^t$, where $x_i^t \in O^t$ is *agent i's type-t allotment*. Alternatively, we sometimes denote an allotment as a subset of objects,

[3] One agent and two agents multiple-type housing market problems are rather trivial cases due to the fact that no-trade or pairwise trade are the only possibilities. The real complexity of the model, however, arises from the possibility that agents are simultaneously trading in different coalitions.

i.e., $x_i = \{x_i^1, \ldots, x_i^m\} \subsetneq O$, and refer to a subset of an allotment as a *partial allotment*. We assume that each agent i has *complete*, *antisymmetric*, and *transitive preferences* R_i over all possible allotments, i.e., R_i is a linear order over $\Pi_{t \in T} O^t$.[4] For two allotments x_i and y_i, x_i is *weakly better than* y_i if $x_i \, R_i \, y_i$, and x_i is *strictly better than* y_i if $[x_i \; R_i \; y_i$ and not $y_i \; R_i \; x_i]$, denoted $x_i \; P_i \; y_i$. Finally, since preferences over allotments are strict, x_i is indifferent to y_i only if $x_i = y_i$. We denote preferences as ordered lists, e.g., $R_i : x_i, \; y_i, \; z_i$ instead of $x_i \, P_i \, y_i \, P_i \, z_i$. The *set of all preferences* is denoted by $\mathcal{R}$, which we will also refer to as the *strict preference domain*.

A *preference profile* specifies preferences for all agents and is denoted by a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We use the following standard notation $R_{-i} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ to denote the list of all agents' preferences, except for agent i's preferences. Furthermore, for each coalition S we define $R_S = (R_i)_{i \in S}$ and $R_{-S} = (R_i)_{i \in N \setminus S}$ to be the lists of preferences of coalitions S and $N \setminus S$, respectively.

In addition to the domain of strict preferences, we are considering several preference subdomains based on agents' "marginal preferences": assume that for each agent $i \in N$ and for each type $t \in T$, i has complete, antisymmetric, and transitive preferences R_i^t over the set of type-t objects O^t. We refer to R_i^t as *agent i's type-t marginal preferences*, and denote by $\mathcal{R}^t$ the *set of all type-t marginal preferences*. Then, we can define the following two preference domains.

Separability. Agent i's preferences $R_i \in \mathcal{R}$ are *separable* if for each $t \in T$ there exist type-t marginal preferences $R_i^t \in \mathcal{R}^t$ such that for any two allotments x_i and y_i,

$$\text{if for all } t \in T, \; x_i^t \, R_i^t \, y_i^t, \; \text{then } x_i \, R_i \, y_i.$$

$\mathcal{R}_s$ denotes the *domain of separable preferences*.

Before introducing our next preference domain, we introduce some notation. We use a bijective function $\pi_i : T \to T$ to order types according to agent i's "(subjective)

[4] Preferences R_i are *complete* if for any two allotments x_i, y_i, $x_i \; R_i \; y_i$ or $y_i \; R_i \; x_i$; they are *antisymmetric* if $x_i \; R_i \; y_i$ and $y_i \; R_i \; x_i$ implies $x_i = y_i$; and they are *transitive* if for any three allotments x_i, y_i, z_i, $x_i \; R_i \; y_i$ and $y_i \; R_i \; z_i$ imply $x_i \; R_i \; z_i$.

importance", with $\pi_i(1)$ being the most important and $\pi_i(m)$ being the least important object type. We denote π_i as an ordered list of types, e.g., by $\pi_i = (2,3,1)$, we mean that $\pi_i(1) = 2$, $\pi_i(2) = 3$, and $\pi_i(3) = 1$. So for each agent $i \in N$ and each allotment $x_i = (x_i^1, \ldots, x_i^m)$, by $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \ldots, x_i^{\pi_i(m)})$ we denote the allotment after rearranging it with respect to the *object-type importance order* π_i.

Lexicographic separability. Agent i's preferences $R_i \in \mathcal{R}$ are *lexicographically separable* if they are separable with type-t marginal preferences $(R_i^t)_{t \in T}$ and there exists an object-type importance order $\pi_i : T \to T$ such that for any two allotments x_i and y_i,

if $x_i^{\pi_i(1)} P_i^{\pi_i(1)} y_i^{\pi_i(1)}$ or

if there exists a positive integer $k \leq m-1$ such that

$x_i^{\pi_i(1)} = y_i^{\pi_i(1)}, \ \ldots, \ x_i^{\pi_i(k)} = y_i^{\pi_i(k)}$, and $x_i^{\pi_i(k+1)} P_i^{\pi_i(k+1)} y_i^{\pi_i(k+1)}$,

then $x_i \, P_i \, y_i$.

$\mathcal{R}_l$ denotes the *domain of lexicographically separable preferences*.

Remark 1 (**Representation of lexicographically separable preferences**). Note that any lexicographically separable preference relation $R_i \in \mathcal{R}_l$ is uniquely determined by agent i's marginal preferences $(R_i^t)_{t \in T}$ and an object-type importance order π_i. For example, consider a situation with $T = \{H(ouse), C(ar)\}$ and $N = \{1,2,3\}$ with each agent i's endowment equal to $o_i = (H_i, C_i)$. Assume that agent i has lexicographically separable preferences $\boldsymbol{R_i}$: (H_1, C_1), (H_1, C_2), (H_1, C_3), (H_2, C_1), (H_2, C_2), (H_2, C_3), (H_3, C_1), (H_3, C_2), (H_3, C_3). Then, agent i's type importance order is $\boldsymbol{\pi_i}$: H, C, and his marginal preferences are $\boldsymbol{R_i^H}$: H_1, H_2, H_3, and $\boldsymbol{R_i^C}$: C_1, C_2, C_3. Hence, agent i's preferences R_i can alternatively be written as $\boldsymbol{R_i = (R_i^H, R_i^C, \pi_i)}$.

For an even compacter description of agent i's lexicographically separable preferences, we can also rely on the strict ordering of objects that is induced by the object-type importance order together with his marginal preferences:

$$\boldsymbol{R_i} : H_1, \ H_2, \ H_3, \ C_1, \ C_2, \ C_3. \quad \square$$

An *allocation* x partitions the set of all objects O into agents' allotments, i.e., $x = \{x_1, \ldots, x_n\}$ is such that for each $t \in T$, $\cup_{i \in N} x_i^t = O^t$ and for each pair $i \neq j$, $x_i^t \neq x_j^t$. For simplicity, sometimes we will restate an allocation as a list $x = (x_1, \ldots, x_n)$. The *set of all allocations* is denoted by X, and the *endowment allocation* is denoted by $e = (o_1, \ldots, o_n)$. Given x, we define $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ to be the list of all agents' allotments, except for agent i's allotment; and $x_S = (x_i)_{i \in S}$ to be the list of allotments of coalition S.

We assume that when facing an allocation x, there are no consumption externalities and each agent $i \in N$ only cares about his own allotment x_i. Hence, each agent i's preferences over allocations X are essentially equivalent to his preferences over allotments $\Pi_{t \in T} O^t$. With some abuse of notation, we use notation R_i to denote an agent i's preferences over allotments as well as his preferences over allocations, i.e., for each agent $i \in N$ and for any two allocations $x, y \in X$, $x \; R_i \; y$ if and only if $x_i \; R_i \; y_i$.[5]

A *(multiple-type housing market) problem* is a triple (N, e, R); as the set of agents N and the endowment allocation e remain fixed throughout, we will simply denote problem (N, e, R) by R. Thus, the strict preference profile domain $\mathcal{R}^N$ also denotes the *set of all problems*.

Solutions / mechanisms and their properties

Note that all following definitions for the domain of strict preferences $\mathcal{R}$ can alternatively be formulated for any subdomain $\hat{\mathcal{R}} \subseteq \mathcal{R}$.

A *solution* is a set-valued function $F : \mathcal{R}^N \to 2^X$ that assigns to each problem $R \in \mathcal{R}^N$ a (possibly empty) set of allocations $F(R) \subseteq X$. A *mechanism* is a function $f : \mathcal{R}^N \to X$ that assigns to each problem $R \in \mathcal{R}^N$ an allocation $f(R) \in X$, and for each $i \in N$, $f_i(R)$ is *agent i's allotment under mechanism f at R*.

[5]Note that when extending strict preferences over allotments to preferences over allocations without consumption externalities, strictness is lost because any two allocations where an agent gets the same allotment are indifferent to that agent.

We next introduce and discuss some well-known properties for allocations, solutions, and mechanisms. First we consider a voluntary participation condition for an allocation x to be implementable without causing agents' any harm: no agent will be worse off than at his endowment.

Individual rationality. An allocation $x \in X$ is *individually rational* if for each agent $i \in N$, $x_i \; R_i \; o_i$. A solution / mechanism is *individually rational* if for each problem $R \in \mathcal{R}^N$, it assigns only individually rational allocations.

Next, we consider a well-known efficiency criterion.

Pareto efficiency. An allocation $y \in X$ *Pareto dominates* allocation $x \in X$ if for each agent $i \in N$, $y_i \; R_i \; x_i$, and for at least one agent $j \in N$, $y_j \; P_j \; x_j$. An allocation $x \in X$ is *Pareto efficient* if there is no allocation $y \in X$ that Pareto dominates it. A solution / mechanism is *Pareto efficient* if for each problem $R \in \mathcal{R}^N$, it assigns only Pareto efficient allocations.

Next, we define an incentive property for mechanisms that models that no agent can benefit from misrepresenting his preferences.

Strategy-proofness. A mechanism f is *strategy-proof* if for each problem $R \in \mathcal{R}^N$, each agent $i \in N$, and each preference relation $R'_i \in \mathcal{R}$, we have $f_i(R_i, R_{-i}) \; R_i \; f_i(R'_i, R_{-i})$.

Next, in order to introduce the standard cooperative solutions of the weak and the strict core, we introduce two blocking notions: for problem $R \in \mathcal{R}^N$, an *allocation $x \in X$ is strictly blocked by coalition $S \subseteq N$* if there exists an allocation $y \in X$ such that

(1) at allocation y agents in S reallocate their endowments, i.e., for each $i \in S$ and each $t \in T$, $y_i^t \in \Pi_{j \in S} o_j^t$ and

(2) all agents in S are strictly better off, i.e., for each $i \in S$, $y_i \; P_i \; x_i$.

An *allocation $x \in X$ is weakly blocked by coalition $S \subseteq N$* if there exists an allocation $y \in X$ such that (1) and

(2') all agents in S are weakly better off with at least one of them being strictly better off, i.e., for each $i \in S$, $y_i \, R_i \, x_i$, and for some $j \in S$, $y_j \, P_j \, x_j$.

Given the blocking notions above, we can restate individual rationality and Pareto efficiency as follows. An allocation is individually rational if it is not weakly or strictly blocked by any singleton coalition $\{i\}$ and an allocation is Pareto efficient if it is not weakly blocked by the set of all agents N.

We now introduce the first type of (possibly empty- or multi-valued) solution to multiple-type housing market problems that we will consider: core solutions.

Strict / weak core-stability. An allocation is a *strict / weak core allocation* if it is not weakly / strictly blocked by any coalition; the set of all strict / weak core allocations is the *strict / weak core*. Given a problem $R \in \mathcal{R}^N$, let $\text{SC}(R)$ / $\text{WC}(R)$ denote its strict / weak core. A mechanism f is *strictly / weakly core-stable* if for any problem, it assigns only strict / weak core allocations.

Note that for all problems $R \in \mathcal{R}^N$, $\text{SC}(R) \subseteq \text{WC}(R)$, and that all strict core allocations satisfy individual rationality and Pareto efficiency. So, if a mechanism is strictly core-stable, then it is individually rational and Pareto efficient as well. Furthermore, for some problems $R \in \mathcal{R}^N$, $\text{WC}(R)$ may be empty.[6]

We next focus on the domain of lexicographically separable preferences ($\mathcal{R}_l$) and extend Gale's famous top trading cycles (TTC) algorithm to multiple-type housing market problems. More specially, we adapt the multiple-type top-trading-cycles algorithm (MTTC) introduced by Sikdar et al. (2017, 2019) to $\mathcal{R}_l$.[7]

The multiple-type top trading cycles (MTTC) algorithm / mechanism.
Input. A multiple-type housing market problem $R \in \mathcal{R}_l^N$.

Step 1. ***Building step.*** Let $N(1) = N$ and $U(1) = O$. We construct a directed graph $G(1)$ with the set of nodes $N(1) \cup U(1)$. For each $o \in U(1)$, we add an edge from the object to its owner and for each $i \in N(1)$, we add an edge from the agent

[6]See Konishi et al. (2001, Example 2.3) for details.
[7]The preference domain that Sikdar et al. (2017, 2019) consider is larger than ours.

to his most preferred object in O (according to the linear representation of R_i we explained in Remark 1). For each edge $(i, o) \in N \times O$ we say that agent i points to object o.

Implementation step. A *trading cycle* is a directed cycle in graph $G(1)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(1)$; we denote the corresponding set of objects assigned through trading cycles by $A(1)$. If agent $i \in N$ was part of a trading cycle, then his partial allotment $x_i(1) = \{a_i(1)\}$; otherwise, $x_i(1) = \emptyset$.

Removal step. We remove all objects that were assigned through trading cycles from set O and set $U(2) := O \setminus A(1)$, which are the objects that have not been allocated yet. For each agent $i \in N$, we derive the set of *feasible continuation objects* $U_i(2)$ by removing all objects in $U(2)$ that are of a type that is already present in agent i's partial allotment $x_i(1)$. Since $m \geq 2$, no agents are removed in this step and we let $N(2) := N$. Go to Step 2.

In general, at Step q (≥ 2) we have the following:

Step *q*. If $U(q)$ (or equivalently $N(q)$) is empty, then stop; otherwise do the following.

Building step. We construct a directed graph $G(q)$ with the set of nodes $N(q) \cup U(q)$. For each $o \in U(q)$, we add an edge from the object to its owner and for each $i \in N$, we add an edge from the agent to his most preferred feasible continuation object in $U_i(q)$ (according to the linear representation of R_i we explained in Remark 1).

Implementation step. A *trading cycle* is a directed cycle in graph $G(q)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(q)$; we denote the corresponding set of objects assigned through trading cycles by $A(q)$. If agent $i \in N$ was part of a trading cycle, then his partial allotment $x_i(q) = x_i(q-1) \cup \{a_i(q)\}$; otherwise, $x_i(q) = x_i(q-1)$.

Removal step. We remove all agents that have received a (complete) allotment and denote the set of remaining agents by $N(q+1)$. Next, we remove all objects that were assigned through trading cycles from set $U(q)$ and set $U(q+1) := U(q) \setminus A(q)$. For each agent $i \in N(q)$, we derive the set of *feasible continuation objects* $U_i(q+1)$ by removing all objects in $U(q+1)$ that are of a type that is already present in agent i's partial allotment $x_i(q)$. Go to Step $q+1$.

Output. The MTTC algorithm terminates when all objects in O are assigned (it takes at most $n \cdot m$ steps). Assume that the final step is Step q^*. Then, the final allocation is $x(q^*) = \{x_1(q^*), \ldots, x_n(q^*)\}$.

The *multiple-type top trading cycles (MTTC) mechanism*, f^{MTTC}, assigns to each problem $R \in \mathcal{R}_l^N$ the allocation $x(q^*)$ obtained by the MTTC algorithm.

Sikdar et al. (2017, Theorem 1) proved that f^{MTTC} is strictly core-stable but not strategy-proof on the domain of lexicographically separable preferences $\mathcal{R}_l^N$. We illustrate the MTTC mechanism with the following example.

Example 1 (**The MTTC mechanism**).
Consider $R \in \mathcal{R}_l^N$ with $N = \{1,2,3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : H_2, H_3, \boldsymbol{H_1}, C_3, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, C_3, H_3, \boldsymbol{H_2}, H_1,$$

$$\boldsymbol{R_3} : H_2, H_1, \boldsymbol{H_3}, C_1, \boldsymbol{C_3}, C_2.$$

The allocation of the MTTC mechanism f^{MTTC} at R is obtained as follows.

Step 1. ***Building step.*** $G(1) = (N \cup O, E(1))$ with set of directed edges $E(1) = \{(H_1, 1), (H_2, 2), (H_3, 3), (C_1, 1), (C_2, 2), (C_3, 3), (1, H_2), (2, C_1), (3, H_2)\}$.

Implementation step. The trading cycle $1 \to H_2 \to 2 \to C_1 \to 1$ forms. Then, $a_1(1) = H_2$ and $a_2(1) = C_1$; $x_1(1) = \{H_2\}$, $x_2(1) = \{C_1\}$, and $x_3(1) = \emptyset$; and $A(1) = \{H_2, C_1\}$.

Removal step. $N(2) = N$, $U(2) = O \setminus A(1) = \{H_1, H_3, C_2, C_3\}$, $U_1(2) = \{C_2, C_3\}$, $U_2(2) = \{H_1, H_3\}$, and $U_3(2) = \{H_1, H_3, C_2, C_3\}$.

Step 2. ***Building step.*** $G(2) = (N(2) \cup U(2), E(2))$ with set of directed edges $E(2) = \{(H_1, 1), (H_3, 3), (C_2, 2), (C_3, 3), (1, C_3), (2, H_3), (3, H_1)\}$.

Implementation step. The trading cycle $1 \to C_3 \to 3 \to H_1 \to 1$ forms. Then, $a_1(2) = C_3$ and $a_3(2) = H_1$; $x_1(2) = \{H_2, C_3\}$, $x_2(2) = \{C_1\}$, and $x_3(2) = \{H_1\}$; and $A(2) = \{H_1, C_3\}$.

Removal step. $N(3) = \{2, 3\}$, $U(3) = U(2) \setminus A(2) = \{H_3, C_2\}$, $U_1(3) = \emptyset$, $U_2(3) = \{H_3\}$, and $U_3(3) = \{C_2\}$.

Step 3. ***Building step.*** $G(3) = (N(3) \cup U(3), E(3))$ with set of directed edges $E(3) = \{(H_3, 3), (C_2, 2), (2, H_3), (3, C_2)\}$.

Implementation step. The trading cycle $2 \to H_3 \to 3 \to C_2 \to 2$ forms. Then, $a_2(3) = H_3$ and $a_3(3) = C_2$; $x_1(3) = \{H_2, C_3\}$, $x_2(3) = \{H_3, C_1\}$, and $x_3(3) = \{H_1, C_2\}$; and $A(3) = \{H_3, C_2\}$.

Removal step. $N(4) = \emptyset$ and $U(4) = \emptyset$.

Thus, the MTTC algorithm yields the strict core allocation $x = ((H_2, C_3), (H_3, C_1), (H_1, C_2))$. □

Preference revelation games

We now formulate a natural preference revelation game for the domain of lexicographically separable preferences ($\mathcal{R}_l$).

Given a multiple-type housing market problem represented by $R \in \mathcal{R}_l^N$ and a mechanism $f : \mathcal{R}_l^N \to X$, the *preference revelation game* induced by f is the *strategic game* $\Gamma_f(R) = (\mathcal{R}_l^N, f, R)$, where $\mathcal{R}_l$ is each agent's *strategy space*, f is the *outcome function*, and each agent i evaluates outcomes with R_i.

Finding suitable solutions of strategic games is an important task in game theory. The most studied solution concept is the Nash equilibrium. However, Nash equilibria may not be Pareto efficient (e.g., in prisoner's dilemma or tragedy of the commons situations). To reestablish Pareto efficiency, Aumann (1959, 1960) introduced strong Nash equilibria, a strengthening of Nash equilibria that requires an equilibrium strategy profile to be robust against coalitional deviations (see Footnote 8). By requiring Pareto efficiency for each coalition, Dubey (1986) introduced an even stronger refinement of the set of Nash equilibria: strict strong Nash equilibria.

Nash / strict strong Nash equilibria. Let $R \in \mathcal{R}_l^N$ be a multiple-type housing market problem and consider the preference revelation game $\Gamma_f(R)$.

A strategy profile $R^* \in \mathcal{R}_l^N$ is a *Nash equilibrium* of $\Gamma_f(R)$ if for each agent $i \in N$ and each strategy $R'_i \in \mathcal{R}_l$, $f_i(R^*) = f_i(R^*_i, R^*_{-i}) \; R_i \; f_i(R'_i, R^*_{-i})$. We denote the *set of Nash equilibria* by $\text{Nash}(\Gamma_f(R))$ and the *set of Nash equilibrium outcomes* by $f(\text{Nash}(\Gamma_f(R)))$.

A strategy profile $R^* \in \mathcal{R}_l^N$ is a *strict strong Nash equilibrium*[8] of $\Gamma_f(R)$ if for each coalition $S \subseteq N$ and each strategy list $R'_S \in \mathcal{R}_l^S$,

$$[\text{for each agent } i \in S,\; f_i(R'_S, R^*_{-S}) \; R_i \; f_i(R^*_S, R^*_{-S})] \text{ implies}$$

$$[\text{for each agent } i \in S,\; f_i(R'_S, R^*_{-S}) = f_i(R^*_S, R^*_{-S})].$$

We denote the *set of strict strong Nash equilibria* by $\text{sNash}(\Gamma_f(R))$ and the *set of strict strong Nash equilibrium outcomes* by $f(\text{sNash}(\Gamma_f(R)))$.

Note that $\text{sNash}(\Gamma_f(R)) \subseteq \text{Nash}(\Gamma_f(R)) \subseteq \mathcal{R}_l^N$.

Given a preference revelation game $\Gamma_f(R)$, we say that agent i plays a *truth-telling strategy* if he truthfully reports his preferences R_i. If all agents play truth-telling strategies, then $R = (R_i)_{i \in N}$ is a *truth-telling strategy profile* at $\Gamma_f(R)$. Note that if f is strategy-proof, then truth-telling is a weakly dominant strategy for each

[8] The set of strict strong Nash equilibria is a refinement of the set of strong Nash equilibria: a strategy profile $R^* \in \mathcal{R}_l^N$ is a *strong Nash equilibrium* of $\Gamma_f(R)$ if for each coalition $S \subseteq N$ and each strategy list $R'_S \in \mathcal{R}_l^S$, [for each agent $i \in S$, $f_i(R'_S, R^*_{-S}) \; R_i \; f_i(R^*_S, R^*_{-S})$] implies [for some agent $j \in S$, $f_j(R'_S, R^*_{-S}) = f_j(R^*_S, R^*_{-S})$]. For a discussion of existence of strict strong Nash equilibria we refer to Remark 2.

agent and the truth-telling strategy profile is a weakly dominant strategy Nash equilibrium.[9]

3 Results

3.1 Motivating examples

As mentioned in the introduction, for Shapley-Scarf housing markets with strict preferences, the unique strict core allocation can be obtained by a unique individually rational, Pareto efficient, and strategy-proof mechanism (Ma, 1994), namely the top-trading cycles (TTC) mechanism. Later, Sönmez (1999) considered a generalization of Shapley and Scarf (1974)'s housing market problems, *generalized indivisible goods allocation problems* (see Appendix A), and showed that, whenever the preference domain satisfies a certain condition of richness and if there exists a mechanism satisfying individual rationality, Pareto efficiency, and strategy-proofness, then for any problem having a non-empty strict core, the strict core must be essentially single-valued[10] and the mechanism must choose a strict core allocation. Takamiya (2003) showed the following converse result: whenever the preference domain satisfies a certain condition of richness and if the strict core solution is essentially single-valued, then any selection from the strict core solution is strategy-proof.

However, for multiple-type housing market problems, these results do not hold anymore: Konishi et al. (2001) (Sikdar et al., 2017, respectively) showed that on the domain of separable preferences (lexicographically separable preferences, respectively), no mechanism satisfies individual rationality, Pareto efficiency, and strategy-proofness. Note that neither the domain of separable preferences nor the domain of lexicographically separable preferences satisfies the domain richness condition of

[9]For $\Gamma_f(R)$ and $i \in N$, a strategy $R_i^* \in \mathcal{R}_l$ is *weakly dominant* if for each $R' \in \mathcal{R}_l^N$, $f_i(R_i^*, R'_{-i})\ R_i\ f_i(R')$. A Nash equilibrium R^* is a *weakly dominant strategy Nash equilibrium* if for each $i \in N$, R_i^* is a weakly dominant strategy.

[10]The strict core is essentially single-valued if each agent is indifferent between any two strict core allocations.

Sönmez (1999) (see Appendix A).

The following example shows that on the one hand an individually rational and Pareto efficient mechanism can pick an allocation at which no agent has an incentive to misrepresent his preferences while on the other hand the strict core may be multi-valued (without being essentially single-valued).

Example 2 (**Non-manipulability and a multi-valued strict core**). Consider $R \in \mathcal{R}_l^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, H_3, C_3, \boldsymbol{C_1}, C_2,$$

$$\boldsymbol{R_2} : H_3, \boldsymbol{H_2}, H_1, C_1, \boldsymbol{C_2}, C_3,$$

$$\boldsymbol{R_3} : H_2, \boldsymbol{H_3}, H_1, C_1, \boldsymbol{C_3}, C_2.$$

Applying the MTTC algorithm to R, at Step 1, the trading cycle $2 \to H_3 \to 3 \to H_2 \to 2$ forms; the trading cycle at Step 2 is $1 \to H_1 \to 1$; the trading cycle at Step 3 is $1 \to C_3 \to 3 \to C_1 \to 1$; and at Step 4, we have $2 \to C_2 \to 2$. The final outcome is the strict core allocation $x = ((H_1, C_3), (H_3, C_2), (H_2, C_1))$.

Note that at problem R, no agent has an incentive to misrepresent his preferences: agent 3 has no incentive to misreport his preferences because he receives his best allotment. Agent 1 cannot obtain his best house H_2 by misreporting his preferences (it is traded in Step 1 between agents 2 and 3). Given that, he receives the best possible allotment and has no incentive to misreport his preferences. Finally, agent 2 already obtains his best house and if he tries to obtain his best car by misreporting his preferences, he cannot obtain his best house; thus, he has no incentive to misreport his preferences. Finally, the strict core is not unique: $((H_1, C_3), (H_3, C_1), (H_2, C_2))$ is also a strict core allocation. □

Recall that for multiple-type housing market problems with lexicographically separable preferences, no mechanism satisfies individual rationality, Pareto efficiency, and strategy-proofness (Sikdar et al., 2017). Hence, strict core stability and strategy-proofness are also not compatible. Thus, in our context, strategy-proofness, or truth-telling being a weakly dominant strategy Nash equilibrium in the corresponding preference revelation game, is a very strong requirement. Therefore, we next consider implementation through a different equilibrium concept: strict strong Nash equilibrium.

For generalized indivisible goods allocation problems, Takamiya (2009) studied the relationship between coalitional equilibria and the strict core. Takamiya's main result implies that for Shapley-Scarf housing market problems and for a preference revelation game induced by an individually rational and Pareto efficient mechanism f, the set of strict strong Nash equilibrium outcomes equals the strict core.

Corollary 1 (Takamiya, 2009). *For each Shapley-Scarf housing market* $R \in \mathcal{R}^N$ *and each individually rational and Pareto efficient mechanism* f*,* $f(\text{sNash}(\Gamma_f(R))) = \text{SC}(R)$*.*

The following example shows that Corollary 1 does not extend to multiple-type housing markets with lexicographically separable preferences.

Example 3 (Corollary 1 does not extend to $\boldsymbol{R_l^N}$).
Consider $R \in \mathcal{R}_l^N$, $N = \{1,2,3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, H_3, C_3, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : H_1, \boldsymbol{H_2}, H_3, C_1, \boldsymbol{C_2}, C_3,$$

$$\boldsymbol{R_3} : H_1, \boldsymbol{H_3}, H_2, C_1, \boldsymbol{C_3}, C_2.$$

Applying the MTTC algorithm to R, at Step 1, the trading cycle $1 \to H_2 \to 2 \to H_1 \to 1$ forms; the trading cycle at Step 2 is $3 \to H_3 \to 3$; the trading cycle at Step 3 is $1 \to C_3 \to 3 \to C_1 \to 1$; and at Step 4 we have $2 \to C_2 \to 2$. The final outcome is the strict core allocation

$$x = ((H_2, C_3), (H_1, C_2), (H_3, C_1)).$$

There is another strict core allocation

$$x' = ((H_2, C_2), (H_1, C_1), (H_3, C_3)).$$

We show that

$$f^{\text{MTTC}}(\text{sNash}(\Gamma_{f^{\text{MTTC}}}(R))) = \{x\} \subsetneq \{x, x'\}.$$

First, we prove that truth-telling, i.e., reporting preference profile R, is a strict strong Nash equilibrium. Suppose that truth-telling is not a strict strong Nash equilibrium. Then, a profitable deviation from R and x exists, i.e., there exist $S \subseteq N$ and $R'_S \in \mathcal{R}_l^S$ such that for each agent $i \in S$, $f_i^{\text{MTTC}}(R'_S, R_{-S})\, R_i\, x_i$ and for some agent $j \in S$, $f_j^{\text{MTTC}}(R'_S, R_{-S})\, P_j\, x_j$. We now show that no such $S \subseteq N$ exists.

Note that at x, agent 1 receives his best allotment and thus coalition $\{1\}$ has no profitable deviation from R. Furthermore, if agent 1 takes part in a profitable deviation, then he must still receive (H_2, C_3).

For coalitions $\{2\}$ or $\{1, 2\}$, agent 2 can only be better off by receiving (H_1, C_1). However, he can only receive C_1 at Step 1 of the MTTC algorithm by misreporting his preferences as $R'_2 : C_1, \cdots$. However, with R'_2, at Step 2, agent 3 would receive H_1 and agent 2 would not be better off.

For coalitions $\{3\}$ or $\{1, 3\}$, agent 3 can only be better off by receiving (H_1, C_1). However, then H_3 has to be assigned to agent 1 or agent 2, which would violate individual rationality (recall that agent 1 only participates in a deviating coalition if he still receives H_2 and agent 2 then would receive H_3).

Next, consider coalition $\{2, 3\}$, which has a conflict of interest. Agent 2 can only be better off by receiving (H_1, C_1), which leaves agent 3 with an allotment that is worse than $x_3 = (H_3, C_1)$, and agent 3 can only be better off by receiving (H_1, C_1), which leaves agent 2 with an allotment that is worse than $x_2 = (H_1, C_2)$.

Finally, the grand coalition $\{1, 2, 3\}$ cannot profitably deviate because the MTTC mechanism is Pareto efficient.

Hence, no profitable deviation from R and x exists.

Second, we prove that x' is not a strict strong Nash equilibrium outcome. Assume that there is a strict strong Nash equilibrium $R' = (R'_1, R'_2, R'_3)$ such that

$f^{\mathrm{MTTC}}(R') = x'$. We show that there is a profitable deviation for coalition $\{1, 3\}$, i.e., there exists $R'' = (R''_1, R'_2, R''_3)$ such that for each agent $i \in \{1, 3\}$, $f_i^{\mathrm{MTTC}}(R'')\, R_i\, x'_i$ and for some agent $j \in \{1, 3\}$, $f_j^{\mathrm{MTTC}}(R'')\, P_j\, x'_j$.

There are two cases depending on agent 2's object-type importance order at R'_2.

Case 1. Agent 2 misreported his importance order at R'_2, i.e., $\pi'_2 : C, H$.

Recall that $f_2^{\mathrm{MTTC}}(R') = (H_1, C_1)$. Hence, by individual rationality of f at R', agent 2 ranked C_1 above C_2 and

$$\boldsymbol{R'_2} : C_3, C_1, \boldsymbol{C_2}, \cdots, \quad \text{or}$$

$$\boldsymbol{R'_2} : C_1, \boldsymbol{C_2}, C_3, \cdots, \quad \text{or}$$

$$\boldsymbol{R'_2} : C_1, C_3, \boldsymbol{C_2}, \cdots .$$

Next, consider strategy profile $R'' = (R''_1, R'_2, R_3)$ obtained by agents 1 and 3 deviating from R' such that

$$\boldsymbol{R''_1} : C_3, \boldsymbol{C_1}, C_2, H_2, \boldsymbol{H_1}, H_3.$$

Applying the MTTC algorithm to R'', at Step 1, the trading cycle $1 \to C_3 \to 3 \to H_1 \to 1$ forms; and at Step 2 we have $2 \to C_1 \to 1 \to H_2 \to 2$. The final outcome is $f^{\mathrm{MTTC}}(R'') = y = ((H_2, C_3), (H_3, C_1), (H_1, C_2))$. Since $y_1\, P_1\, x'_1$ and $y_3\, P_3\, x'_3$, coalition $\{1, 3\}$ has an incentive to deviate from R' to R'', which implies that R' is not a strict strong Nash equilibrium; a contradiction.

Case 2. Agent 2 truthfully reported his importance order at R'_2, i.e., $\pi'_2 : H, C$.

Recall that $f^{\mathrm{MTTC}}(R') = (H_1, C_1)$. Hence, by individual rationality of f at R', agent 2 ranked H_1 above H_2 and

$$\boldsymbol{R'_2} : H_3, H_1, \boldsymbol{H_2}, \cdots, \quad \text{or}$$

$$\boldsymbol{R'_2} : H_1, \boldsymbol{H_2}, H_3, \cdots, \quad \text{or}$$

$$\boldsymbol{R'_2} : H_1, H_3, \boldsymbol{H_2}, \cdots \quad .$$

Next, consider strategy profile $R'' = (R_1, R'_2, R''_3)$ obtained by agents 1 and 3 deviating from R' such that

$$\boldsymbol{R''_3} : \boldsymbol{H_3}, H_1, H_2, C_1, \boldsymbol{C_3}, C_2.$$

Applying the MTTC algorithm to R'', at Step 1, the trading cycle $3 \to H_3 \to 3$ forms; and we also have $1 \to H_2 \to 2 \to H_1 \to 1$ (this cycle, depending on R'_2, occurs at Step 1 or Step 2). Subsequently, we have the trading cycle $1 \to C_3 \to 3 \to C_1 \to 1$. The final outcome is $f^{\text{MTTC}}(R'') = x = ((H_2, C_3), (H_1, C_2), (H_3, C_1))$. Since $x_1 \; P_1 \; x'_1$ and $x_3 \; P_3 \; x'_3$, coalition $\{1, 3\}$ has an incentive to deviate from R' to R'', which implies that R' is not a strict strong Nash equilibrium; a contradiction. □

Based on Corollary 1 and Example 3 one could now conjecture that for each multiple-type housing market $R \in \mathcal{R}_l^N$ and each individually rational and Pareto efficient mechanism f, we have $f(\text{sNash}(\Gamma_f(R))) \subseteq \text{SC}(R)$. That conjecture is almost correct; however, we need to strengthen individual rationality and Pareto efficiency to strict core-stability (see Example 5).

3.2 Main results

We show that for lexicographically separable preferences, if a mechanism is strictly core-stable, then any strict strong Nash equilibrium of the corresponding preference revelation game will induce a strict core allocation. However, for some lexicographically separable multiple-type housing markets, there exist strict core allocations that cannot be implemented in strict strong Nash equilibrium.

Theorem 1. *Let f be a strictly core-stable mechanism on $\mathcal{R}_l^N$.*
Then, for each problem $R \in \mathcal{R}_l^N$ and the corresponding preference revelation game $\Gamma_f(R) = (\mathcal{R}_l^N, f, R)$, the set of strict strong Nash equilibrium outcomes is a subset of the strict core, that is, $f(\text{sNash}(\Gamma_f(R))) \subseteq \text{SC}(R)$.

Furthermore, there exist problems $R \in \mathcal{R}_l^N$ such that $f(\text{sNash}(\Gamma_f(R))) \subsetneq \text{SC}(R)$.

We would like to emphasize that the strict core-stability of f is key for this result. Clearly, if for some preference profiles the strict core is empty, then a strictly core-stable mechanism f cannot exist. Thus, in this first result, we restrict the preference domain to $\mathcal{R}_l^N$ with the intent to generalize Theorem 1 later on.

Proof. Let f be a strictly core-stable mechanism on $\mathcal{R}_l^N$.

First, let $R \in \mathcal{R}_l^N$ and assume by contradiction, that $f(\text{sNash}(\Gamma_f(R))) \not\subseteq \text{SC}(R)$. Let $R' \in \mathcal{R}_l^N$ be such that $R' \in \text{sNash}(\Gamma_f(R))$ and $f(R') = x \notin \text{SC}(R)$. Hence, x can be weakly blocked by a coalition S and there exists an allocation y such that (1) for each $i \in S$ and each $t \in T$, $y_i^t \in \{o_j^t\}_{j \in S}$, and (2') for each $i \in S$, $y_i \, R_i \, x_i$, and for some $j \in S$, $y_j \, P_j \, x_j$.

Now we consider the profile $(\hat{R}_S, R'_{-S}) \in \mathcal{R}_l^N$ such that each agent $i \in S$ ranks allotment y_i as his best allotment; for each $i \in S$, it then holds that $\hat{R}_i : y_i, \cdots$, i.e., each agent i, for each object type t, ranks y_i^t as best type-t object. We want to show that coalition S has an incentive to deviate from R'_S to $\hat{R}_S$. To this end, we first prove the following claim.

Claim 1. For each $i \in S$, we have $f_i(\hat{R}_S, R'_{-S}) = y_i$.

Let $z = f(\hat{R}_S, R'_{-S})$. Suppose that for some agent $j \in S$, $z_j \neq y_j$. We show that z is not a strict core allocation at $(\hat{R}_S, R'_{-S})$, i.e., $z \notin \text{SC}(\hat{R}_S, R'_{-S})$.

At $(\hat{R}_S, R'_{-S})$, for each agent $i \in S$, $y_i \, \hat{R}_i \, z_i$ because y_i is his best allotment. Since $z_j \neq y_j$, $y_j \, \hat{P}_j \, z_j$. Therefore, at $(\hat{R}_S, R'_{-S})$, allocation z can also be weakly blocked by coalition S via allocation y. Thus, $f(\hat{R}_S, R'_{-S}) \notin \text{SC}(\hat{R}_S, R'_{-S})$, which contradicts that f is strictly core-stable. $\square$

Strictly speaking, by Claim 1, we now only know that $f(\hat{R}_S, R'_{-S}) = y'$ such that $y'_S = y_S$. However, since allotments to agents in $N \setminus S$ play no role in our proof, it is without loss of generality to assume that $y' = y$. Hence, when coalition S deviates from R'_S to $\hat{R}_S$, by Claim 1 and without loss of generality, $f(\hat{R}_S, R'_{-S}) = y$. Thus, since $f(R')$ is weakly blocked by S via y, for each $i \in S$, $f_i(\hat{R}_S, R'_{-S}) \, R_i \, f_i(R')$ and for $j \in S$, $f_j(\hat{R}_S, R'_{-S}) \; P_j \; f_j(R')$; contradicting that R' is a strict strong Nash equilibrium.

Example 3 exhibits a problem $R \in \mathcal{R}_l^N$ such that $f(\text{sNash}(\Gamma_f(R))) \subsetneq \text{SC}(R)$ (recall that in Example 3 there is a unique strict strong Nash equilibrium outcome while multiple strict core allocations exist). □

Remark 2 (**Existence of strict strong Nash equilibria, an open problem**). The existence of (strict) strong Nash equilibria has been proven for specific classes of games, such as social choice / voting (Dutta and Sen, 1991), congestion games (Holzman and Law-Yone, 1997), cost sharing games (Epstein et al., 2009), and continuously convex games (Nessah and Tian, 2014). However, in general, (strict) strong Nash equilibria do not need to exist.[11]

> **Question:** Let f be a strictly core-stable mechanism on $\mathcal{R}_l^N$. For each problem $R \in \mathcal{R}_l^N$, do we have $f(\text{sNash}(\Gamma_f(R))) \neq \emptyset$?

For Shapley-Scarf housing markets and the TTC mechanism, truth-telling is a strict strong Nash equilibrium. Thus, for higher-dimensional multiple-type housing markets, one could conjecture that for f^{MTTC}, MTTC allocations can always be implemented in strict strong Nash equilibrium. The following example shows that the MTTC allocation cannot always be implemented truthfully in strict strong Nash equilibrium.

Consider $R \in \mathcal{R}_l^N$ with $N = \{1,2,3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$R_1 : H_2, \boldsymbol{H_1}, H_3, C_3, \boldsymbol{C_1}, C_2,$$

$$R_2 : H_3, H_1, \boldsymbol{H_2}, \boldsymbol{C_2}, C_1, C_3,$$

$$R_3 : H_1, H_2, \boldsymbol{H_3}, \boldsymbol{C_3}, C_1, C_2.$$

Applying the MTTC algorithm to R, at Step 1, the trading cycle $1 \to H_2 \to 2 \to H_3 \to 3 \to H_1 \to 1$ forms; the trading cycles at Step 2 are $2 \to C_2 \to 2$ and

[11] Hoefer and Skopalik (2013) pointed out the following technical difficulty of finding strong Nash equilibria: "a strong Nash equilibrium must be the optimal solution of multiple non–convex optimization problems."

$3 \to C_3 \to 3$; and at Step 3 we have $1 \to C_1 \to 1$. The final outcome is the strict core allocation $x = ((H_2, C_1), (H_3, C_2), (H_1, C_3))$.

However, the truth-telling profile R is not a strict strong Nash equilibrium: agent 1 has an incentive to misreport the following preferences

$$R'_1 : C_3, \boldsymbol{C_1}, C_2, H_2, \boldsymbol{H_1}, H_3.$$

For profile $R' = (R'_1, R_2, R_3)$, the MTTC algorithm yields allocation $x' = ((H_2, C_3), (H_3, C_2), (H_1, C_1))$. Since $x'_1 = (H_2, C_3)\ P_1\ (H_2, C_1) = x_1$, R is not a strict strong Nash equilibrium.

The above example illustrates that an implementation of the MTTC allocation in strict strong Nash equilibrium might require some agents to (possibly mutually) change their object type sequences. We neither found a systematic way for agents to change their object type sequences to show existence of strict strong Nash equilibria, nor did we manage to construct a counter example. □

A more general result

Note that the proof of Theorem 1 did not use many properties of the lexicographically separable preference domain. It turns out that our result can easily be extended to other preference domains. Consider a subdomain of preferences $\hat{\mathcal{R}} \subseteq \mathcal{R}$ that satisfies the following two assumption.

Assumption 1 (**Strict core existence and minimal preference domain richness**)**.** Preference domain $\hat{\mathcal{R}} \subseteq \mathcal{R}$ satisfies

(a) *strict core existence* if for each problem $R \in \hat{\mathcal{R}}^N$, $\mathrm{SC}(R) \neq \emptyset$; and

(b) *minimal preference domain richness* if for each allocation $x \in X$, each agent i can position x_i as his best allotment; i.e., for each $x \in X$, there exists a profile $\hat{R} \in \hat{\mathcal{R}}^N$ such that for each $i \in N$, $\hat{R}_i : x_i, \cdots$.

Assumption 1 is simple and reasonable. Assumption 1 (a) allows us to focus on the solution of the strict core and for that the strict core should always be nonempty. Assumption 1 (b) is a very weak preference domain richness condition that is different from the one used by Sönmez (1999, Assumption B) and weaker than the one imposed by Takamiya (2009, Condition A). We discuss the preference domain richness conditions of Sönmez (1999) and Takamiya (2009) in Appendix A.

Remark 3 (Preference domains satisfying Assumption 1). The domains of weak and strict preferences for Shapley-Scarf housing markets and the lexicographically separable preference domain for multiple-type housing markets all satisfy Assumption 1. There are various larger lexicographic domains, e.g., those of Monte and Tumennasan (2015, generalized lexicographical preferences) and Sikdar et al. (2017, lexicographical preferences), that satisfy Assumption 1. Hence, our Theorem 1 applies to these settings as well (see the following Theorem 2). □

We now show that Theorem 1 can be extended to any preference domain $\hat{\mathcal{R}} \subseteq \mathcal{R}$ satisfying Assumption 1.

Theorem 2. *Let $\hat{\mathcal{R}}$ satisfy Assumption 1 and let f be a strictly core-stable mechanism on $\hat{\mathcal{R}}^N$. Then, for each problem $R \in \hat{\mathcal{R}}^N$ and the corresponding preference revelation game $\Gamma_f(R) = (\hat{\mathcal{R}}^N, f, R)$, the set of strict strong Nash equilibrium outcomes is a subset of the strict core, that is, $f(\text{sNash}(\Gamma_f(R))) \subseteq \text{SC}(R)$.*

Furthermore, there exist problems $R \in \hat{\mathcal{R}}^N$ such that $f(\text{sNash}(\Gamma_f(R))) \subsetneq \text{SC}(R)$.

***Proof*.** The proof is the same as that of Theorem 1 since in that proof the only properties of the preference domain that were (implicitly) used were strict core existence and minimal domain richness. □

The role of assumptions in Theorems 1 and 2

In Theorems 1 and 2, we make three sufficient assumptions: (a) strict core existence, (b) minimal preference domain richness, and (c) strict core-stability of f. We now show that if assumptions (a) and (c) do not hold, then our result(s) need not be true. We do not discuss the role of minimal preference domain richness for our result since we believe that, once ranking certain allotments first is not possible for the agents, one starts to discuss very unstructured preference domains.

(a) For some $R \in \mathcal{R}_s^N$ it is possible that $\mathrm{SC}(R) = \emptyset$ and $\mathrm{sNash}(\Gamma_f(R)) \neq \emptyset$. See Example 4 below.

(c) If f is individually rational and Pareto efficient but not strictly core-stable (even if f is defined on $\mathcal{R}_l^N$), then the allocation induced by f may not be a strict core allocation. See Example 5 below.

Example 4 (Strict core existence is important for our result to hold). Consider Example 2.2 in Konishi et al. (2001), i.e., consider $R \in \mathcal{R}_s^N$ with $N = \{1,2,3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$\boldsymbol{R_1}$: $(H_1,C_3),(H_3,C_3),(H_1,C_2),\boldsymbol{(H_1,C_1)}, \cdots$,

$\boldsymbol{R_2}$: $(H_2,C_3),(H_2,C_1),(H_3,C_3),(H_3,C_1),\boldsymbol{(H_2,C_2)}, \cdots$,

$\boldsymbol{R_3}$: $(H_2,C_1),(H_2,C_2),(H_3,C_1),(H_1,C_1),(H_3,C_2),(H_1,C_2),(H_2,C_3),\boldsymbol{(H_3,C_3)},(H_1,C_3)$.

Note that $\mathrm{SC}(R) = \emptyset$ (see Konishi et al., 2001, Example 2.2).

Now, consider a mechanism f that chooses a strict core allocation whenever the strict core is nonempty and otherwise it determines an allocation by serial dictatorship based on the sequence of agents $1 \rhd 3 \rhd 2$: agent 1 moves first and chooses his most preferred allotment; then agent 3 moves and, considering only the remaining objects, chooses his most preferred allotment; finally agent 2 receives the remaining objects. By definition, f is strictly core-stable whenever this is possible.

At the preference revelation game $\Gamma_f(R)$, by truth-telling agents 1 and 3 get their best allotments, and hence, agent 2's strategy does not influence the outcome. Thus, the truth-telling strategy profile is a strict strong Nash equilibrium, i.e., $R \in \mathrm{sNash}(\Gamma_f(R))$, and its outcome is $f(R) = ((H_1,C_3),(H_3,C_2),(H_2,C_1))$. Since the strict core is empty, we thus have $f(\mathrm{sNash}(\Gamma_f(R))) \not\subseteq \mathrm{SC}(R)$. □

Example 5 (Strict core-stability of f is important for our result to hold). We introduce the so-called *trading cycles and chains (TCC) algorithm / mechanism* on $\mathcal{R}_l^N$. The first step of the TCC algorithm is the same as that of the MTTC algorithm: let each agent point to his most preferred object and clear all trading cycles. Furthermore, “deactivate” all agents who have received an object from *another agent*; a deactivated agent cannot point to an object in the next steps of the algorithm.

The main difference between TCC and MTTC now arises: only agents who did not receive any object from another agent yet are active and can proceed to the next step and point to their most preferred object among those objects that have not been assigned yet. In the following steps, if there is a trading cycle, then we assign objects to the agents in the trading cycle.

However, if there is no trading cycle (this can happen due to the fact that some agents are inactive), then we clear a so-called trading chain where the last object belongs to a de-activated agent - we select a trading chain according to a tie-breaking rule, e.g., if two agents i and j point to the same object, then we break the tie in favor of agent $j > i$. Agents along the chain receive the object they point to (except the last agent in the chain who was inactive). Again, deactivate all agents who have received an object from another agent.

Continue clearing first trading cycles then trading chains until all agents are deactivated. Then, we activate all agents and repeat the TCC algorithm to allocate the remaining objects.

Since at each step of the TCC algorithm, each agent who receives an object receives his most preferred object among all of his feasible continuation objects, the TCC mechanism is individually rational and Pareto efficient. The following example shows that the TCC mechanism is not strictly core-stable.

Consider $R \in \mathcal{R}_l^N$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment (H_i, C_i), and

$$\boldsymbol{R_1} : C_2, \boldsymbol{C_1}, C_3, H_2, \boldsymbol{H_1}, H_3,$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, C_3, H_1, H_3, \boldsymbol{H_2},$$

$$\boldsymbol{R_3} : H_1, H_2, \boldsymbol{H_3}, C_2, C_1, \boldsymbol{C_3}.$$

Applying the MTTC algorithm to R yields the unique strict core allocation $x = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$.

The TCC algorithm at R proceeds as follows. At Step 1 trading cycle $1 \to C_2 \to 2 \to C_1 \to 1$ forms, is executed, and agents 1 and 2 are deactivated. At Step 2, agent 3 points to his most preferred object H_1 and since this forms a trading chain, agent 3 receives H_1 and is deactivated.

Then, all agents are activated again. At Step 3, trading cycle $3 \to C_3 \to 3$ forms, is executed, and agent 3 is deactivated. Finally, agent 1 points to H_2 and agent 2 points to H_3, and since this forms a trading chain, agent 1 and 2 receive H_2 and H_3, respectively. The final allocation is $y = ((H_2, C_2), (H_3, C_1), (H_1, C_3))$. Allocation y is individually rational and Pareto efficient, but it is weakly blocked by coalition $\{1, 2\}$ via allocation x.

Let f^{TCC} be the TCC mechanism. We prove that truth-telling, i.e., reporting preference profile R, is a strict strong Nash equilibrium. Suppose that truth-telling is not a strict strong Nash equilibrium. Then, there exist $S \subseteq N$ and $R'_S \in \mathcal{R}_l^S$ such that for each agent $i \in S$, $f_i^{\text{TCC}}(R'_S, R_{-S}) \; R_i \; y_i$ and for some agent $j \in S$, $f_j^{\text{TCC}}(R'_S, R_{-S}) \; P_j \; y_j$.

Note that at y, agent 1 receives his best allotment and thus, if agent 1 takes part in a profitable deviation, then he must still receive (H_2, C_2). This implies that

(a) there is no profitable deviation for coalition $\{1\}$.

(b) We next show that there is no profitable deviation for coalitions $\{2\}$ and $\{1, 2\}$. Note that agent 2 can only be better off by receiving (H_1, C_1). Consider (R'_1, R'_2) (possibly $R'_1 = R_1$) and note that for this to be a profitable deviation, agents 1 and 2 still need to trade in the first step of the TCC algorithm, which means that they are not active in the second step where agent 3 now can choose an object.

If agents 1 and 2 traded cars C_1 and C_2 or car C_1 and house H_2, then agent 3 will choose H_1 in the second step. Thus, agent 2 cannot receive (H_1, C_1) and hence would not be better off.

If agents 1 and 2 traded house H_1 and car C_2, then agent 3 will choose H_2 in the second step. Thus, the third step is a normal trading step in which agent 1 will point to the only remaining house H_3, agent 3 points at car C_1, and hence the final allocation is $((H_3, C_2), (H_1, C_3), (H_2, C_1))$. Thus, agent 2 would not be better off.

If agents 1 and 2 traded houses H_1 and H_2, then both of them participated in the preference deviation. Then, agent 3 will choose H_3. However, since agent 3 still did not receive any object from another agent yet, he remains the only active agent and will choose car C_2 in the third step. Now, the final allocation is either $((H_2, C_1), (H_1, C_3), (H_3, C_2))$ or $((H_2, C_3), (H_1, C_1), (H_3, C_2))$. Thus, agent 1 or agent 2 is worse off.

(c) We next show that there is no profitable deviation for coalitions $\{3\}$ and $\{1, 3\}$. Note that agent 3 can only be better off by receiving (H_1, C_1) or (H_1, C_2). Consider (R'_1, R'_3) (possibly $R'_1 = R_1$) and note that agents 1 and 2 still trade in the first step of the TCC algorithm (either because agents 1 and 2 report true preferences, or because agent 1 participates in the deviation and hence must still receive (H_2, C_2)).

If agent 1 did not participate in the deviation, then agents 1 and 2 trade cars C_1 and C_2 in the first step of the TCC algorithm. But then, agent 3 cannot be bettor off since C_1 and C_2 are already assigned.

Assume that agent 1 participated in the deviation. Then, agent 1 must still receive (H_2, C_2). Hence, since agent 2 reports preference truthfully, agent 2 will receive car C_1 in the first step. Thus, agent 3 can only be better off by receiving (H_1, C_2), which is not compatible with agent 1 receiving (H_2, C_2). Hence, agent 3 cannot be better off.

(d) We next show that there is no profitable deviation for coalition $\{2, 3\}$. Note that agent 2 can only be better off by receiving (H_1, C_1), but this would make agent 3 worse off. Thus, agent 2 would have to receive $y_2 = (H_3, C_1)$ at a profitable deviation and in order for agent 3 to be better off, he would have to receive (H_1, C_2). This implies that agent 1 receives (H_2, C_3), which violates individual rationality.

(e) Finally, the grand coalition $\{1, 2, 3\}$ cannot profitably deviate because the TCC mechanism is Pareto efficient. □

4 Conclusion

We consider multiple-type housing market problems when agents have lexicographically separable preferences $\mathcal{R}_l$; or alternatively, preferences are drawn from a preference domain $\hat{\mathcal{R}}$ that guarantees strict core existence and that satisfies a minimal preference domain richness condition (see Assumption 1). We show that if a mechanism is strictly core-stable, then any strict strong Nash equilibrium outcome of its corresponding preference revelation game is a strict core allocation (Theorems 1 and 2). The converse statement is not true, i.e., there exist problems with strict core allocations that cannot be implemented in strict strong Nash equilibrium (Example 3). We also demonstrated the necessity of two crucial assumptions (strict core non-emptiness and strict core-stability of mechanisms) in our results (Examples 4 and 5).

Comparing our results to Takamiya's result for Shapley-Scarf housing markets, Corollary 1 (see Appendix A for the generalized individual goods allocation model considered in Takamiya, 2009), our results (Theorems 1 and 2) have two differences with his main result.

First, we show that not all strict core allocations may be implementable through strict strong Nash equilibria of the preference revelation game while Takamiya (2009) showed full implementation for Shapley-Scarf housing markets. The main reason for our partial implementation versus his full implementation result is that our preference domains are less rich than the ones he considers. Neither separable nor lexicographically separable preferences satisfy Takamiya's preference domain richness condition (Takamiya, 2009, Condition A, see Appendix A). For example, for multiple-type housing market problems with lexicographically separable preferences, no agent can protect an allotment by positioning it as his first best and his endowment as his second best allotment (an argument that is crucial in Takamiya's proof).

Second, we require strict core-stability for our mechanisms while Takamiya (2009) only required individual rationality and Pareto efficiency. In Takamiya's model, each agent only demands one object. Thus, each agent will trade within only one coalition. Therefore, once the induced allocation (the equilibrium outcome) is individually rational and Pareto efficient, no coalition can block it.[12] However, the same is not true for multiple-type housing market problems because each agent may trade different objects with different coalitions. That is, a multiple-type housing market problem cannot be easily transformed into a coalition formation game.

A Appendix: the generalized indivisible goods allocation model

A *generalized indivisible goods allocation problem* (as first introduced by Sönmez, 1999) is a list $(N, \omega, \mathcal{A}^f, R)$ where $N = \{1, \ldots, n\}$ is a finite *set of agents* and for each $i \in N$, $\omega(i)$ denotes the *endowment of agent* i; we will interpret $\omega(i)$ as a set of indivisible objects. An *allocation* is a multi-valued function $x : N \rightrightarrows \bigcup_{i \in N} \omega(i)$ satisfying (i) for any $i, j \in N$, $i \neq j$, $x(i) \cap x(j) = \emptyset$ (no two agents can receive the same object) and (ii) $\bigcup_{i \in N} x(i) = \bigcup_{i \in N} \omega(i)$ (there is no free disposal); $\mathcal{A}$ denotes the set of all allocations. Next, a subset $\mathcal{A}^f \subseteq \mathcal{A}$ is exogenously fixed as the set of all *feasible allocations*. For each agent $i \in N$, agent i's preference relation R_i is a *complete* and *transitive* binary relation on $\mathcal{A}^f$. The set $\mathcal{R}$ of preferences over allocations in $\mathcal{A}^f$ is assumed to satisfy the following properties.

Assumption A (Sönmez, 1999): An agent is indifferent between an allocation and the endowment allocation if and only if he keeps his endowment, i.e., for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x \in \mathcal{A}^f$,

$$x \, I_i \, \omega \text{ if and only if } x(i) = \omega(i).$$

Assumption B (Sönmez, 1999): For any preference relation R_i ($i \in N$), and any allocation x that is at least as good as the endowment allocation, there exists a preference relation R'_i such that (i) all allocations that are better than x at R_i are better than x at R'_i, (ii) all allocations that are worse than x at R_i are worse than

[12] See Takamiya's (2009) proof of Theorem (B) for details.

x at R_i', and (iii) the endowment allocation ranks right after (or indifferent to) x. Formally, for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x \in \mathcal{A}^f$ with $x \, R_i \, \omega$, there is R_i' such that

1. for all $y \in \mathcal{A}^f \setminus \{x\}$, $y \, R_i \, x$ if and only if $y \, R_i' \, x$,
2. for all $y \in \mathcal{A}^f \setminus \{x\}$, $x \, R_i \, y$ if and only if $x \, R_i' \, y$,
3. for all $y \in \mathcal{A}^f \setminus \{x\}$, $x \, P_i \, y$ if and only if $x \, P_i' \, y$ and $x \, R_i' \, \omega \, R_i' \, y$.

Assumption B is a *preference domain richness condition*.

The above general model can be specified to a range of well-known models such as Shapley-Scarf housing markets (Shapley and Scarf, 1974), marriage and roommate markets (Gale and Shapley, 1962), hedonic coalition formation problems (Banerjee et al., 2001), and network formation problems (Jackson and Wolinsky, 1996).

Apart from preference domain richness Assumption B, our multiple-type housing market problems also fits the generalized indivisible goods allocation model as introduced by Sönmez (1999): for each $i \in N$, $\omega(i) = \{o_i^1, \ldots, o_i^m\}$, $\mathcal{A}^f = X$, and agents' preferences over allotments are extended straightforwardly to preferences over feasible allocations by assuming that there are no consumption externalities, i.e., any agent is indifferent between two allocations at which he receives the same allotment. Furthermore, by our assumption that agents' preferences over allotments are strict, Assumption A is satisfied. However, the preference domain richness Assumption B is violated for our marginal type-preference based domains, even for the larger preference domain of separable preferences:

Consider $R \in \mathcal{R}_s^N$ with $N = \{1, 2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, each agent i's endowment is (H_i, C_i), and agent 1's marginal preferences are

$$R_1^H : H_2, H_1$$

and

$$R_1^C : C_2, C_1.$$

Thus, separability implies that either

$$\boldsymbol{R_1} : (H_2, C_2),\ (H_1, C_2),\ (H_2, C_1),\ \boldsymbol{(H_1, C_1)},\ \cdots$$

or

$$\boldsymbol{R_1} : (H_2, C_2),\ (H_2, C_1),\ (H_1, C_2),\ \boldsymbol{(H_1, C_1)},\ \cdots .$$

It is not possible to derive separable preferences R_1' over agent 1's allotments such that (H_2, C_2) is the best allotment (this is only the case when both objects are acceptable and ranked first in the marginal preferences) and the endowment is ranked right behind, i.e.,

$$\boldsymbol{R_1'} : (H_2, C_2),\ \boldsymbol{(H_1, C_1)},\ \cdots$$

is not possible. This implies that no separable preference relation R_1' over allocations satisfying Assumption B can be derived.

Takamiya (2009) also considered generalized indivisible goods allocation problems but he imposed the following conditions on preferences, which are slightly different from Assumptions A and B in Sönmez (1999).

Condition A (Takamiya, 2009):
(i) There are no consumption externalities, i.e., for each $i \in N$, each $R_i \in \mathcal{R}$, and each $x, y \in \mathcal{A}^f$, if $x(i) = y(i)$ then $x\ I_i\ y$;
(ii) For any allocation x, each agent i can rank allocations that contain $x(i)$ as his most preferred allocations, and rank allocations that contain his endowment right behind, i.e., for each $i \in N$ and each $x \in \mathcal{A}^f$, there exists a preference relation $R_i \in \mathcal{R}$ such that
(a) for all $y \in \mathcal{A}^f$ such that $y(i) \neq x(i)$, $x\ P_i\ y$; and
(b) for all $y \in \mathcal{A}^f$ such that $y(i) \notin \{x(i), \omega(i)\}$, $\omega\ P_i\ y$.[13]

Condition A(ii) is a *preference domain richness condition.*

[13]In our model, where Takamiya's Condition A(i) is satisfied, we can restate Condition A in terms of strict preferences over allotments as follows: for each $i \in N$ and each allotment x_i, there exists a preference relation $R_i \in \mathcal{R}$ such that for all allotments $x_i' \notin \{o_i, x_i\}$, $x_i\ R_i\ o_i\ R_i\ x_i'$.

Condition B (Takamiya, 2009): If a coalition S is autarkic at two feasible allocations x and y, which means that in both allocations they reallocated their endowments among themselves, then a new feasible allocation is obtained by allocating allotments according to x to all agents in S and by allocating allotments according to y to all agents in $N \setminus S$, i.e., if there are two feasible allocations $x, y \in \mathcal{A}^f$ and a coalition $S \subseteq N$ such that $\cup_{i \in S} x(i) = \cup_{i \in S} y(i) = \cup_{i \in S} \omega(i)$, then $((x(i))_{i \in S}, (y(i))_{i \in N \setminus S}) \in \mathcal{A}^f$.

Condition B is a richness condition on the set of feasible allocations.

Apart from preference domain richness Condition A(ii), our multiple-type housing market problems also fits the generalized indivisible goods allocation model as considered by Takamiya (2009): we extend agents' preferences over allotments to feasible allocations by assuming Condition A(i) and since coalitions can freely reallocate endowments among each other, Condition B is satisfied as well. The same example that shows that for separable preferences Sönmez's preference domain richness Assumption B is not satisfied, shows that Takamiya's preference domain richness Condition A(ii) is not satisfied either.

To summarize, in our model with separable preferences, both preference domain richness conditions that were used before require a certain flexibility to position an agent's endowment right behind an allotment. The intuitive use of this condition in corresponding proofs is to be able to truncate an agent's preferences just right behind a specific allotment and guarantee that by individual rationality, the agent receives his endowment, the previous allotment, or a better allotment. In our model, neither preference domain richness condition is satisfied; hence, the corresponding results of Sönmez (1999) and Takamiya (2009) need not hold anymore. We show how the lack of preference domain richness in our model changes the results that we can obtain compared to Takamiya's original results.

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