

Minimal-Access Rights in School Choice and the Deferred Acceptance Mechanism*

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Abstract

A classical school choice problem consists of a set of schools with priorities over students and a set of students with preferences over schools. Schools' priorities are often based on multiple criteria, e.g., merit-based test scores as well as minimal-access rights (siblings attending the school, students' proximity to the school, etc.). Traditionally, minimal-access rights are incorporated into priorities by always giving minimal-access students higher priority over non-minimal-access students. However, stability based on such adjusted priorities can be considered unfair because a minimal-access student may be admitted to a popular school while another student with higher merit-score but without minimal-access right is rejected, even though the former minimal-access student could easily attend another of her minimal-access schools.

We therefore weaken stability to minimal-access stability: minimal-access rights only promote access to at most one minimal-access school. Apart from minimal-access stability, we also would want a school choice mechanism to satisfy strategy-proofness and minimal-access monotonicity, i.e., additional minimal-access rights for a student do not harm her. Our main result is that the student-proposing deferred acceptance mechanism is the only mechanism that satisfies minimal-access stability, strategy-proofness, and minimal-access monotonicity. Since this mechanism is in fact stable, our result can be interpreted as an impossibility result: fairer outcomes that are made possible by the weaker property of minimal-access stability are incompatible with strategy-proofness and minimal-access monotonicity.

Keywords: school choice, priorities, minimal-access rights, justified envy, stability, deferred acceptance.

JEL-Numbers: C78; D47; D63; D78.

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1 Introduction

A classical school choice problem¹ consists of a set of schools that have priorities over students and a set of students who have preferences over schools. Priorities are often determined by various components such as a merit-based component (e.g., entrance exam scores or existing grade point averages) and a normative component (e.g., having a sibling already attending a school, living in walking distance to a school, or public transport accessibility). However, these components are fundamentally different since academic merit applies to all schools equally while aspects due to a sibling attending a school and easy logistics to get to school only apply to some schools. We therefore refer to the first component as “absolute priority” and the second (augmenting) component as “minimal-access rights.” Traditionally, a school’s final priority ranking over students is such that students who have minimal-access rights are ranked above those who do not have minimal-access rights, and within each of these two groups of students the absolute priority (i.e., the merit-based ranking) applies.

More specifically, if only one minimal-access criterion, e.g., walk-zone accessibility, is considered, then one way to adjust absolute priorities is, at each school, to always give walk-zone students higher priority over non-walk-zone students. However, stability based on such minimal-access adjusted priorities can be criticized as giving students with walk-zone rights at several schools advantages that go beyond granting a minimal-access to a walk-zone school: for example, a walk-zone student may be admitted to a popular school while another student with higher merit-based (absolute) priority but without walk-zone right is rejected, even though the former walk-zone student could easily attend another walk-zone school. Such an outcome, while stable with respect to minimal-access adjusted priorities, might be considered unfair.

This criticism is first mentioned and illustrated by Duddy (2019, page 362), who writes that the priority profile of schools

“... can fail to capture important aspects of the information from which it is derived.

In particular, important information is lost when a student satisfies a priority criterion across multiple schools. This loss of information means that matching mechanisms must treat situations that are substantially different from one another as though they were identical.”

Duddy (2019) then offers various examples to illustrate his point of view and suggests a model extension that allows to treat additional priority criteria across multiple schools in a more differentiated way. In addition to multiple types of minimal-access rights (walk-zone rights, siblings-at-a-

¹See Pathak (2011) and Abdulkadiroğlu (2013) for surveys on mechanism and market design in school choice.

school rights, etc.), Duddy (2019) considers probabilistic matchings. In contrast, we consider only one type of minimal-access criterion, e.g., walk-zone rights, and focus on deterministic matchings.² However, we do adopt Duddy’s differential treatment of minimal-access rights and weaken the standard notion of stability with respect to minimal-access adjusted priorities to minimal-access stability: minimal-access rights only matter to guarantee access to one minimal-access school (if possible).

To be more precise, stability is classically based on (minimal-access) adjusted priorities, and it requires, in addition to non-wastefulness and individual rationality, the absence of justified envy: student i would justifiably envy student j if she would like to attend student j ’s school and she has a higher adjusted priority at that school than student j (Balinski and Sönmez, 1999). The interpretation is that minimal-access rights apply across *all* minimal-access schools, which is why we refer to the derived property in our model as no justified *max* envy. If minimal-access rights are interpreted as minimal guarantees, then a situation where student i is matched to a minimal-access school (or better) and envies student j only because of the minimal-access right (that is, student j is ranked higher in merit and has no minimal-access right for his school, while student i does) does no longer justify a complaint; we call the associated notion no justified *min* envy. Using no justified min envy instead of no justified max envy weakens stability to minimal-access stability.

Apart from minimal-access stability, we would want a school choice mechanism to satisfy strategy-proofness, that is, no student can obtain a better match by misrepresenting her preferences. Apart from being a strategic robustness property, strategy-proofness in matching models represents a certain notion of fairness. Former Boston Public Schools superintendent Thomas Payzant (Payzant, 2005),³ in a memo to the Boston School Committee on May 25, 2005, describes the rationale for switching away from a manipulable school choice mechanism as follows:

“A strategy-proof algorithm levels the playing field by diminishing the harm done to parents who do not strategize or do not strategize well.”

Finally, we introduce a natural monotonicity property for the school choice model with minimal-access rights: minimal-access monotonicity requires that additional minimal-access rights for a student do not harm her.

The student-proposing deferred acceptance mechanism that is based on adjusted priorities satisfies all the desirable properties discussed above; in fact, it even satisfies the stronger stability

²Since we interpret our main result as an impossibility result to derive matching mechanisms that can in fact accommodate the differential treatment that Duddy (2019) calls for, it suffices to show that impossibility result for a less general model.

³A direct on-line reference for this quote does not seem available anymore but we refer, for instance, to Pathak and Sönmez (2008, page 1637).

property with respect to adjusted priorities. Based on Duddy's (2019) critique, however, a different mechanism, one that can treat minimal-access rights in a more differentiated way, could be desirable. To further explore this line of thought, we first need to answer the question:

“Which mechanisms satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity?”

Our answer to this question is perhaps disappointing: apart from the student-proposing deferred acceptance mechanism that is based on adjusted priorities, there exists no other mechanism that satisfies the three properties (Theorem 1). Hence, it is impossible for a school-choice mechanism to satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity while treating minimal-access rights in a differentiated way, as demanded by Duddy (2019).

2 Model and main result

A standard school choice problem consists of a population of students and a set of schools. Students are defined by their preferences over schools, and schools are defined by their capacities and priorities over students. Priorities are often determined by various components such as a merit-based component (e.g., entrance exam scores or existing grade point averages) and a normative component (e.g., having a sibling already attending a school, living in walking distance, or public transport accessibility). We refer to the first component as “absolute priority” and the second (augmenting) component as “minimal-access rights” (see our discussion at the beginning of the Introduction).

We define an **(extended school choice) problem** as a sextuple (I, S, q, P, \succ, w) with

- a finite set **I of students**;
- a finite set **S of schools**;
- a list of **capacities** $\mathbf{q} \equiv (q_s)_{s \in S}$ where for each $s \in S$, $q_s \in \mathbb{N}$;
- a list of strict **preferences** $\mathbf{P} \equiv (P_i)_{i \in I}$ over $S \cup \{\emptyset\}$, where \emptyset represents the “no-school option”;
- a list of strict **(absolute) priority relations** $\succ \equiv (\succ_s)_{s \in S}$ over I ; and
- a list of **minimal-access rights** $\mathbf{r} \equiv (r(i))_{i \in I}$ where for each $i \in I$, $r(i) \subseteq S$.

For each $i \in I$, we call $r(i)$ student i 's minimal-access schools. For each $s \in S$, let $r(s) \equiv \{i \in I : s \in r(i)\}$. Let \mathcal{P}_i denote the set of possible preferences of student i . Let $P_i \in \mathcal{P}_i$ and $s, s' \in S \cup \{\emptyset\}$. We write $s R_i s'$ if $s P_i s'$ or $s = s'$. A school $s \in S$ is **acceptable** for student i if $s P_i \emptyset$. In the

sequel, since the set of students and schools and the schools' capacities remain fixed, a problem is more compactly denoted by (P, \succ, r) .

A **matching** is a mapping $\mu : I \cup S \rightarrow 2^I \cup S$ such that (i) for each $i \in I$, $\mu(i) \in S$ or $\mu(i) = \emptyset$, (ii) for each $s \in S$, $\mu(s) \subseteq I$ and $|\mu(s)| \leq q_s$, and (iii) for each $(i, s) \in I \times S$, $\mu(i) = s$ if and only if $i \in \mu(s)$. For each $i \in I$, $\mu(i)$ is student i 's "match", i.e., the school or no-school option to which the student is matched. Similarly, for each $s \in S$, $\mu(s)$ is school s 's "match", i.e., the students to which the school is matched.

Matching μ is **individually rational** if for all $i \in I$, $\mu(i) R_i \emptyset$.

Matching μ is **non-wasteful** if for all $i \in I$ and all $s \in S$, $s P_i \mu(i)$ implies $|\mu(s)| = q_s$.

Student $i \in I$ has **justified max envy** at matching μ if there is a student $j \in I$ and a school $s \in S$ such that $\mu(j) = s P_i \mu(i)$ and

- $s \notin r(i)$, $s \notin r(j)$, and $i \succ_s j$; or
- $s \in r(i)$, $s \in r(j)$, and $i \succ_s j$; or
- $s \in r(i)$ and $s \notin r(j)$.

Matching μ is **stable** if it is individually rational, non-wasteful, and no student has justified max envy.

Remark 1 (Stability and adjusted priorities).

A student has justified max envy at a matching μ with respect to (P, \succ, r) if and only if she has justified envy (Abdulkadiroğlu and Sönmez, 2003) at μ with respect to (P, \succ^r) where $\succ^r \equiv (\succ_s^r)_{s \in S}$ are the **adjusted priorities**: for each $s \in S$, the priority relation \succ_s^r is such that

- for all $i, j \notin r(s)$, $i \succ_s^r j$ if and only if $i \succ_s j$;
- for all $i, j \in r(s)$, $i \succ_s^r j$ if and only if $i \succ_s j$; and
- for all $i \in r(s)$ and all $j \notin r(s)$, $i \succ_s^r j$.

Therefore, a matching is stable with respect to (P, \succ, r) if and only if it is "classically" stable with respect to (P, \succ^r) , i.e., as in college admissions (see, e.g. Balinski and Sönmez, 1999, p. 79). \diamond

Remark 2 (Stability and schools' responsive preferences).

By assuming that schools have priorities over all students, together with our stability notion, we implicitly assume that each school finds all students acceptable and has responsive priority preferences over sets of students. More precisely, school $s \in S$ with capacity q_s and priority relation \succ_s compares sets of students as follows. Let $2_{q_s}^I$ denote the set of all subsets of I that do not exceed the capacity q_s , i.e., $2_{q_s}^I \equiv \{I' \subseteq I : |I'| \leq q_s\}$. Let P_s denote a *priority preference*

relation on $2_{q_s}^I$, i.e., P_s strictly orders all sets in $2_{q_s}^I$. Then, P_s is responsive to \succ_s if the following two conditions hold:

- (a) for all $I' \in 2_{q_s}^I$ such that $|I'| < q_s$ and all $i \in I \setminus I'$, $I' \cup \{i\} P_s I'$ and
- (b) for all $I' \in 2_{q_s}^I$ such that $|I'| < q_s$ and all $i, j \in I \setminus I'$, $(I' \cup \{i\}) P_s (I' \cup \{j\})$ if and only if $i \succ_s j$.

When formulating (a), we implicitly assume that each school finds all students acceptable. Note that a model extension by allowing schools to find some students unacceptable while still having responsive priority preference relations would not change our results (but require additional notation in order to adjust individual rationality and stability when unacceptable students are concerned).

◇

The set of stable matchings is non-empty. A stable matching can be obtained by adapting Gale and Shapley's (1962) **(student-proposing) deferred acceptance (DA) algorithm** (see also Roth, 2008) to the context of extended school choice problems as follows. Let (P, \succ, r) be a problem.

Step 0. Using \succ and r , compute \succ^r .

Step 1. Each student i proposes to the acceptable school she most prefers or the no-school option (according to P_i). Among all proposals it receives, each school s tentatively assigns its seats to the students who have highest priority according to \succ_s^r and rejects all other proposals.

Step 2,.... Each student i who was rejected at the previous step proposes to her next most preferred acceptable school or the no-school option (according to P_i). Each school s considers the students it tentatively assigned a seat to (if any) and all students who have just proposed to it. Among these students, school s tentatively assigns its seats to the students who have highest priority according to \succ_s^r and rejects all other proposals.

The algorithm stops when each student is either tentatively matched or has been rejected by all her acceptable schools. It follows from Gale and Shapley (1962) (see also Abdulkadiroğlu and Sönmez, 2003, Proposition 1) that the resulting matching is classically stable with respect to (P, \succ^r) . Hence, by Remark 1, the deferred acceptance algorithm yields a matching that is stable with respect to (P, \succ, r) . Moreover, the resulting matching is student-optimal in the sense that all students weakly prefer it to any other stable matching.

Stability and no justified max envy are key properties when allocating school seats to students and both notions crucially depend on how priorities of students are adjusted to minimal-access rights. In particular, when using adjusted priorities, a student who has minimal-access rights for

several schools obtains higher priority for *all* these schools. Duddy (2019) points out that stability based on these adjusted priorities can be considered unfair because instead of just guaranteeing minimal-access rights to *one* of these schools, it could create unfair situations where a minimal-access student with low grades is admitted to a popular school while a student with higher grades is rejected in spite of the fact that the minimal-access student could easily have been admitted to another (but potentially less preferred) minimal-access school. In fact, as soon as a school (or even the no-school option) that is at least as good as a minimal-access school is offered to a student, one could consider a claim to be assigned to a better school based on minimal-access rights as unjustifiable. In other words, when using adjusted priorities, a student may receive more minimal-access rights than needed to guarantee a minimal-access school welfare level.

In order to take the above criticism into account, we introduce the following stricter envy concept that declares envy due to minimal-access rights unjustified if the student is already matched to a minimal-access school or one that is at least as good as a minimal-access school. Student $i \in I$ has **justified min envy**⁴ at matching μ if there is a student $j \in I$ and a school $s \in S$ such that $\mu(j) = s P_i \mu(i)$ and

- $s \notin r(i)$, $s \notin r(j)$, and $i \succ_s j$; or
- $s \in r(i)$, $s \in r(j)$, and $i \succ_s j$; or
- $s \in r(i)$, $s \notin r(j)$, and there is no school $s' \in S$ with $s' \in r(i)$ and $\mu(i) R_i s'$.

The only (but important) difference with justified max envy lies in the third condition. A matching is **minimal-access stable** if it is individually rational, non-wasteful, and no student has justified min envy. Since it is harder to achieve justified min envy than justified max envy, the set of minimal-access stable matchings contains the set of stable matchings. We illustrate the difference between stable and minimal-access stable matchings in the following example. In particular, the example demonstrates that two classical results for the set of stable matchings cannot be extended to the set of minimal-access stable matchings: neither does the set of minimal-access stable matchings form a distributive lattice (Blair, 1988) nor does it permit a “Rural Hospital Theorem.”⁵

⁴Our definition of justified min envy is based on Duddy’s (2019) notion of strongly justified envy.

⁵A basic version of the Rural Hospital Theorem in the school choice context states that for each school the number of filled seats is invariant across all stable matchings. Thus, the number of students assigned to the no-school option does not vary across stable matchings. The first versions of the theorem appear in Gale and Sotomayor (1985), Roth (1984), and Roth (1986).

Example 1 (Stability versus minimal-access stability).

Consider the extended school choice problem (I, S, q, P, \succ, r) where $I = \{1, 2, 3\}$ and $S = \{A, B, C\}$ such that for each $s \in S$, $q_s = 1$ and where preferences P and priorities \succ are given in Table 1. More precisely, student 1 finds all schools acceptable with $A \ P_1 \ B \ P_1 \ C \ P_1 \emptyset$; for student 2 only school A is acceptable; and for student 3 only school C is acceptable. The minimal-access rights are given by $r(1) = \{A, C\}$, $r(2) = \emptyset$, and $r(3) = \{C\}$ and the resulting adjusted priorities \succ^r are also given in Table 1.

P_1	P_2	P_3	\succ_A	\succ_B	\succ_C	\succ_A^r	\succ_B^r	\succ_C^r
A	A	C	3	3	3	1	3	3
	B		2	2	2	3	2	1
	C		1	1	1	2	1	2

Table 1: Students' preferences P , priorities \succ , and adjusted priorities \succ^r where $r(1) = \{A, C\}$, $r(2) = \emptyset$, and $r(3) = \{C\}$.

By applying the two versions⁶ of the deferred acceptance algorithm, one immediately verifies that the unique stable matching is

$$\mu^* : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ A & \emptyset & C \end{array}$$

which is the boldfaced matching in Table 1. The stable matching μ^* is by definition also minimal-access stable. However, there is exactly one other minimal-access stable matching, namely

$$\mu : \begin{array}{ccc} 1 & 2 & 3 \\ | & | & | \\ B & A & C \end{array}$$

which is the boxed matching in Table 1. To see that μ is minimal-access stable, first note that only student 1 does not get her most preferred school. Second, the only school that student 1 prefers to her match is school A . Note that $2 \in \mu(A)$, $1 \in r(A)$, and $2 \notin r(A)$. However, student 1 does not have justified min envy because $\mu(1) = B \ P_1 \ C$ and $C \in r(1)$. One easily verifies that apart from μ^* and μ there is no other minimal-access stable matching.

⁶The second version of the deferred acceptance algorithm, the school-proposing deferred acceptance algorithm (see Remark 3), is obtained by switching the roles of students and schools (i.e., proposers and receivers) and yields the stable matching that is student-pessimistic, i.e., all students weakly prefer any other stable matching.

Since students 1 and 2 both most prefer school A and $\mu^*(1) = A = \mu(2)$, there does not exist a minimal-access stable matching that is unanimously most preferred by all students. In particular, the set of minimal-access stable matchings does not form a distributive lattice.

Finally, note that different minimal-access stable matchings may have different numbers of students assigned to the no-school option. So, the set of minimal-access stable matchings does not permit a “Rural Hospital Theorem.” Interestingly, at the unique stable matching μ^* one student is assigned to the no-school option, while at the only other minimal-access stable matching, no student is assigned to the no-school option. \diamond

A mechanism φ is a function that selects for each problem (P, \succ, r) a matching $\varphi(P, \succ, r)$. For each student i , $\varphi_i(P, \succ, r)$ denotes the school to which the student is assigned by φ . Mechanism φ is **individually rational / non-wasteful / (minimal-access) stable** if for each problem (P, \succ, r) , $\varphi(P, \succ, r)$ is individually rational / non-wasteful / (minimal-access) stable.

A mechanism is minimal-access monotonic if for each student an expansion of her minimal-access rights induces the mechanism to assign her to a weakly more preferred school. Formally, mechanism φ is **minimal-access monotonic** if for each student i and for each pair of problems (P, \succ, r) and (P, \succ, r') with $r'(i) \subseteq r(i)$ and for each student $j \neq i$, $r'(j) = r(j)$, we have $\varphi_i(P, \succ, r) R_i \varphi_i(P, \succ, r')$. Assuming students can renounce / hide minimal-access rights, a mechanism is minimal-access monotonic if whenever a student renounces / hides some of her minimal-access rights, the mechanism assigns her to a weakly less preferred school. Put differently, it is always optimal for a student to not hide any of her minimal-access rights. Thus, a minimal-access monotonic mechanism is strategically simple and hence levels the playing field. In the context of classical exchange economies, Postlewaite (1979) is the first to introduce and study hiding-proofness and destruction-proofness with respect to individual endowments; minimal-access monotonicity is a natural version of these properties in our model.

The well-known non-manipulability property strategy-proofness requires that no student can ever benefit from misrepresenting her preferences. Formally, mechanism φ is **strategy-proof** if for each problem (P, \succ, r) , for each student i , and for all preferences $P'_i \in \mathcal{P}_i$, $\varphi_i(P, \succ, r) R_i \varphi_i(P'_i, P_{-i}, \succ, r)$ where $P_{-i} \equiv (P_j)_{j \neq i}$.

The mechanism that always assigns the matching obtained by the deferred acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) deferred acceptance (DA) mechanism** and denoted by γ . As mentioned earlier, γ is stable and hence also minimal-access stable. The following lemma also shows that γ inherits strategy-proofness from the deferred acceptance mechanism in the standard setting. Finally, the lemma shows that γ is minimal-access monotonic; this follows from the deferred acceptance mechanism in the standard setting “respecting

improvements” (Balinski and Sönmez, 1999)—the intuition being that since more minimal-access rights for a student improve her position in the priority ranking of some schools, chances to be matched to a more desirable school throughout the DA algorithm increase.

Lemma 1 (Properties of the deferred acceptance mechanism).

The deferred acceptance mechanism γ is stable, strategy-proof, and minimal-access monotonic.

Proof. We only have to prove strategy-proofness and minimal-access monotonicity. Let (P, \succ, r) be a problem. Let $i \in I$ and $P'_i \in \mathcal{P}_i$. With a slight abuse of notation, let γ denote the deferred acceptance mechanism in the standard setting. Then, it follows from Dubins and Freedman (1981) and Roth (1982) that⁷ $\gamma_i(P, \succ, r) = \gamma_i(P, \succ^r) R_i \gamma_i(P'_i, P_{-i}, \succ^r) = \gamma_i(P'_i, P_{-i}, \succ, r)$. Hence, γ is strategy-proof.

Let $i \in I$ and let (P, \succ, r) and (P, \succ, r') be two problems with $r'(i) \subseteq r(i)$ and for each student $j \neq i$, $r'(j) = r(j)$. By Remark 1, r and r' induce adjusted priorities \succ^r and $\succ^{r'}$ such that for each $j \in I$ and each $s \in S$, $i \succ_s^{r'} j$ implies $i \succ_s^r j$. Hence, \succ^r is a so-called improvement of $\succ^{r'}$ for student i and from Balinski and Sönmez (1999, Theorem 5) it follows that $\gamma_i(P, \succ, r) = \gamma_i(P, \succ^r) R_i \gamma_i(P, \succ^{r'}) = \gamma_i(P, \succ, r')$. \square

The following example demonstrates that picking another minimal-access stable matching than the matching obtained by γ can lead to a violation of both strategy-proofness and minimal-access monotonicity.

Example 2 (A minimal-access stable mechanism that is neither strategy-proof nor minimal-access monotonic). Consider again the extended school choice problem of Example 1. Let φ be a minimal-access stable mechanism such that $\varphi(P, \succ, r) = \mu$. We show first that φ is not strategy-proof. Let P'_1 be the preference relation with $C \ P'_1 \ A \ P'_1 \ B \ P'_1 \ \emptyset$. The unique minimal-access stable matching at profile $P' \equiv (P'_1, P_2, P_3)$ is μ^* . To see this, note first that at any minimal-access stable matching student 3 is assigned to school C and second that μ is not minimal-access stable because student 1 has justified min envy with respect to student 2 and school A . Hence, $\varphi_1(P', \succ, r) = A \ P_1 \ B = \varphi_1(P, \succ, r)$ and φ is not strategy-proof.

Next, we show that φ is not minimal-access monotonic. Consider the minimal-access rights r' where $r'(1) = \{A\} \subsetneq \{A, C\} = r(1)$, $r'(2) = r(2) = \emptyset$, and $r'(3) = r(3) = \{C\}$. Then, by the same arguments as before, $\varphi(P, \succ, r') = \mu^*$. Hence, $\varphi_1(P, \succ, r') = \mu^*(1) = A \ P_1 \ B = \varphi_1(P, \succ, r)$ and φ is not minimal-access monotonic. \diamond

⁷See also Abdulkadiroğlu and Sönmez (2003, Proposition 2).

The following theorem shows that the fact that minimal-access stable mechanism φ in Example 2 fails to satisfy strategy-proofness and walk-some monotonicity is not a coincidence: the only mechanism satisfying all three properties is the deferred acceptance mechanism γ .

Theorem 1 (Characterization).

A mechanism φ is minimal-access stable, strategy-proof, and minimal-access monotonic if and only if $\varphi = \gamma$.

Theorem 1 demonstrates that apart from the student-proposing deferred acceptance mechanism that is based on adjusted priorities, there exists no other mechanism that satisfies the three normatively appealing properties we have considered. Hence, it is impossible for a school-choice mechanism to satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity while treating minimal-access rights in a differentiated way, as demanded by Duddy (2019).

Proof of Theorem 1. From Lemma 1 it follows that γ satisfies the three properties. We prove that there is no other mechanism that satisfies the three properties.

Let φ satisfy minimal-access stability, strategy-proofness, and minimal-access monotonicity. Suppose there is a problem (P, \succ, r) such that $\mu \equiv \varphi(P, \succ, r) \neq \gamma(P, \succ, r) \equiv \mu^*$. Then, there is a student i such that $\mu(i) \neq \mu^*(i)$. Note that since φ and γ are individually rational, $\mu(i) R_i \emptyset$ and $\mu^*(i) R_i \emptyset$. We distinguish between two cases.

CASE 1: $\mu^*(i) P_i \mu(i) R_i \emptyset$. Let \bar{P}_i be a preference relation where $\mu^*(i)$ is the only acceptable school. Let $\bar{P} \equiv (\bar{P}_i, P_{-i})$, $\bar{\mu} \equiv \varphi(\bar{P}, \succ, r)$, and $\bar{\mu}^* \equiv \gamma(\bar{P}, \succ, r)$. Using the definition of the deferred acceptance algorithm⁸ it is easy to verify that $\bar{\mu}^*(i) = \mu^*(i)$. Since φ is individually rational, $\bar{\mu}(i) \in \{\emptyset, \mu^*(i)\}$. By strategy-proofness of φ , $\bar{\mu}(i) \neq \mu^*(i)$. Hence, $\bar{\mu}(i) = \emptyset$. Moreover, since $\mu^*(i) P_i \mu(i) R_i \emptyset$, $\bar{\mu}^*(i) \neq \emptyset = \bar{\mu}(i)$.

CASE 2: $\mu(i) P_i \mu^*(i) R_i \emptyset$. Let \bar{P}_i be a preference relation where $\mu(i)$ is the only acceptable school. Let $\bar{P} \equiv (\bar{P}_i, P_{-i})$, $\bar{\mu} \equiv \varphi(\bar{P}, \succ, r)$, and $\bar{\mu}^* \equiv \gamma(\bar{P}, \succ, r)$. Since γ is individually rational, $\bar{\mu}^*(i) \in \{\emptyset, \mu(i)\}$. By strategy-proofness of γ , $\bar{\mu}^*(i) \neq \mu(i)$. Hence, $\bar{\mu}^*(i) = \emptyset$. Since φ is strategy-proof, $\bar{\mu}(i) = \mu(i)$. Moreover, since $\mu(i) P_i \mu^*(i) R_i \emptyset$, $\bar{\mu}(i) \neq \emptyset = \bar{\mu}^*(i)$.

If there is a student j such that (i) $\bar{\mu}(j) \neq \bar{\mu}^*(j)$ and (ii) at least two different schools are acceptable under \bar{P}_j , then transform the preferences $\bar{P}_j = P_j$ in the same way as in Case 1 (if $\mu^*(j) P_j \mu(j) R_j \emptyset$) or Case 2 (if $\mu(j) P_j \mu^*(j) R_j \emptyset$). With a slight abuse of notation, let \bar{P} denote the new preference profile. We iterate the transformation of preferences until we obtain a preference

⁸Or alternatively, using stability and student-optimality directly (see, e.g., Roth and Sotomayor, 1990, Lemma 4.8).

profile \bar{P} such that for $\bar{\mu} \equiv \varphi(\bar{P}, \succ, r)$ and $\bar{\mu}^* \equiv \gamma(\bar{P}, \succ, r)$ and for each student j with $\bar{\mu}^*(j) \neq \bar{\mu}(j)$ only one school is acceptable under \bar{P}_j . Then, there are (at most) three types of students:

TYPE 1: $i \in I$ is such that $\bar{\mu}^*(i) \in S$ is the only acceptable school under \bar{P}_i and $\bar{\mu}(i) = \emptyset$.

TYPE 2: $i \in I$ is such that $\bar{\mu}(i) \in S$ is the only acceptable school under \bar{P}_i and $\bar{\mu}^*(i) = \emptyset$.

TYPE 3: $i \in I$ is such that $\bar{\mu}^*(i) = \bar{\mu}(i) \bar{R}_i \emptyset$.

Suppose $\bar{\mu}$ is stable at (\bar{P}, \succ, r) . Then, since $\bar{\mu}^*$ is also stable at (\bar{P}, \succ, r) , it follows from Remark 1 and Roth (1984, Theorem 9) that all students are of Type 3. However, this contradicts the fact that after each transformation of the preference profile there is some student i with $\bar{\mu}^*(i) \neq \bar{\mu}(i)$ (as shown at the end of Cases 1 and 2 above). Hence, $\bar{\mu}$ is not stable at (\bar{P}, \succ, r) .

Since φ is minimal-access stable, $\bar{\mu}$ is minimal-access stable at (\bar{P}, \succ, r) . Since $\bar{\mu}$ is not stable at (\bar{P}, \succ, r) this means that any justified max envy at $\bar{\mu}$ is not justified min envy. Let $i \in I$ be a student who has justified max envy at $\bar{\mu}$ with respect to some student $j \in I \setminus \{i\}$ and some school $s \in S$. Then, $\bar{\mu}(j) = s \bar{P}_i \bar{\mu}(i)$, $s \in r(i)$, $s \notin r(j)$, and there is a school $s' \in S$ with $s' \in r(i)$ and $\bar{\mu}(i) \bar{R}_i s'$. Since $s \bar{P}_i \bar{\mu}(i)$, student i is not of Type 2. We distinguish between the two cases where student i is of Type 1 or 3.

CASE A. Student i is of Type 1. Then, since $\bar{\mu}(i) = \emptyset$ and the only acceptable school under \bar{P}_i is $\bar{\mu}^*(i)$, it follows from $s \bar{P}_i \bar{\mu}(i)$ that $\bar{\mu}^*(i) = s$. Let r' be the minimal-access rights defined by $r'(i) \equiv \{s\}$ and for each $j \in I \setminus \{i\}$, $r'(j) \equiv r(j)$. Let $\bar{\nu} \equiv \varphi(\bar{P}, \succ, r')$ and $\bar{\nu}^* \equiv \gamma(\bar{P}, \succ, r')$. Since φ is minimal-access monotonic, $\emptyset = \bar{\mu}(i) \bar{R}_i \bar{\nu}(i)$. Hence, by individual rationality, $\bar{\nu}(i) = \emptyset$. Note that now s is the only acceptable school under \bar{P}_i , $r'(s) = r(s)$, and for each school $\tilde{s} \in S \setminus \{s\}$, $r'(\tilde{s}) \cap I \setminus \{i\} = r(\tilde{s}) \cap I \setminus \{i\}$. Thus, it follows from the definition of γ that $\bar{\nu}^*(i) = \bar{\mu}^*(i) = s$ (because under $\succ^{r'}$, student i retained his minimal-access right at the school he was matched to under \succ^r while other students' minimal-access rights did not change). In particular, $\bar{\nu}^*(i) \neq \bar{\nu}(i)$. Note that since $s' \neq s$, $s' \in r(i) \setminus r'(i)$. Therefore, $1 = |r'(i)| < |r(i)|$. Finally, we again iterate the transformation of preferences until each student is of Type 1, 2, or 3. (Possibly no transformation is required.)

CASE B. Student i is of Type 3. So, $\bar{\mu}^*(i) = \bar{\mu}(i)$. Since $\bar{\mu}(j) = s \neq \emptyset$, student j is not of Type 1. Suppose student j is of Type 3. Then, $\bar{\mu}^*(j) = \bar{\mu}(j) = s$. But then, since student i has justified max envy with respect to student j and school s at matching $\bar{\mu}$, student i has justified max envy with respect to student j and school s at matching $\bar{\mu}^*$ as well. Since this contradicts the stability of $\bar{\mu}^*$, student j is not of Type 3. So, student j is of Type 2. Since $s = \bar{\mu}(j)$ is the only acceptable school under \bar{P}_j , $\bar{\mu}^*(j) = \emptyset$, and since γ is non-wasteful, there is a student $k \in I$ such that $\bar{\mu}^*(k) = s$ and $\bar{\mu}(k) \neq s$. Obviously, student k is of Type 1. In particular, $\bar{\mu}(k) = \emptyset$.

Since $\bar{\mu}^*$ is stable at (\bar{P}, \succ, r) , it follows from $\bar{\mu}^*(j) = \emptyset$ and Remark 1 that $k \succ_s^r j$. Since $s \notin r(j)$, it follows that at $\bar{\mu}$ student k has justified max envy with respect to student j and school s . So, there is a student of Type 1 that has justified max envy at $\bar{\mu}$, as in Case A. We now apply the transformation of the minimal-access rights as in Case A (with student k in the role of student i). Finally, we again iterate the transformation of preferences until each student is of Type 1, 2, or 3. (Possibly no transformation is required.)

After the transformation of minimal-access rights and preferences in Case A or B, the minimal-access rights of one Type 1 student have strictly shrunk to minimal-access rights at one school, while the minimal-access rights of all other students have remained unchanged. Moreover, each student is again of Type 1, 2, or 3. Finally, the resulting matchings $\bar{\mu}$ and $\bar{\mu}^*$ are again such that $\bar{\mu} \neq \bar{\mu}^*$ and $\bar{\mu}$ is minimal-access stable but not stable. Thus, we can conduct another round of transformation according to Case A or B, etc. Since the number of students is finite, this finally yields a contradiction (when all Type 1 students who have justified max envy have only one remaining minimal-access school, justified max envy equals justified min envy). Therefore, $\varphi = \gamma$. \square

Before discussing the independence of the properties that characterize the deferred acceptance mechanism in Theorem 1 (see Remark 7), we would like to explore what happens to other well-known mechanisms in the presence of minimal-access rights. Using the adjusted priorities \succ^r we can adapt three more well-known mechanisms: the school-proposing deferred acceptance mechanism, the immediate acceptance (IA) mechanism, and the top trading cycles (TTC) mechanism.

Remark 3 (The school-proposing deferred acceptance mechanism).

Let (P, \succ, r) be a problem.

Step 0. Using \succ and r , compute \succ^r .

Step 1. Each school s proposes to the students with highest priority (under \succ_s^r), up to its capacity. Among all proposals she receives, each student i tentatively accepts the most preferred acceptable school or the no-school option (according to P_i) and rejects all other proposals.

Step 2,.... For each student who rejected s at the previous step, school s proposes to the next highest priority student (according to \succ_s^r) to whom it has not yet made a proposal. Each student i considers the proposal she tentatively accepted (if any) and all proposals she has just received. Among these proposals, student i tentatively accepts the most preferred acceptable school or the no-school option (according to P_i) and rejects all other proposals.

The algorithm stops when students do no longer reject proposals. The obtained matching is stable with respect to (P, \succ, r) . The mechanism that always assigns the matching obtained by the school-

proposing deferred acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) school-proposing deferred acceptance mechanism** and we denote it by γ^S . It is well-known that γ^S is stable but not strategy-proof. We now show that γ^S is not minimal-access monotonic. Consider the extended school choice problem (I, S, q, P, \succ, r) where $I = \{1, 2\}$ and $S = \{A, B\}$ such that for each $s \in S$, $q_s = 1$ and $2 \succ_s 1$. Students' preferences are given by Table 2. The minimal-access rights are given by $r(1) = \{A\}$ and $r(2) = \emptyset$.

P_1	P_2	\succ_A^r		\succ_B^r		$\succ_A^{r'}$		$\succ_B^{r'}$	
B	A	1		2		2		2	
\boxed{A}	\boxed{B}	2	1	1	1	1	1	1	1

Table 2: Students' preferences P and adjusted priorities \succ^r and $\succ^{r'}$ in Remark 3.

One immediately verifies that $\gamma^S(P, \succ, r)$ is the boxed matching in Table 2. Let r' be the minimal-access rights defined by $r'(1) = r'(2) = \emptyset$. Then, $\gamma^S(P, \succ, r')$ is the bold-faced matching in Table 2. Since $\gamma_1^S(P, \succ, r') = B P_1 A = \gamma_1^S(P, \succ, r)$, γ^S is not minimal-access monotonic. \diamond

While the school-proposing deferred acceptance mechanism violates minimal-access monotonicity (Remark 3), the following remark shows that, apart from the student-proposing deferred acceptance mechanism, there do exist other stable mechanisms that satisfy minimal-access monotonicity.

Remark 4 (Stability together with strategy-proofness or minimal-access monotonicity).

In the standard setting, the unique stable mechanism that satisfies strategy-proofness is the (student-proposing) deferred acceptance mechanism (see, e.g., Roth and Sotomayor, 1990, Theorem 4.6). Together with Lemma 1 this implies that in our setting the unique stable mechanism that satisfies strategy-proofness is the (student-proposing) deferred acceptance mechanism.

However, the (student-proposing) deferred acceptance mechanism is not the unique stable mechanism that satisfies minimal-access monotonicity. To see this, let $\bar{\gamma}$ be a mechanism defined as follows:

$$\bar{\gamma}(P, \succ, r) \equiv \begin{cases} \gamma(P, \succ, r) & \text{if for some } k \in I, r(k) \neq \emptyset; \\ \text{any}^9 \text{ stable matching at } (P, \succ, r) & \text{if for each } k \in I, r(k) = \emptyset. \end{cases}$$

By definition, $\bar{\gamma}$ is stable and $\bar{\gamma} \neq \gamma$. We now show that $\bar{\gamma}$ is also minimal-access monotonic. Let $i \in I$ and (P, \succ, r) and let (P, \succ, r') be two problems with $r'(i) \subsetneq r(i)$ and for each student $j \neq i$,

⁹To guarantee $\bar{\gamma} \neq \gamma$, one has to pick some stable matching different from $\gamma(P, \succ, r)$ for some problem (P, \succ, r) .

$r'(j) = r(j)$. Then,

$$\bar{\gamma}_i(P, \succ, r) = \gamma_i(P, \succ, r) R_i \gamma_i(P, \succ, r') R_i \bar{\gamma}_i(P, \succ, r').$$

The first equality follows from the definition of $\bar{\gamma}$. The first R_i -comparison follows from minimal-access monotonicity of γ (Lemma 1). The second R_i -comparison follows from the student-optimality of the stable matching $\gamma(P, \succ, r')$ at problem (P, \succ, r') , i.e., all students weakly prefer stable matching $\gamma(P, \succ, r')$ to any other stable matching at problem (P, \succ, r') . Hence, $\bar{\gamma}$ is minimal-access monotonic.

To see that $\bar{\gamma}$ is not strategy-proof, let (P, \succ, r) be a problem with $\bar{\gamma}(P, \succ, r) \neq \gamma(P, \succ, r)$. Then, there is a student $i \in I$ who can state (truncation) preferences P'_i that are obtained from her preferences P_i by declaring all schools less preferred than $\gamma_i(P, \succ, r)$ as unacceptable while keeping all other acceptable schools in the same order. Thus, $\bar{\gamma}_i(P'_i, P_{-i}, \succ, r) = \gamma_i(P, \succ, r) P_i \bar{\gamma}_i(P, \succ, r)$. Therefore, student i is better off by misrepresenting her preferences. Hence, $\bar{\gamma}$ is not strategy-proof. \diamond

Remark 5 (The immediate acceptance (IA) mechanism).

We adapt the classical **immediate acceptance (IA) algorithm** to our model with minimal-access rights. Let (P, \succ, r) be a problem.

Step 0. Using \succ and r , compute \succ^r .

Step 1. Each student i proposes to the acceptable school she most prefers or the no-school option (according to P_i). Among all proposals it receives, each school s immediately assigns its seats to the students who have highest priority according to \succ_s^r and rejects all other proposals.

Step 2,.... Each student i who was rejected at the previous step proposes to her next most preferred acceptable school or the no-school option (according to P_i). Among all proposals it receives, each school s immediately assigns its remaining seats (if any) to the students who have highest priority according to \succ_s^r and rejects all other proposals.

The algorithm stops when each student is either matched or has been rejected by all her acceptable schools. The resulting matching is not necessarily classically stable with respect to (P, \succ^r) or minimal-access stable with respect to (P, \succ, r) . The mechanism that always assigns the matching obtained by the immediate acceptance algorithm based on adjusted priorities is called **(minimal-access adjusted) immediate acceptance (IA) mechanism**. It follows from Abdulkadiroğlu and Sönmez (2003) that the IA mechanism is neither stable nor strategy-proof. However, since more minimal-access rights for a student improve her position in the priority ranking of some

schools, chances to be matched to a more desirable school earlier in the IA algorithm increase and thus the IA mechanism satisfies minimal-access monotonicity.¹⁰ \diamond

Remark 6 (The top trading cycles (TTC) mechanism).

Inspired by David Gale's **top trading cycle (TTC) algorithm**, Abdulkadiroğlu and Sönmez (2003) introduced the so-called top trading cycles mechanism, which we adapt to our model with minimal-access rights. Let (P, \succ, r) be a problem.

Step 0. Using \succ and r , compute \succ^r .

Step 1. Each student i points to the acceptable school she most prefers or the no-school option (according to P_i). The no-school option points to all students and each school s points to the student who has highest priority according to \succ_s^r . There is at least one cycle. Each student in a cycle is assigned to the school (or the no-school option) she points to and she is removed. The capacity of each school (but not the no-school option) that is in a cycle is reduced by 1. If the capacity of a school is now 0, then the school is removed (the no-school option is not removed).

Step 2, Each remaining student i points to the school she most prefers among the remaining schools or the no-school option (according to P_i). The no-school option points to all students and each remaining school s points to the student who has highest priority according to \succ_s^r among all remaining students. There is at least one cycle. Each student in a cycle is assigned to the school (or the no-school option) she points to and she is removed. The capacity of each school (but not the no-school option) that is in a cycle is reduced by 1. If the capacity of a school is now 0, then the school is removed (the no-school option is not removed).

The algorithm stops when each student has been removed and matched to a school or the no-school option. The resulting matching is not necessarily classically stable with respect to (P, \succ^r) or minimal-access stable with respect to (P, \succ, r) . The mechanism that always assigns the matching obtained by the top trading cycles algorithm based on adjusted priorities is called **(minimal-access adjusted) top trading cycles (TTC) mechanism**. It follows from Abdulkadiroğlu and Sönmez (2003) that the TTC mechanism is strategy-proof but not stable. Furthermore, since more minimal-access rights for a student improve her position in the priority ranking of some schools, chances to form a trading cycle that leads to matching with a more desirable school earlier in the TTC algorithm increase and thus the TTC mechanism satisfies minimal-access monotonicity. A formal proof is relegated to the Appendix. \diamond

¹⁰We omit a formal proof since it is straightforward.

Remark 7 (Independence of properties in Theorem 1).

The following three mechanisms show that the three axioms in Theorem 1 are logically unrelated. We label the following independence examples by the property that is *not* satisfied.

Strategy-proofness: Mechanism $\bar{\gamma}$ in Remark 4 satisfies minimal-access monotonicity and minimal-access stability (in fact, it satisfies stability), but not strategy-proofness.

Minimal-access stability: A serial dictatorship where students sequentially get assigned to their most preferred acceptable school among all schools with remaining seats or the no-school option. Another example is the deferred acceptance mechanism based on *unadjusted* priorities \succ . These mechanisms are strategy-proof and trivially minimal-access monotonic (since minimal-access rights are ignored altogether). A mechanism that is strategy-proof, minimal-access monotonic, and that *does* take minimal-access rights into account is the TTC mechanism (Remark 6).

Minimal-access monotonicity: We define a mechanism $\tilde{\gamma}$ as follows. In the particular situation where all schools have the same (particular) priority order, all students but the lowest priority student have no minimal-access rights, and the lowest priority student has at least 2 minimal-access rights, we apply the associated serial dictatorship with a small twist: as soon as there is only one minimal-access seat left, the deferred acceptance algorithm is applied. In all other situations, the deferred acceptance mechanism is applied directly. Formally, let (I, S, q, P, \succ, r) be a problem. Let $I = \{1, 2, \dots, n\}$. We distinguish between two cases.

CASE 1: For each $s \in S$, $1 \succ_s 2 \succ_s \dots \succ_s n$ and for each $i \in I \setminus \{n\}$, $r(i) = \emptyset$ and $|r(n)| > 1$.

For each $s \in S$, let $q_s(0) \equiv q_s$. The following procedure outputs a matching.

Begin Procedure

Step 1. Student 1 is assigned to her most preferred school or the no-school option (according to P_1), say s_1^* . If $s_1^* \in S$, then $q_{s^*}(1) \equiv q_{s^*}(0) - 1$ and for each $s \in S \setminus \{s_1^*\}$, $q_s(1) \equiv q_s(0)$. If $s_1^* = \emptyset$, then for each $s \in S$, $q_s(1) \equiv q_s(0)$. Go to Step 2.

Step $i > 1$.

- (a) If $i < n$ and $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(i-1) > 1$, then student i is assigned to her most preferred school or the no-school option (according to P_i), say s_i^* , among the schools in the set $\{s \in S : q_s(i-1) > 0\}$. If $s_i^* \in S$, then $q_{s^*}(i) \equiv q_{s^*}(i-1) - 1$ and for each $s \in S \setminus \{s_i^*\}$, $q_s(i) \equiv q_s(i-1)$. If $s_i^* = \emptyset$, then for each $s \in S$, $q_s(i) \equiv q_s(i-1)$. Go to the next step.
- (b) If $i < n$ and $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(i-1) = 1$, then students $\{i, \dots, n\}$ are matched to the remaining seats of the schools in $\{s \in S : q_s(i-1) > 0\}$ and the no-school option by applying the deferred acceptance algorithm (with adjusted priorities based on agent n 's last remaining minimal-access school). The procedure ends.

- (c) If $i = n$, then student n is assigned to her most preferred school or the no-school option (according to P_n), say s_n^* , among the schools in the set $\{s \in S : q_s(n-1) > 0\}$. The procedure ends.

End Procedure

Let $\tilde{\gamma}(P, \succ, r)$ be the matching that is obtained by the above procedure. For later convenience, we refer to steps $i(a)$ and $i(c)$ in the procedure ($i = 1, [i < n \text{ and } \sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(i-1) > 1]$, and $i = n$) as the “serial dictatorship (SD) steps.” Step $i(b)$ in the procedure is referred to as the DA step.

CASE 2: Otherwise. In this case, the mechanism coincides with the deferred acceptance mechanism, i.e., $\tilde{\gamma}(P, \succ, r) \equiv \gamma(P, \succ, r)$.

It is easy to see that mechanism $\tilde{\gamma}$ is strategy-proof. In Case 2 this follows immediately from strategy-proofness of the deferred acceptance mechanism. In Case 1 this is due to the SD steps (in particular, by misstating her preferences, no student can change the set of schools that is available to her) and strategy-proofness of the deferred acceptance mechanism.

Mechanism $\tilde{\gamma}$ is also minimal-access stable. This is obvious in Case 2 since the deferred acceptance mechanism always yields a stable matching. We now show that in Case 1 mechanism $\tilde{\gamma}$ always yields a minimal-access stable matching. Let $\mu = \tilde{\gamma}(P, \succ, r)$. First, since the no-school option is available at each SD and DA step, μ is individually rational. Second, μ is non-wasteful because (i) at the SD steps, unoccupied seats are always available and (ii) the deferred acceptance mechanism is non-wasteful. Third, there is no justified min envy:

- (a) At each SD step $i < n$, student i does not have justified min envy with respect to any student $j \in I \setminus \{i\}$ because $j \succ_{\mu(j)} i$ or $\mu(i) R_i \mu(j)$.
- (b) At DA step i , no student in $k \in \{i, \dots, n-1\}$ has justified min envy with respect to any student $j \in \{1, \dots, i-1\}$ because $j \succ_{\mu(j)} k$. Student n does not have min envy with respect to any student $j \in \{1, \dots, i-1\}$ since the deferred acceptance mechanism is minimal-access stable and $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(i-1) = 1$. Finally, since the deferred acceptance mechanism is minimal-access stable, no student in $k \in \{i, \dots, n\}$ has justified min envy with respect to any other student $j \in \{i, \dots, n\} \setminus \{k\}$.
- (c) At SD step n , student n does not have justified min envy with respect to any other student. To see this, note that at this step, student n is assigned to her most preferred school or the no-school option (according to P_n) among the schools in the set $\{s \in S : q_s(n-1) > 0\}$. Since $\sum_{\tilde{s} \in r(n)} q_{\tilde{s}}(n-1) \geq 1$, there is a minimal-access school $\tilde{s} \in r(n)$ with $\mu(n) R_n \tilde{s}$.

Finally, mechanism $\tilde{\gamma}$ is not minimal-access monotonic. For an illustration, consider the extended school choice problem (I, S, q, P, \succ, r) where $I = \{1, 2\}$ and $S = \{A, B\}$ such that for each $s \in S$, $q_s = 1$ and $1 \succ_s 2$. Students' preferences are given by Table 3. The minimal-access rights are given by $r(1) = \emptyset$ and $r(2) = \{A, B\}$.

	P_1	P_2
	\boxed{A}	\circled{A}
	\circled{B}	\boxed{B}

Table 3: Students' preferences P .

One immediately verifies that $\tilde{\gamma}(P, \succ, r)$ is the boxed matching in Table 3. Let r' be the minimal-access rights defined by $r'(1) = r(1) = \emptyset$ and $r'(2) = \{A\} \subsetneq \{A, B\} = r(2)$. Then, $\tilde{\gamma}(P, \succ, r')$ is the encircled matching in Table 3. Since $\tilde{\gamma}_2(P, \succ, r') = A P_2 B = \tilde{\gamma}_2(P, \succ, r)$, $\tilde{\gamma}$ is not minimal-access monotonic. \diamond

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Appendix: Proof that TTC is minimal-access monotonic

We first prove that TTC is minimal-access monotonic in the “unit setting” where each school has 1 seat, i.e., for each $s \in S$, $q_s = 1$. Let τ denote the TTC mechanism.

Let $i \in I$. Let (P, \succ, r) and (P, \succ, r') be two problems such that $r'(i) \subseteq r(i)$ and for each $j \in I \setminus \{i\}$, $r'(j) = r(j)$. With a slight abuse of notation we write $\tau(\succ)$ for $\tau(P, \succ, r)$ and $\tau(\succ')$ for $\tau(P, \succ, r')$. Step $t \geq 1$ in the TTC algorithm applied to (P, \succ, r) and (P, \succ, r') is referred to as step t of $\tau(\succ)$ and $\tau(\succ')$, respectively. In addition, let t_i and t'_i denote the step of $\tau(\succ)$ and $\tau(\succ')$ at which student i is assigned to a school (or the no-school option \emptyset), respectively.

For each $t \in \{1, \dots, t_i\}$, let $A(\succ, t)$ denote the set of agents (students and schools)¹¹ that are present at step t of $\tau(\succ)$. Similarly, for each $t' \in \{1, \dots, t'_i\}$, let $A(\succ', t')$ denote the set of agents (students and schools) that are present at step t' of $\tau(\succ')$. Finally, for each $t \in \{1, \dots, t_i\}$, let $P(i, \succ, t)$ denote the set of predecessors of student i at step t of $\tau(\succ)$, i.e., the agents (students and schools) from which there is a path (that does not involve \emptyset) to student i .¹² For convenience, we

¹¹The no-school option \emptyset is not considered an agent.

¹²Here, each directed edge in the path refers to the “pointing” as described in the TTC algorithm (see Remark 6).

always exclude student i and (obviously also) the no-school option \emptyset from the set of predecessors, i.e., $i, \emptyset \notin P(i, \succ, t)$. In particular, it is possible that $P(i, \succ, t) = \emptyset$, i.e., no school points to student i at step t of $\tau(\succ)$. Finally, by a cycle C we here refer to the set of agents (students and schools) *and the no-school option \emptyset* involved in a “pointing” (top trading) cycle.¹³

The following proposition shows that the TTC mechanism is minimal-access monotonic.

Proposition 1.

$$\tau_i(\succ) R_i \tau_i(\succ'). \quad (1)$$

Proof of Proposition 1. If $t_i = 1$, then $\tau_i(\succ)$ is student i ’s most preferred school (or, if all schools are unacceptable, \emptyset), in which case (1) holds trivially. Let $t_i > 1$. Assume that (1) does not hold, i.e.,

$$\tau_i(\succ') P_i \tau_i(\succ). \quad (2)$$

We first prove the following lemma by induction.

Lemma 2. For each step $t \in \{1, \dots, t_i - 1\}$,

- (A) $t < t'_i$;
- (B) if C is a cycle at step t of $\tau(\succ')$, then
 - (1) C is a cycle at step t of $\tau(\succ)$ or
 - (2) $C \subseteq P(i, \succ, t) \cup \{\emptyset\}$;
- (C) if C is a cycle at step t of $\tau(\succ)$, then C is a cycle at step t of $\tau(\succ')$.

Proof of Lemma 2.

Induction basis. Let $t = 1$. We first prove (A). Suppose $t \geq t'_i$. Then, $t'_i = 1$. So, i is in a cycle at step 1 of $\tau(\succ')$. But then i is in the same cycle at step 1 of $\tau(\succ)$, which contradicts $t_i > 1$. Hence, $t < t'_i$. This proves (A).

Next, we prove (B). Let C be a cycle at step 1 of $\tau(\succ')$. From (A) it follows that $i \notin C$. Suppose C is not a cycle at step 1 of $\tau(\succ)$. Then, $C \subseteq I \cup S$ and there is a non-empty set of schools $S^* \subseteq S \cap C$ that point to student i at step 1 of $\tau(\succ)$, and each of the other agents in $C \setminus S^*$ points to the same agent at step 1 of $\tau(\succ)$ and $\tau(\succ')$. Hence, $C \subseteq P(i, \succ, t)$. This proves (B).

Finally, we prove (C). Let C be a cycle at step 1 of $\tau(\succ)$. Since $1 < t_i$, $i \notin C$. Hence, C is a cycle at step 1 of $\tau(\succ')$, which proves (C).

¹³Note that whenever the no-school option \emptyset is in a top trading cycle the cycle is trivial in the sense that it only contains one student (and no school).

Induction hypothesis. Suppose (A), (B), and (C) hold for each step $1, \dots, t-1$ with $t < t_i$ ($t-1 < t_i - 1$).

Induction step. We prove that (A), (B), and (C) also hold for step t . Note first that at each step of the TTC algorithm the only agents that are removed from the problem are the agents that are part of a cycle. Hence, by the induction hypothesis ((B) and (C) for steps $1, \dots, t-1$) it follows that

- (a) $A(\succ, t) \supseteq A(\succ', t)$ and
- (b) $A(\succ, t) \setminus A(\succ', t) \subseteq P(i, \succ, t)$.

We first prove (A) for step t . Since $t-1 < t < t_i$, it follows from (A) for step $t-1$ that $t-1 < t'_i$. So, $t \leq t'_i$. Suppose $t = t'_i$. Then, i is in a cycle, say C , at step t of $\tau(\succ')$.

We claim that $C \subseteq P(i, \succ, t) \cup \{i, \emptyset\}$. This is obviously true if C is a trivial cycle (consisting of i and \emptyset only).¹⁴ Suppose C is a non-trivial cycle. Then, $\emptyset \notin C$. Since $C \subseteq A(\succ', t)$, by (a), $C \subseteq A(\succ, t)$. However, since $t < t_i$, C is not a cycle at step t of $\tau(\succ)$. Let $a^* \neq i$ be an agent in C that points to different objects in cycle C and at step t of $\tau(\succ)$, say b' in cycle C and $b \neq b'$ at step t of $\tau(\succ)$. Since $\emptyset \notin C$, $b' \neq \emptyset$. Since $b' \in C \subseteq A(\succ, t)$, agent a^* points to b either because of a minimal-access right at Step t of $\tau(\succ)$ ($b = i$) or because b is preferred to b' but it is not present at step t of $\tau(\succ')$, i.e., it follows that $b \in \{i\} \cup [A(\succ, t) \setminus A(\succ', t)]$. If $b = i$, then $a^* \in P(i, \succ, t)$. Suppose $b \neq i$. Since by (b), $A(\succ, t) \setminus A(\succ', t) \subseteq P(i, \succ, t)$, we have $b \in P(i, \succ, t)$. Hence, by definition of $P(i, \succ, t)$, $a^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t) \cup \{i, \emptyset\}$.¹⁵

Now note that for each $s \in [S \cap P(i, \succ, t)] \cup \{\emptyset\}$, $\tau_i(\succ) R_i s$. Since $\tau_i(\succ') \in [S \cap C] \cup \{\emptyset\} \subseteq [S \cap P(i, \succ, t)] \cup \{\emptyset\}$, we have $\tau_i(\succ) R_i \tau_i(\succ')$, which contradicts assumption (2) we made at the beginning of the proof of Proposition 1. Hence, $t < t'_i$ and (A) for step t holds.

Next, we prove (B) for step t . Let C be a cycle at step t of $\tau(\succ')$. From (A) for step t it follows that $i \notin C$. Suppose C is not a cycle at step t of $\tau(\succ)$. It suffices to show that $C \subseteq P(i, \succ, t) \cup \{\emptyset\}$.

Suppose C is a trivial cycle. Then, C consists of some student $j^* \neq i$ and \emptyset only. Then, by (a), $j^* \in A(\succ', t) \subseteq A(\succ, t)$. However, C is not a cycle at step t of $\tau(\succ)$. Hence, j^* points to different objects in cycle C and at step t of $\tau(\succ)$. Then, j^* points to some school $s^* \in S$ at step t of $\tau(\succ)$. In particular, $s^* \in A(\succ, t) \setminus A(\succ', t)$. Since by (b), $A(\succ, t) \setminus A(\succ', t) \subseteq P(i, \succ, t)$, we have $s^* \in P(i, \succ, t)$. Hence, by definition of $P(i, \succ, t)$, $j^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t) \cup \{\emptyset\}$.

¹⁴A cycle that contains multiple instances of \emptyset , i.e., $i_1 \rightarrow \emptyset \rightarrow i_2 \rightarrow \emptyset \dots \rightarrow i_p \rightarrow \emptyset \rightarrow i_1$ will be interpreted as p trivial cycles.

¹⁵It was sufficient to consider agents $a^* \neq i$ in C that point to different objects in cycle C and at step t of $\tau(\succ)$. To see this, let agent $a \neq i$ be an agent that points to the *same* agent in C and at step t of $\tau(\succ)$. Then, there is a path at step t of $\tau(\succ)$ from a to i or to an agent a^* in C that points to different objects in C and at step t of $\tau(\succ)$. Since we have shown that $a^* \in P(i, \succ, t)$, it immediately follows that $a \in P(i, \succ, t)$ as well.

Now suppose C is a non-trivial cycle. Then, $\emptyset \notin C$. Since $C \subseteq A(\succ', t)$, by (a), $C \subseteq A(\succ, t)$. However, C is not a cycle at step t of $\tau(\succ)$. Let a^* be an agent in C that points to different objects in cycle C and at step t of $\tau(\succ)$, say b' in cycle C and $b \neq b'$ at step t of $\tau(\succ)$. Since $\emptyset \notin C$, $b' \neq \emptyset$. Since $b' \in C \subseteq A(\succ, t)$, it follows that $b \in \{i\} \cup [A(\succ, t) \setminus A(\succ', t)]$. If $b = i$, then $a^* \in P(i, \succ, t)$. Suppose $b \neq i$. Since by (b), $A(\succ, t) \setminus A(\succ', t) \subseteq P(i, \succ, t)$, we have $b \in P(i, \succ, t)$. Hence, by definition of $P(i, \succ, t)$, $a^* \in P(i, \succ, t)$ as well. This shows that $C \subseteq P(i, \succ, t)$.¹⁶ This completes the proof of (B) for step t .

Finally, we prove (C). Let C be a cycle at step t of $\tau(\succ)$. Since $t < t_i$, $i \notin C$. So, $C \subseteq A(\succ, t) \cup \{\emptyset\}$ and $C \cap P(i, \succ, t) = \emptyset$. Since by (b), $A(\succ, t) \setminus A(\succ', t) \subseteq P(i, \succ, t)$, we have $C \subseteq A(\succ', t) \cup \{\emptyset\}$. Since by (a), $A(\succ', t) \subseteq A(\succ, t)$ and $i \notin C$, C is also a cycle at step t of $\tau(\succ')$. This proves (C) and completes the proof of Lemma 2. \square

With the result of Lemma 2 we can now complete the proof of Proposition 1.

From Lemma 2 it follows that $t_i \leq t'_i$ and that $A(\succ, t_i) \supseteq A(\succ', t_i)$. From the TTC algorithm it follows that $A(\succ', t_i) \supseteq A(\succ', t'_i)$. Hence, $A(\succ, t_i) \supseteq A(\succ', t'_i)$. When student i is removed she is assigned to the school that she most prefers among the schools that are still present (or the no-school option \emptyset if all present schools are unacceptable to i). Hence, $\tau_i(\succ) R_i \tau_i(\succ')$. \square

We now consider the general setting where schools can have multiple seats, i.e., for each $s \in S$, $q_s \geq 1$. The TTC mechanism is also minimal-access monotonic in the general setting. This can be easily seen by applying minimal-access monotonicity from the unit setting as follows. First, make q_s copies of each school $s \in S$ and label them $1, 2, \dots, q_s$. Second, let each copy of a school inherit the priority ordering of the school. Third, students' new preferences are obtained from their original preferences by replacing each school s by its q_s copies (in the strict order of increasing labels). Fourth, note that the TTC matching for the original problem “coincides” with the TTC matching for the new problem (by lumping together the students who are matched to copies of the same school). Fifth, by applying minimal-access monotonicity to problems in the unit setting we obtain minimal-access monotonicity in the general setting.

¹⁶It was sufficient to consider agents a^* in C that point to different objects in cycle C and at step t of $\tau(\succ)$. To see this, let agent a be an agent that points to the *same* agent in C and at step t of $\tau(\succ)$. Then, there is a path at step t of $\tau(\succ)$ from a to i or to an agent a^* in C that points to different objects in C and at step t of $\tau(\succ)$. Since we have shown that $a^* \in P(i, \succ, t)$, it immediately follows that $a \in P(i, \succ, t)$ as well.