Stable partitions for proportional generalized claims problems[*]

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March 24, 2022

Abstract

We consider a set of agents, e.g., a group of researchers, who have claims on an endowment, e.g., a research budget from a national science foundation. The research budget is not large enough to cover all claims. Agents can form coalitions and coalitional funding is proportional to the sum of the claims of its members, except for singleton coalitions which receive no funding. We analyze the structure of stable partitions when coalition members use well-behaved rules to allocate coalitional endowments, e.g., the well-known constrained equal awards rule (CEA) or the constrained equal losses rule (CEL).

For continuous, (strictly) resource monotonic, and consistent rules, stable partitions with (mostly) pairwise coalitions emerge. For CEA and CEL we provide algorithms to construct such a stable pairwise partition. While for CEL the resulting stable pairwise partition is assortative and sequentially matches up lowest-claims pairs, for CEA the resulting stable pairwise partition is obtained sequentially by matching up in each step either a highest-claims pair or a highest-lowest-claims pair.

More generally, we also assume that the minimal coalition size to have a positive endowment is $\theta \geq 2$. We then show how all results described above are extended to this general case.

Keywords: claims problems, coalition formation, stable partitions.

JEL codes: C71, C78, D63, D71, D74.

[*]We gratefully acknowledge financial support from the Swiss National Science Foundation (SNSF) through Project 100018_192583 and thank William Thomson, Mehmet Karakaya, and Elena Inarra for helpful comments and suggestions.

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1 Introduction

The formation of coalitions is a widespread aspect of social, economic, or political environments. Agents form coalitions in very different situations in order to achieve some joint benefits. Cooperation between agents is sometimes hampered by the existence of two opposing fundamental forces: on the one hand, the increasing returns to scale, which incentivizes agents to cooperate and, therefore, to form large coalitions and, on the other hand, the heterogeneity of agents, which causes instability and pushes towards the formation of only small coalitions.

Gallo and Inarra (2018) introduce *generalized claims problems*[1] to deal with coalition formation in a bankruptcy framework. A generalized claims problem consists of a group of agents, each of them with a claim and a set of *coalitional endowments*, one for each possible coalition, which are not sufficient to meet the claims of their members. Coalitional endowments are divided among their members according to a pre-specified rule, which thus is a decisive element of the coalition formation process. Their main result (Gallo and Inarra, 2018, Theorem 2) states that, given a generalized claims problem, there is a stable partition for each coalition formation problem that is induced by a continuous rule if and only if it also satisfies resource monotonicity and consistency. In this paper, we study the structure of stable partitions under different resource monotonic and consistent rules to answer two types of questions: What coalition sizes can emerge? And, who are the coalition partners?

The model proposed by Gallo and Inarra (2018) does not impose any restriction on coalitional endowments and, consequently, answering the above questions is not really possible in their general model. In contrast, we consider *non-singleton proportional generalized claims problems* where singleton coalitions have zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of their members' claims. Proportionality is justified in many situations such as the funding of research projects where the budgets are often divided proportionally to funding needs or according to other funding criteria such as project quality.[2] Moreover, in many situations, institutions are interested in sparking cooperation and hence, discouraging singleton coalitions. Note that for didactic reasons we first focus on non-singleton

[1]Note that these authors use the term *coalition formation problem with claims* instead of *generalized claims problems*.

[2]Other examples can be found in a bankruptcy situation where assets have to be allocated proportionally among creditors according to their claims or, in a legislature, where seats are distributed proportionally among the parties according to voting shares.

proportional generalized claims problems and later on extend our results to address minimal coalition sizes $\theta > 2$ for positive coalitional endowments (see Barberà et al., 2015, for another model whith a minimal size for coalitions to be productive).

Non-singleton proportional generalized claims problems are a subclass of the class of generalized claims problems studied by Gallo and Inarra (2018) and hence their results hold. Then, given a non-singleton proportional generalized claims problem, we first characterize the structure of any possible stable partition when the rule applied satisfies continuity, strict resource monotonicity, and consistency. We show that at most one singleton coalition belongs to each stable partition and that for each coalition in the stable partition with size larger than two, each agent of the coalition receives a proportional payoff (Theorem 1). Furthermore, considering resource monotonicity instead of its strict version, even though we do not characterize all stable partitions, we show that a stable partition formed by pairwise coalitions (i.e., coalitions of size two) always exists, with the exception of at most one singleton coalition if the set of agents is odd (Theorem 2).

With the result of Theorem 2 as the departure point, we analyze how agents sort themselves into pairwise coalitions under some parametric rules (see Young, 1987; Stovall, 2014). These parametric rules are well-studied in the literature because the payoff of each agent is given by a function that depends only on the claim of the agent and a parameter that is common to all agents. We focus on two well-known parametric rules that represent two egalitarian principles: the constrained equal awards rule (CEA) and the constrained equal losses rule (CEL). On the one hand, CEA divides the endowment as equally as possible subject to no agent receiving more than her claim (e.g., rationing toilet paper when shortage occurs). On the other hand, CEL divides the losses as equally as possible subject to no agent receiving a negative amount (e.g., equal sacrifice taxation when utility is measured linearly[3]).

We propose two algorithms, one for each rule, to find a stable and (almost) pairwise partition. The *CEA algorithm* sequentially pairs off either two highest-claims agents or a highest-claim with a lowest-claim agent (Theorem 3). Examples of the first type of cooperation are found in social environments where agents tend to join other agents with similar characteristics. In contrast, the second type of cooperation may

[3]The idea of the equal sacrifice principle in taxation is that all tax payers end up sacrificing equally, according to some cardinal utility function. Young (1988) provides a characterization of the family of equal-sacrifice rules based on a few compelling principles and, more recently, Chambers and Moreno-Ternero (2017) generalize the previous family.

be interpreted as a transfer of knowledge between agents as happens, for instance, between apprentices and advisors. While the CEA algorithm produces stable partitions that can contain *assortative* as well as *extremal pairwise coalitions*, the *CEL algorithm* is purely assortative and sequentially pairs off two lowest-claims agents (Theorem 4).

Continuing with our example, we consider the fact that some funding calls may require a larger minimal number of agents in a group to generate a positive endowment. Therefore, we introduce another subclass of the class of generalized claims problems, *θ-minimal proportional generalized claims problems.* In these problems, coalitions of size lower than θ have zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of their members' claims. We generalize our results from $\theta = 2$, the non-singleton proportional generalized claims problems, to any $\theta \in \mathbb{N}$. More specifically, we first show that when the rule applied satisfies continuity, strict resource monotonicity, and consistency, then there are fewer than θ agents in coalitions of size smaller than θ and that for each coalition in the stable partition with size larger than θ, each agent of the coalition receives a proportional payoff (Theorem 5). Moreover, if the rule satisfies resource monotonicity instead of its strict version, we show that a stable partition formed by the maximal possible number of coalitions of size θ and one coalition (of size lower than θ) formed by the remaining agents exists (Theorem 6).

In a similar way as in the non-singleton proportional generalized claims model, with the result of Theorem 6 as the departure point, we analyze how agents sort themselves into coalitions of size θ under the CEA and the CEL rules. We propose two algorithms, one for each rule, to find a stable partition. The *θ-CEA algorithm* generates a stable partition formed by coalitions of size θ constructed by sequentially adding a lowest-claim agent or a highest-claim agent (Theorem 7). For the CEL rule, an assortative stable partition is obtained by sequentially pairing off θ lowest-claims agents (Theorem 8).

There is a large number of papers that pay attention to the structure of the coalitions that form. Becker (1973) and Greenberg and Weber (1986) introduce the notion of assortative coalitions.[4] Observe that in both our algorithms assortative coalitions (in terms of claims) may form. In our conclusion (Section 5), we discuss some papers in which similar results concerning assortative stable coalitions are obtained.

[4]Assortativeness is based on an ordering of agents according to a specific variable such as claims, productivity, or location. Alternative terminology includes that of consecutive coalitions.

Our paper is organized as follows. In Section 2 we introduce all ingredients needed to define the class of generalized claims problems of Gallo and Inarra (2018) and our subclasses of non-singleton and θ-minimal proportional generalized claims problems. We also introduce the class of parametric claims rules (including CEA and CEL) and their key properties (continuity, (strict) resource monotonicity, and consistency). Section 3 introduces non-singleton proportional generalized claims problems and the results we have discussed above with subsections dedicated to CEA (Subsection 3.1) and CEL (Subsection 3.2). Section 4 introduces θ-minimal proportional generalized claims problems and the results we have discussed above, again with subsections dedicated to CEA (Subsection 4.1) and CEL (Subsection 4.2). We conclude in Section 5.

2 Model and preliminaries

Consider a **coalition of agents**, e.g., a group of researchers, who have claims on an **endowment**, e.g., a research budget from a national science foundation (NSF). The researchers' **claims** could be related to the past performance / productivity of researchers or be an estimate of the research costs. The research budget is not large enough to cover all claims. Now assume that the NSF prefers to subsidize research projects or teams instead of individual researchers and that then researchers will need to allocate the research funding within the research teams. Furthermore, anticipating this method of allocating funding, researchers might prefer to be members of certain research teams over others. This situation was analyzed by Gallo and Inarra (2018) under the name of *coalition formation problem with claims*. Before fully specifying this class of problems, we present the preliminaries of the two classical type of problems it is based on: **claims problems** and **coalition formation problems**.

First, we introduce some notation. Let $\mathbb{N}$ be the set of potential agents and $\mathcal{N}$ the set of all nonempty finite subsets or **coalitions** of $\mathbb{N}$. Given $N \in \mathcal{N}$ and $x, y \in \mathbb{R}^N$, if for each $i \in N$, $x_i > y_i$, then $x \gg y$ and if for each $i \in N$, $x_i \geq y_i$, then $x \geq y$. Furthermore, for each $x \in \mathbb{R}^N$ and each $S \subseteq N$, $x_S := (x_j)_{j \in S}$ denotes the **restriction of $\boldsymbol{x}$ to coalition $\boldsymbol{S}$**.

Generalized claims problems

Consider a coalition of agents who have claims on a certain endowment, this endowment being insufficient to satisfy all the claims. A primary example is bankruptcy, where agents are the creditors of a firm and the endowment is its liquidation value; however, we have a more general interpretation of the data in mind.

Formally, let $N \in \mathcal{N}$. For $i \in N$, let c_i be agent i's claim and $c = (c_j)_{j \in N}$ the **claims vector**. Let $E \in \mathbb{R}_+$ be the **endowment**. A **claims problem with coalition $\boldsymbol{N}$** is a pair $(c, E) \in \mathbb{R}^N_{++} \times \mathbb{R}_+$ such that $\sum_{j \in N} c_j \geq E$. Let $\boldsymbol{\mathcal{C}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{C}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{C}^N$.

An **allocation for $\boldsymbol{(c, E)}$** $\in \mathcal{C}^N$ is a (payoff) vector $x = (x_i)_{i \in N} \in \mathbb{R}^N_+$ that satisfies the non-negativity and claims boundedness conditions $0 \leq x \leq c$, and the efficiency condition $\sum_{j \in N} x_j = E$. A **rule** is a function F defined on $\mathcal{C}$ that associates with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ an allocation for (c, E). Let $\boldsymbol{\mathcal{F}}$ denote the set of rules.

A rule is **continuous** if small changes in the data of the problem do not lead to large changes in the chosen allocation.

Continuity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each sequence $\{(c^k, E^k)\}$ of elements of $\mathcal{C}^N$, if (c^k, E^k) converges to (c, E) then $F(c^k, E^k)$ converges to $F(c, E)$.

Consider a claims problem and the allocation given by the rule for it. We require that if the endowment increases, then each agent should receive at least as much as (more than, respectively) initially.

Resource monotonicity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{j \in N} c_j \geq E'$, then $F(c, E') \geq F(c, E)$.

Strict resource monotonicity. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' > E$, if $\sum_{j \in N} c_j \geq E'$, then $F(c, E') \gg F(c, E)$.

Consider a claims problem and the allocation given by the rule for it. Imagine that some agents leave with their payoffs. At that point, reassess the situation of the remaining agents, that is, consider the problem of dividing what remains of the endowment among them. We require that the rule should assign the same payoffs to each of them as initially.

Consistency. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $S \subsetneq N$, let $x \equiv F(c, E)$ and recall that $c_S = (c_i)_{i \in S}$. Then, $x_S = F(c_S, E - \sum_{j \in N \setminus S} x_j)$ or, equivalently, $x_S = F(c_S, \sum_{j \in S} x_j)$.

For claims problems many rules are continuous, resource monotonic, and consistent. The most important ones are the so-called "parametric" rules. For a rule in this class, there is a function of two variables such that for each problem, each agent's payoff is the value taken by this function when the first argument is her claim and the second one is parameter λ, which is the same for all agents. This parameter is chosen so that the sum of agents' payoffs is equal to the endowment. Young (1987) characterizes parametric rules on the basis of symmetry,[5] continuity, and bilateral consistency[6]. Stovall (2014) characterizes the family of possibly asymmetric parametric rules on the basis of continuity, resource monotonicity, bilateral consistency, and two additional axioms, "N-continuity" and "intrapersonal consistency."

A **parametric rule** for $N \in \mathcal{N}$ is defined as follows: Let f be a collection of functions $\{f_i\}_{i \in N}$,[7] where each $f_i : \mathbb{R}_{++} \times [a, b] \longrightarrow \mathbb{R}_+$ is continuous and weakly increasing in λ, $\lambda \in [a, b]$, $-\infty \leq a < b \leq \infty$ and for each $i \in N$ and $c_i \in \mathbb{R}_{++}$, $f_i(c_i, a) = 0$ and $f_i(c_i, b) = c_i$. Hence, for each f, a rule F is defined as follows. For each $(c, E) \in \mathcal{C}^N$ and each $i \in N$,

$$F_i(c, E) = f_i(c_i, \lambda) \text{ where } \lambda \text{ is chosen so that } \sum_{j \in N} f_j(c_j, \lambda) = E.$$

Then, f is said to be a **parametric representation of parametric rule $\boldsymbol{F}$**.

The proportional, constrained equal awards, constrained equal losses, Talmud, reverse Talmud, and Piniles rules are symmetric parametric rules while the sequential priority rule associated with a strict priority $\succ$ on agents is an asymmetric parametric rule. We define the first three rules and refer to Thomson (2003, 2015, 2019) for the definition of the other rules mentioned above.

The most commonly used rule in practice makes awards proportional to claims.

Proportional rule, $\boldsymbol{P}$. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $P_i(c, E) = \lambda c_i$, where λ is chosen so that $\sum_{j \in N} \lambda c_j = E$.

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[5] Two agents with equal claims should receive equal payoffs.

[6] Bilateral consistency requires consistency only when $|S| = 2$.

[7] When the parametric rule is symmetric, f_i is the same for all agents.

Our next rule assigns the endowment as equally as possible among agents subject to no one receiving more than her claim.

Constrained equal awards rule, *CEA*. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEA_i(c, E) = \min\{c_i, \lambda\}$, where λ is chosen so that $\sum_{j \in N} \min\{c_j, \lambda\} = E$.

An alternative to the constrained equal awards rule is obtained by focusing on the losses agents incur (what they do not receive), as opposed to what they receive, and to assign losses as equally as possible among agents subject to no one receiving a negative amount.

Constrained equal losses rule, *CEL*. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $CEL_i(c, E) = \max\{0, c_i - \lambda\}$, where λ is chosen so that $\sum_{j \in N} \max\{0, c_j - \lambda\} = E$.

Next, we generalize the notion of a claims problem. Consider $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Then, each coalition of agents $S \subseteq N$ has the **reduced claims vector** $\boldsymbol{c_S} = (c_i)_{i \in S}$. Next, assume that for each coalition $S \subseteq N$ there is a **coalitional endowment** $\boldsymbol{E_S}$ such that $(c_S, E_S) \in \mathcal{C}^S$ and $E_N = E$. Formally, given $N \in \mathcal{N}$, a **generalized claims problem with coalition** $\boldsymbol{N}$ is a pair $(c, (E_S)_{S \subseteq N}) \in \mathbb{R}^N_{++} \times \mathbb{R}^{2^{|N|}-1}_{+}$, such that for each coalition $S \subseteq N$, $(c_S, E_S) \in \mathcal{C}^S$. Let $\boldsymbol{\mathcal{G}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{G}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{G}^N$.

We study the following subclass of generalized claims problems. First, for each coalition $S \subseteq N$, we define a **coalitional claim**, $\boldsymbol{c^S}$, that is equal to the sum of the claims of the members of the coalition, i.e., $c^S := \sum_{j \in S} c_j$. Then, given $(c, E) \in \mathcal{C}^N$ and $\alpha := \frac{E}{c^N} \in (0, 1)$, a **proportional generalized claims problem** is a tuple $(c, (E_S)_{S \subseteq N})$ such that, for each coalition $S \subseteq N$, the coalitional endowment $E_S = \alpha c^S$. Let $\boldsymbol{\mathcal{P}^N}$ denote the class of such problems and $\boldsymbol{\mathcal{P}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{P}^N$. Since coalitional endowments E_S, $S \subseteq N$, are completely determined by c and E, we will simplify notation and denote a proportional generalized claims problem $(c, (E_S)_{S \subseteq N}) \in \mathcal{P}^N$ by $(c, E) \in \mathcal{P}^N$.

An **allocation configuration for** $\boldsymbol{(c, E)} \in \mathcal{P}^N$ is a list $(x_S)_{S \subseteq N}$ such that for each $S \subseteq N$, x_S is an allocation for the claims problem derived from (c, E) for coalition S, (c_S, E_S).[8] Any rule $F \in \mathcal{F}$ can be extended to a **generalized rule** defined on $\mathcal{P}$ by associating with each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{P}^N$ an allocation configuration

[8]The notion of allocation (payoff) configurations was introduced by Hart (1985) in his characterization of the Harsanyi non-transferable utility solution.

$F(c, E) = (F(c_S, E_S))_{S \subseteq N}$. Since it should not lead to any confusion we use $\boldsymbol{\mathcal{F}}$ to also denote the set of generalized rules.

Coalition formation problems

Consider a society where each agent can rank the coalitions that she may belong to. Some well-known examples of such problems are matching problems, in particular, marriage and roommate problems.

Formally, let $N \in \mathcal{N}$. For each agent $i \in N$, $\succsim_i$ is a complete and transitive preference relation over the coalitions of N containing i. Given $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i S'$ means that agent i finds coalition S at least as desirable as coalition S'. Let $\mathcal{R}_i$ be the set of such preference relations for agent i and $\mathcal{R}^N \equiv \Pi_{i \in N} \mathcal{R}_i$. A **coalition formation problem with agent set $\boldsymbol{N}$** consists of a list of preference relations, one for each $i \in N$, $\boldsymbol{\succsim} = (\succsim_i)_{i \in N} \in \mathcal{R}^N$. Let $\boldsymbol{\mathcal{D}^N}$ be the class of such problems and $\boldsymbol{\mathcal{D}} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{D}^N$.

A partition of a set of agents $N \in \mathcal{N}$ is a set of disjoint coalitions of N whose union is N and whose pairwise intersections are empty. Formally, a **partition** of $N \in \mathcal{N}$ is a list $\pi = \{S_1, \ldots, S_m\}$, ($m \leq |N|$ is a positive integer) such that (i) for each $l = 1, \ldots, m$, $S_l \neq \emptyset$, (ii) $\bigcup_{l=1}^m S_l = N$, and (iii) for each pair $l, l' \in \{1, \ldots, m\}$ with $l \neq l'$, $S_l \cap S_{l'} = \emptyset$. Let $\boldsymbol{\Pi(N)}$ denote the set of all partitions of N. For each $\pi \in \Pi(N)$ and each $i \in N$, let $\boldsymbol{\pi(i)}$ denote the unique coalition in π which contains agent i. We refer to the partition $\boldsymbol{\pi^0}$ at which each agent forms a singleton coalition, i.e., for each $i \in N$, $\pi^0(i) = \{i\}$, as the **singleton partition**.

An important question for coalition formation problems is the existence of partitions from which no agent wants to deviate. Let $\succsim \in \mathcal{D}^N$ and consider a partition $\pi \in \Pi(N)$. Then, coalition $T \subseteq N$ **blocks** π if for each agent $i \in T$, $T \succ_i \pi(i)$. A partition $\pi \in \Pi(N)$ is **stable** for $\succsim$ if it is not blocked by any coalition $T \subseteq N$. Let $\boldsymbol{St(\succsim)}$ denote the set of all stable partitions of $\succsim$. We refer to the coalitions that are part of a stable partition as **stable coalitions**.

Coalition formation problems induced by a generalized claims problem and a rule

Given a generalized claims problem and a rule, each agent, by calculating her payoff in each coalition, can form preferences over coalitions, giving rise to a coalition formation problem. Examples of this situation can be the formation of jurisdictions and research groups (see Gallo and Inarra, 2018, for further examples).

Formally, given $N \in \mathcal{N}$, $(c, (E_S)_{S\subseteq N}) \in \mathcal{G}^N$, and $F \in \mathcal{F}$, the **coalition formation problem induced by $((c, (E_S)_{S\subseteq N}), F)$**[9] consists of the list of preferences $\succsim^{((c,E),F)} = (\succsim_j)^{((c,E),F)}_{j\in N}$ defined as follows: for each $i \in N$, and each pair $S, S' \subseteq N$ such that $i \in S \cap S'$, $S \succsim_i^{((c,E),F)} S'$ if and only if $F_i(c_S, E_S) \geq F_i(c_{S'}, E_{S'})$.

Gallo and Inarra (2018, Proposition 1) show that, given any generalized claims problem, parametric rules always induce coalition formation problems that have stable partitions. More generally, they show that a coalition formation problem induced by any generalized claims problem and any continuous rule has a stable partition if and only if the rule F is resource monotonic and consistent (Gallo and Inarra, 2018, Theorem 2).

3 Stable partitions for non-singleton proportional generalized claims problems

We first restrict attention to the class of proportional generalized claims problems $\mathcal{P}$. Thus, consider $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Even without any further assumptions on rule F, the coalition formation problem induced by $((c, E), F)$ has a stable partition. Due to the assumption that the generalized claims problems we consider are proportional, the singleton-partition is always stable and all stable partitions are payoff equivalent to the singleton partition.

Proposition 1. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, the singleton-partition π^0 is stable and each stable partition π induces the proportional allocation configuration where each agent $i \in N$ receives αc_i.*

[9] Note that Gallo and Inarra (2018) refer to this coalition formation problems with claims.

Proof. Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $F \in \mathcal{F}$. Consider the coalition formation problem with agent set N induced by $((c, E), F)$.

First, assume, by contradiction, that there exists a stable partition π that does not induce the proportional allocation configuration. Hence, there exists some agent $i \in N$ who receives an under-proportional payoff $F_i(c_{\pi(i)}, E_{\pi(i)}) < \alpha c_i$. However, since $(c, E) \in \mathcal{P}^N$, agent i by forming the singleton coalition $\{i\}$ obtains $\alpha c_i > F_i(c_{\pi(i)}, E_{\pi(i)})$ and blocks π; contradicting the stability of π. Thus, each stable partition π induces the proportional allocation configuration.

Second, consider the singleton partition π^0 at which each agent $i \in N$ obtains αc_i. Since $(c, E) \in \mathcal{P}^N$, for each $S \subseteq N$, $E_S = \alpha c^S$. Thus, no coalition $S \subseteq N$ can achieve payoffs larger than αc_i for its members $i \in S$ and block π^0. Hence, π^0 is stable. $\square$

Proposition 1 illustrates that for proportional generalized claims problems essentially only the "trivial" singleton partition is stable. For our motivating example, the allocation of research funding to research teams, Proposition 1 predicts the following consequences. If the main principle of research funding allocation from the funding agency to teams is proportionality, then, since essentially only individual proportional funding is stable, the formation of larger research collaborations is unlikely. However, many scientific funding schemes are aimed at the promotion of cooperation of researchers from different countries or disciplines and require research teams of at least size two; see, for instance the international programs of the Swiss National Sciences Foundation (http://www.snf.ch/en/funding/directaccess/international/), and the synergy grants of the European Research Council (https://erc.europa.eu/funding/synergy-grants).

We take this as the departure point to modify the class of proportional generalized claims problems such that only coalitions of a specified minimal size have an incentive to form. For didactic reasons (the intuition of our results will be more easily accessible), we first only disincentivize singleton coalitions by assuming that they will not receive funding. In the next section, we extend our model and results to address all minimal coalition sizes $\theta \geq 2$ for positive coalitional endowments. Hence, the second subclass of generalized claims problems that we consider is the following adjustment of the class of proportional generalized claims problems where, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$, for each $S \subseteq N$ such that $|S| \geq 2$, $E_S = \alpha c^S$ (where $\alpha := \frac{E}{c^N} \in (0, 1)$) and for each $S \subseteq N$ such that $|S| = 1$, $E_S = 0$. Let $\widetilde{\boldsymbol{\mathcal{P}}}^{\boldsymbol{N}}$ be the class of such problems. We refer to this subclass of generalized claims problems as **non-singleton propor-**

tional generalized claims problems and denote it by $\widetilde{\boldsymbol{\mathcal{P}}} \equiv \bigcup_{N\in\mathcal{N}} \widetilde{\mathcal{P}}^N$. We again simplify notation and denote a non-singleton proportional generalized claims problem $(c,(E_S)_{S\subseteq N}) \in \widetilde{\mathcal{P}}^N$ by $(c,E) \in \widetilde{\mathcal{P}}^N$.

Next, we describe the structure of stable partitions for coalition formation problems that are induced by non-singleton proportional generalized claims problems if the underlying rule is continuous, strictly resource monotonic, and consistent. By Gallo and Inarra (2018) the set of stable partitions is nonempty. Furthermore, for each stable partition, there is at most one singleton coalition and each other coalition either allocates proportional payoffs or is of size two. Hence, in comparison to the benchmark result of proportional stable sharing that is equivalent to the singleton-partition, non-proportional cooperation can be sustained in a stable way in pairwise research teams, i.e., research teams formed by two researchers.

Theorem 1. *Let $N \in \mathcal{N}$ and $(c,E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying continuity, strict resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c,E),F)$, the set of stable partitions is nonempty and each stable partition π is such that*

(i) *there is at most one singleton coalition and*

(ii) *if $S \in \pi$ such that $|S| > 2$, then for all $i \in S$, $F_i(c_S,E_S) = \alpha c_i$.*

Proof. Let N, (c,E), and F as specified in the theorem. First, by Gallo and Inarra (2018), each coalition formation problem with agent set N induced by $((c,E),F)$ has at least one stable partition.

(i) Let $\pi \in St(\succsim^{((c,E),F)})$ and assume, by contradiction, that there exist $i,j \in N$, $i \neq j$, such that $\pi(i) = \{i\}$ and $\pi(j) = \{j\}$. For the trivial claims problem $(c_{\{i,j\}},0) \in \mathcal{C}^{\{i,j\}}$, we have $F_i(c_{\{i,j\}},0) = F_j(c_{\{i,j\}},0) = 0$. Then, since $E_{\{i,j\}} > 0$, by strict resource monotonicity, $F_i(c_{\{i,j\}},E_{\{i,j\}}), F_j(c_{\{i,j\}},E_{\{i,j\}}) > 0$. Given that $(c,E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{i\}} = E_{\{j\}} = 0$. Then, $F_i(c_{\{i\}},0) = F_j(c_{\{j\}},0) = 0$. Thus, coalition $\{i,j\}$ blocks π, which is a contradiction.

(ii) Assume that $\pi \in St(\succsim^{((c,E),F)})$ is such that $S \in \pi$, $|S| > 2$, and for some $i \in S = \pi(i)$, $F_i(c_S,E_S) \neq \alpha c_i$. Without loss of generality, assume that agent i receives an over-proportional payoff $F_i(c_S,E_S) > \alpha c_i$. By consistency, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, \sum\nolimits_{k\in S\setminus\{i\}} F_k(c_S,E_S)\right) = F_j(c_S,E_S).$$

Furthermore, $E_{S\setminus\{i\}} = \alpha c^{S\setminus\{i\}} > \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)$ and the agents in coalition $S\setminus\{i\}$ have a larger endowment to share among themselves if they get rid of agent i. Then, by strict resource monotonicity, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}\right) > F_j\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right).$$

Hence, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}\right) > F_j(c_S, E_S)$$

and coalition $S \setminus \{i\}$ blocks π, which is a contradiction. □

We now weaken the property of strict resource monotonicity to resource monotonicity and show that among all possible stable partitions that can exist in a coalition formation problem induced by a non-singleton proportional generalized claims problem, there is always one stable partition that is composed of pairwise research teams (with at most one singleton coalition if the number of agents is odd).

Theorem 2. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, there exists a stable partition π such that*

(i) *if $|N|$ is even, then for each $i \in N$, $|\pi(i)| = 2$ and*

(ii) *if $|N|$ is odd, then there exists an agent $j \in N$, such that $\pi(j) = \{j\}$ and for each $i \in N \setminus \{j\}$, $|\pi(i)| = 2$.*

We prove Theorem 2 in Appendix A.

Now, taking Theorem 2 as departure point, we focus on stable pairwise coalitions, i.e., coalitions of size two. Note that Theorem 2 does not give information about how agents sort themselves into those stable pairwise coalitions. In fact, the stable partitions may differ depending on how endowments are divided among agents. Next, we will study the specific structure of stable partitions under two egalitarian parametric rules: the constrained equal awards (CEA) and the constrained equal losses (CEL) rule. Egalitarianism is a natural principle applied in many economic environments and hence, studying CEA and CEL in our context seems a natural first step to understand stable partitions.

3.1 Stable coalitions under the constrained equal awards rule

We first analyze how agents organize themselves into stable pairwise coalitions when endowments are distributed under the constrained equal awards rule, CEA. Recall that this rule divides endowments as equally as possible subject to no agent receiving more than her claim. Hence, under CEA, agents who get their claims receive over-proportional payoffs (more so the lower the claims are) while some agents with higher claims receive under-proportional payoffs (more so the higher the claims are). Furthermore, the higher an agent's claim, the higher her contribution towards the endowment of any coalition she is part of. So intuitively, in order to form a stable pairwise coalition, one could suspect that an agent with a very high claim will pair up with another high-claim agent. Indeed, high-claim agents play a special role in our construction of stable pairwise coalitions.

Let $N = \{1, \dots n\}$ and assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. First, consider a highest-claim agent, e.g., agent n. As explained above, agent n in a coalition with agent $n-1$ would be a contender to be part of a stable partition with the following possible justification: agent n needs to team up with some agent to obtain a positive payoff and agent $n-1$ provides the highest possible contribution to coalition $\{n-1, n\}$ while requiring a smaller transfer from the proportional payoff compared to other agents. This reasoning is correct, unless a lowest-claim agent, e.g., agent 1, has such a small claim that conceding this small claim to agent 1 is a smaller loss from the proportional payoff for agent n than the transfer from the proportional payoff of n to agent $n-1$. We capture this intuition in an algorithm that determines a stable partition by sequentially pairing off either two highest-claims agents or a highest-claim with a lowest-claim agent. Note that if $|N| \leq 2$, then the grand coalition N forms a stable partition. We therefore define the algorithm for agent sets with at least three agents.

CEA Algorithm

Input: $N \in \mathcal{N}$ such that $|N| > 2$ and $(c, E) \in \widetilde{\mathcal{P}}^N$.

Step 1. Let $N_1 := N$ and

$$\delta_1 := \frac{2c_1}{2c_1 - c_{n-1} + c_n}.$$

We distinguish two cases:

(i) If $\alpha \leq \delta_1$, then $\{n-1, n\} \succsim_n^{((c,E),CEA)} \{1, n\}$ and set $S_1 := \{n-1, n\}$.

(ii) If $\alpha > \delta_1$, then $\{1, n\} \succ_n^{((c,E),CEA)} \{n-1, n\}$ and set $S_1 := \{1, n\}$.

Set $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step *k* (*k* > 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. We relabel the agents in N_k such that $N_k = \{1', \ldots, n'\}$ and $c_{1'} \leq \ldots \leq c_{n'}$. Let

$$\delta_k := \frac{2c_{1'}}{2c_{1'} - c_{(n-1)'} + c_{n'}}.$$

We distinguish two cases:

(i) If $\alpha \leq \delta_k$, then $\{(n-1)', n'\} \succsim_{n'}^{((c,E),CEA)} \{1', n'\}$ and set $S_k := \{(n-1)', n'\}$.

(ii) If $\alpha > \delta_k$, then $\{1', n'\} \succ_{n'}^{((c,E),CEA)} \{(n-1)', n'\}$ and set $S_k := \{1', n'\}$.

Set $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \ldots, S_l\}$ for the coalition formation problem with agent set N induced by $((c,E), CEA)$ such that for each $k \in \{1, \ldots, l\}$, $|S_k| \leq 2$. If $|N|$ is even, then partition π is constructed in $l-1 = \frac{n-2}{2}$ steps. If $|N|$ is odd, then partition π is constructed in $l-1 = \frac{n-1}{2}$ steps.

The following result states that the partition obtained by the CEA algorithm is stable.

Theorem 3. *Let $N \in \mathcal{N}$ such that $|N| > 2$ and $(c,E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c,E), CEA)$. Then, the partition obtained by the CEA algorithm is stable.*

We prove Theorem 3 in Appendix B but we explain the key intuition of the proof here. Observe that in each step of the CEA algorithm either two agents with the highest claims are paired off or an agent with the highest claim is paired with an agent with the lowest claim. We first show (Appendix B, Lemma 3) that each agent weakly prefers to form a pairwise coalition with a highest-claim agent instead of with any other agent, e.g., agent n. Hence, by matching agent n with her most desirable pairwise coalition partner, a stable pairwise coalition is formed (in fact, a pairwise *top coalition* is formed, see Appendix B).

Second, we show (Appendix B, Lemmas 4 and 5) that agents 1 and $n-1$ are potential "stable partners" for agent n. Furthermore, we show that when $\alpha \le \frac{2c_1}{c_1+c_n}$, then $\{n-1, n\}$ is the candidate for a stable pairwise coalition, and when $\alpha \ge \frac{2c_{n-1}}{c_{n-1}+c_n}$, then $\{1, n\}$ is the candidate for a stable pairwise coalition. Note that $\frac{2c_1}{c_1+c_n} \le \frac{2c_{n-1}}{c_{n-1}+c_n}$. Thus, we next need to determine a threshold value for parameter $\alpha \in [\frac{2c_1}{c_1+c_n}, \frac{2c_{n-1}}{c_{n-1}+c_n}]$ to see when agent n's partner of choice is $n-1$ and when it is 1. We then show that the threshold we are looking for is exactly $\frac{2c_1}{2c_1-c_{n-1}+c_n}$ (Appendix B, Lemma 6), the value specified in Step 1 of the CEA algorithm to trigger either Case (i) with stable coalition candidate $\{n-1, n\}$ or Case (ii) with stable coalition candidate $\{1, n\}$. Using the steps of the CEA algorithm, we then show that the resulting CEA partition is stable.

Finally, note that the results in Appendix B (Lemmas 4, 5, and 6) can be used to show that for values of α low enough to trigger Case (i) in each step of the CEA algorithm, starting with agent n with the highest claim, a stable partition is formed by assortative pairwise coalitions (we call this an *assortative stable partition*). This implies that if n is odd, then the singleton coalition will be formed by agent 1. Similarly, for values of α high enough to trigger Case (ii) in each step of the CEA algorithm, a stable partition is formed by pairwise coalitions in which a highest-claim and a lowest-claim agent are matched in each step (we call this an *extremal stable partition*). So, depending on α, the CEA algorithm constructs a stable partition that is either assortative, extremal, or a mix of both type of pairwise coalitions.

Assortative matching of high types has been observed in other contexts, for instance, the neoclassical marriage model by Becker (1973). In our example of research team formations, assortative matching of high types can indeed be observed in practice but it can also be observed in other situations such as the formation of pairs of students for class projects or other social environments. However, we also observe extremal research team formations, for example in mentor-mentee relationships such as between a PhD student and her advisor.

Finally, the stable partition obtained by the CEA algorithm is not unique (even beyond tie-breaking between Cases (i) and (ii) in the algorithm). The following example demonstrates that stable partitions with larger coalition sizes are possible.

Example 1. Let $N = \{1, 2, 3\}$, $c = (1, 2, 3)$, $E = 5.4$, and $(c, E) \in \widetilde{\mathcal{P}}^N$ (hence, $\alpha = \frac{9}{10}$). Let $F = CEA$. Hence,

Coalition	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Endowment	0	0	0	2.7	3.6	4.5	5.4
Allocation	(0)	(0)	(0)	$(1,1.7)$	$(1,2.6)$	$(2,2.5)$	$(1,2,2.4)$

The coalition formation problem induced, $\succsim^{((c,E),CEA)}$, is

$$\begin{aligned}
\succsim_1^{((c,E),CEA)}&: \quad \{1,2\} \sim_1 \{1,3\} \sim_1 \{1,2,3\} \succ_1 \{1\},\\
\succsim_2^{((c,E),CEA)}&: \quad \{2,3\} \sim_2 \{1,2,3\} \succ_2 \{1,2\} \succ_2 \{2\},\\
\succsim_3^{((c,E),CEA)}&: \quad \{1,3\} \succ_3 \{2,3\} \succ_3 \{1,2,3\} \succ_3 \{3\}.
\end{aligned}$$

The stable partition obtained by our algorithm in Theorem 3 is $\{\{13\},\{2\}\}$. However, it can be easily verified that partition $\{\{123\}\}$ is also stable.

3.2 Stable coalitions under the constrained equal losses rule

We next analyze how agents organize themselves into stable pairwise coalitions when endowments are distributed under the constrained equal losses rule, CEL. Recall that this rule allocates losses as equally as possible subject to no agent receiving a negative amount. Hence, under CEL, the agents who get a zero payoff (if they exist) receive under-proportional payoffs (more so the lower the claims are) while some agents with higher claims receive over-proportional payoffs (more so the higher the claims are). Furthermore, the lower an agent's claim, the lower her contribution towards the loss of any coalition she is part of. So intuitively, in order to form a stable pairwise coalition, one could suspect that an agent with a very low claim will pair up with another low-claim agent. So in contrast to the CEA rule, when the CEL rule is used, low-claim agents play a special role in our construction of stable pairwise coalitions.

Let $N = \{1,\dots n\}$ and assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. First, consider a lowest-claim agent, e.g., agent 1. As explained above, agent 1 in a coalition with agent 2 would be a contender to be part of a stable partition with the following possible justification: agent 1 needs to team up with some agent to obtain a positive payoff (if possible) and agent 2 provides the lowest possible loss to coalition $\{1,2\}$ while requiring a smaller transfer from the proportional payoff compared to other agents. We capture this intuition in an algorithm that determines a stable partition by sequentially pairing off two lowest-claims agents. Note that if $|N| \leq 2$, then the grand coalition N forms a stable partition. We therefore define the algorithm for agent sets with at least three agents.

CEL Algorithm

Input: $N \in \mathcal{N}$ such that $|N| > 2$ and $(c, E) \in \widetilde{\mathcal{P}}^N$.

Step 1. Let $N_1 := N$. Set $S_1 := \{1, 2\}$ and $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step $\boldsymbol{k}$ ($\boldsymbol{k > 1}$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Set $S_k := \{2k-1, 2k\}$ and $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \ldots, S_l\}$ for the coalition formation problem with agent set N induced by $((c, E), CEL)$ such that for each $k \in \{1, \ldots, l\}$, $|S_k| \leq 2$. If $|N|$ is even, then partition π is constructed in $l - 1 = \frac{n-2}{2}$ steps. If $|N|$ is odd, then partition π is constructed in $l - 1 = \frac{n-1}{2}$ steps.

The following result states that the partition obtained by the CEL algorithm is stable.

Theorem 4. *Let $N \in \mathcal{N}$ such that $|N| > 2$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$. Then, the partition obtained by the CEL algorithm is stable.*

We prove Theorem 4 in Appendix C but we explain the key intuition of the proof here. Consider $i, j \in N$ such that $c_i < c_j$. Then, for coalition $\{i, j\}$, $E_{\{i,j\}} = \alpha(c_i + c_j)$ and the associated loss equals $(1 - \alpha)(c_i + c_j)$. Hence, the loss decreases (increases) if either agent i or j switches to a pairwise coalition with a lower-claim (higher-claim) agent. Since losses are split as equally as possible (taking zero as lower bound), sequentially matching pairs of lowest-claims agents will lead to a stable partition.

Observe that in each step of the CEL algorithm two agents with the lowest claims are paired off and we obtain an assortative stable partition starting from an agent with the lowest claim. In particular, this implies that if n is odd, then the singleton coalition will be formed by agent n (this contrasts the assortative case of the CEA algorithm where an agent with the lowest claim would form the singleton coalition).

Finally, the stable partition obtained by the CEL algorithm is not unique. The following example demonstrates that stable partitions with larger coalition sizes are possible.

Example 2. Let $N = \{1, 2, 3\}$, $c = (1, 3, 11)$, $E = 7.5$, and $(c, E) \in \widetilde{\mathcal{P}}^N$ (hence, $\alpha = \frac{1}{2}$). Let $F = CEL$. Hence,

Coalition	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
Endowment	0	0	0	2	6	7	7.5
Allocation	(0)	(0)	(0)	(0, 2)	(0, 6)	(0, 7)	(0, 0, 7.5)

The coalition formation problem induced, $\succsim^{((c,E),CEL)}$ is

$$\begin{aligned}
\succsim_1^{((c,E),CEL)}&: \quad \{1,2\} \sim_1 \{1,3\} \sim_1 \{1,2,3\} \sim_1 \{1\},\\
\succsim_2^{((c,E),CEL)}&: \quad \{1,2\} \succ_2 \{1,2,3\} \sim_2 \{2.3\} \sim_2 \{2\},\\
\succsim_3^{((c,E),CEL)}&: \quad \{1,2,3\} \succ_3 \{2.3\} \succ_3 \{1.3\} \succ_3 \{3\}.
\end{aligned}$$

It can be checked that any partition of N is stable.

4 Stable partitions for θ-minimal proportional generalized claims problems

Based on the assumption that singleton coalitions are not funded, in Section 3 we have analyzed the class of non-singleton proportional generalized claims problems. Of course, there are funding schemes where a larger minimal number of researchers is required, i.e., the minimal size for a coalition to generate any positive value could be larger than two. An example are, for instance, European Union funded COST (European Cooperation in Science and Technology) actions that focus on research and innovation networks (https://www.cost.eu/) with at least seven participating countries.

Thus, we introduce the more general subclass of proportional generalized claims problems in which coalitions of size smaller than $\theta \in \mathbb{N}$ ($\mathbb{N}$ denotes the set of positive integers) are disincentivized by assuming that all coalitions of size smaller than θ

receive a zero coalitional endowment. Hence, the next subclass of generalized claims problems that we consider is an adjustment of the class of proportional generalized claims problems where, given $N \in \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$, for each $S \subseteq N$ such that $|S| \geq \theta$, $E_S = \alpha c^S$ (where $\alpha := \frac{E}{c^N} \in (0,1)$) and for each $S \subseteq N$, $S \neq \emptyset$, such that $|S| < \theta$, $E_S = 0$. Let $\boldsymbol{\widetilde{\mathcal{P}}_\theta^N}$ be the class of such problems. We refer to this subclass of generalized claims problems as **θ-minimal proportional generalized claims problems** and denote it by $\boldsymbol{\widetilde{\mathcal{P}}_\theta} \equiv \bigcup_{N \in \mathcal{N}} \widetilde{\mathcal{P}}_\theta^N$. For $\theta = 2$, $\widetilde{\mathcal{P}}_\theta$ corresponds to the class of non-singleton proportional generalized claims problems $\widetilde{\mathcal{P}}$ considered in Section 3. We again simplify notation and denote a θ-minimal proportional generalized claims problem $(c, (E_S)_{S \subseteq N}) \in \widetilde{\mathcal{P}}_\theta^N$ by $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$.

Next, we generalize Theorems 1 and 2 from $\widetilde{\mathcal{P}}$ to $\widetilde{\mathcal{P}}_\theta$, i.e., from $\theta = 2$ to $\theta \in \mathbb{N}$. We first analyze the structure of coalition formation problems that are induced by θ-minimal proportional generalized claims problems and describe the structure of stable partitions when considering a rule that is continuous, strictly resource monotonic, and consistent.

Theorem 5. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider $F \in \mathcal{F}$ satisfying continuity, strict resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, the set of stable partitions is nonempty and each stable partition π is such that*

(i) *there are fewer than θ agents in coalitions of size smaller than θ, and*

(ii) *if $S \in \pi$ such that $|S| > \theta$, then for all $i \in S$, $F_i(c_S, E_S) = \alpha c_i$.*

Proof. Let N, (c, E), and F as specified in the theorem. First, by Gallo and Inarra (2018), each coalition formation problem with agent set N induced by $((c, E), F)$ has at least one stable partition.

(i) Let $\pi \in St(\succsim^{((c,E),F)})$, $T = \{i \in N : \pi(i) < \theta\}$, and assume, by contradiction, that $|T| \geq \theta$. For the trivial claims problem $(c_T, 0) \in \mathcal{C}^T$, we have that for each $i \in T$, $F_i(c_T, 0) = 0$. Then, since $E_T > 0$, by strict resource monotonicity, for each $i \in T$, $F_i(c_T, E_T) > 0$. Given that $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$, for each $i \in T$ we have $E_{\pi(i)} = 0$. Thus, coalition T blocks π, which is a contradiction.

(ii) The proof is identical to the proof of Theorem 1 (ii), by simply replacing $\theta = 2$ with $\theta \in \mathbb{R}$. □

We now weaken the property of strict resource monotonicity to resource monotonicity and show that among all possible stable partitions that can exist in a coalition formation problem induced by a θ-minimal proportional generalized claims problem, there is always one stable partition that is composed of the maximal possible number of coalitions of size θ and one coalition (of size lower than θ) formed by the remaining agents.[10]

Theorem 6. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. Then, for the coalition formation problem with agent set N induced by $((c, E), F)$, there exists a stable partition π such that*

(i) *if $|N|$ is divisible by θ, then for each $i \in N$, $|\pi(i)| = \theta$ and*

(ii) *if $|N|$ is not divisible by θ, then π contains $k = \lfloor \frac{|N|}{\theta} \rfloor$ coalitions of size θ and one coalition of size $|N| - k < \theta$.*

We prove Theorem 6 in Appendix D (the proof follows the same proof structure as the proof of Theorem 2 in Appendix A).

Note that up to now, extending results from non-singleton proportional generalized claims problems (Theorems 1 and 2) to θ-minimal proportional generalized claims problems (Theorems 5 and 6) was relatively easy; that is, proofs, up to some small adjustments to accommodate $\theta \geq 2$ minimal coalition sizes, were essentially the same. For our remaining results, this is going to change: analyzing the stable partitions that can emerge for θ-minimal proportional generalized claims problems when the rules applied are CEA and CEL require new theoretical results. We again provide algorithms to construct a stable partition for each of these rules. Our algorithms are natural extensions of those proposed in Section 3 in that at each step of the algorithm, either a highest-claim agent (for CEA) or a lowest-claim agent (for CEL) myopically adds agents one by one to maximize her own payoff in each step until a coalition of size θ is reached. However, the theoretical challenge in this sequential approach is to show that the step by step myopic optimization procedure in the end also yields a farsightedly optimal stable partition.

4.1 Stable coalitions under the constrained equal awards rule

For the case of the CEA rule, remember first that in Section 3.1, for $\theta = 2$, we have constructed an algorithm to find a stable partition such that in each step a highest-

[10]For convenience, we put all "remaining agents" together in one coalition. However, note that all coalitions formed among remaining agents have sizes smaller than θ and thus all payoffs are zero. Hence, we could have payoff-equivalently partitioned this set of remaining agents differently.

claim agent forms a coalition with either a lowest-claim agent or another highest-claim agent. Therefore, one could suspect that a similar argument could arise for any $\theta \in \mathbb{N}$.

Let $N = \{1, \dots n\}$ and assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. For each $S \subseteq N$, we denote the CEA parameter associated with (c_S, E_S) by λ_{E_S}, i.e., for each $i \in S$, $CEA_i(c_S, E_S) = \min\{c_i, \lambda_{E_S}\}$, where λ_{E_S} is chosen so that $\sum_{j \in S} \min\{c_j, \lambda_{E_S}\} = E_S$.

First, assume that $\theta = 2$ and consider a highest-claim agent, e.g., agent n. By the CEA algorithm, we know that agents 1 and $n-1$ are the possible "stable partners" for agent n. Hence, either coalition $\{1, n\}$ or coalition $\{n-1, n\}$ would form. Consider now $\theta = 3$ and the coalition formed for $\theta - 1 = 2$ (either coalition $\{1, n\}$ or coalition $\{n-1, n\}$), which now plays the role of agent n. Then, the next agent to join will again be either a lowest-claim agent (agents 2 or 1, respectively) or a highest-claim agent (agents $n-1$ or $n-2$, respectively). A similar reasoning can now, step by step, be applied to $\theta + 1$, $\theta + 2$, etc., and a first coalition can be constructed for any value of θ. We now more formally present our step by step algorithm to determine the first coalition of a stable partition by sequentially adding either a lowest-claim agent or a highest-claim agent. Note that if $|N| \leq \theta$, then the grand coalition N forms a stable partition. We therefore define the algorithm for agent sets with at least $\theta + 1$ agents.

$\boldsymbol{\theta}$-CEA set algorithm with agent set $\boldsymbol{N}$

Input: $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}^N_\theta$.

Step 1. Let $\theta' = 2$, $N'_1 := N$, and consider coalition $\{n-1, n\}$ and $\lambda_{E_{\{n-1,n\}}}$. We distinguish two cases:

(i) If $\lambda_{E_{\{n-1,n\}}} \leq (1-\alpha)c_1 + \alpha c_{n-1}$, then $\{n-1, n\} \succsim_n^{((c,E),CEA)} \{1, n\}$ and set $S'_1 := \{n-1, n\}$.

(ii) If $\lambda_{E_{\{n-1,n\}}} > (1-\alpha)c_1 + \alpha c_{n-1}$, then $\{1, n\} \succ_n^{((c,E),CEA)} \{n-1, n\}$ and set $S'_1 := \{1, n\}$.

Note that these cases are equivalent to the cases specified in Step 1 of the CEA algorithm.[11]

[11] By the definition of the CEA rule, $\lambda_{E_{\{n-1,n\}}} \geq \frac{\alpha(c_{n-1}+c_n)}{2}$. Next, $\frac{\alpha(c_{n-1}+c_n)}{2} \lesseqgtr (1-\alpha)c_1 + \alpha c_{n-1}$ is equivalent to $\alpha \lesseqgtr \frac{2c_1}{2c_1 - c_{n-1} + c_n} = \delta_1$. Hence, if $\lambda_{E_{\{n-1,n\}}} = \frac{\alpha(c_{n-1}+c_n)}{2}$ we obtain the equivalence of Cases (i) and (ii) with CEA Algorithm Cases (i) and (ii). Finally, if $\lambda_{E_{\{n-1,n\}}} > \frac{\alpha(c_{n-1}+c_n)}{2}$, then $\lambda_{E_{\{n-1,n\}}} > c_{n-1} = (1-\alpha)c_{n-1} + \alpha c_{n-1} \geq (1-\alpha)c_1 + \alpha c_{n-1}$ and Case (ii) applies. Therefore, $\lambda_{E_{\{n-1,n\}}} = \alpha(c_{n-1} + c_n) - c_{n-1} > \frac{\alpha(c_{n-1}+c_n)}{2}$, which implies $\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \geq \frac{2c_1}{2c_1 - c_{n-1} + c_n}$ (the last inequality is shown as $\gamma_1 \geq \delta_1$ at the end of Appendix B), which coincides with CEA Algorithm Case (ii).

If $\theta = 2$, then set $S_1 := S'_1$ and stop. Otherwise, set $N'_2 := N \setminus S'_1$, and go to Step 2.

Step k ($k = 2, \ldots, \theta - 1$). Consider $\theta' = k + 1$. Recall from Step $k - 1$ that $N'_k := N \setminus S'_{k-1}$. Let agent i be the agent with the lowest label in N'_k, $i = \min N'_k$, and let agent j be the agent with the highest label in N'_k, $j = \max N'_k$. Thus, agent i is a lowest-claim agent and agent j is a highest-claim agent in N'_k. Consider coalition $S'_{k-1} \cup \{j\}$ and $\lambda_{E_{(S'_{k-1} \cup \{j\})}}$. We distinguish two cases:

(i) If $\lambda_{E_{(S'_{k-1} \cup \{j\})}} \leq (1 - \alpha)c_i + \alpha c_j$, then $S'_{k-1} \cup \{j\} \succsim^{((c,E),CEA)}_{S'_{k-1}} S'_{k-1} \cup \{i\}$ and set $S'_k := S'_{k-1} \cup \{j\}$.

(ii) If $\lambda_{E_{(S'_{k-1} \cup \{j\})}} > (1 - \alpha)c_i + \alpha c_j$, then $S'_{k-1} \cup \{i\} \succ^{((c,E),CEA)}_{S'_{k-1}} S'_{k-1} \cup \{j\}$ and set $S'_k := S_{k-1} \cup \{i\}$.

If $\theta = k + 1$, then set $S_1 = S'_k$ and stop. Otherwise, set $N'_{k+1} := N \setminus S'_k$, and go to Step $k + 1$.

Output: A coalition S_1 of size θ, which is obtained in $\theta - 1$ steps.

Note that in each Step k of the above algorithm, a set of agents S'_{k-1} considers to add either the lowest-label or the highest-label agent of the remaining set of agents. In Appendix E, we prove that the agent who is added according to Case (i) or (ii) is weakly preferred by the agents in S'_{k-1} to any other agent who could have been added.

Starting with the set of all agents N, the above algorithm constructs the first coalition S_1 of a stable partition. Then, if the residual set of agents $N_1 = N \setminus S_1$ is such that $|N_1| \leq \theta$, (S_1, N_1) forms a stable partition. Otherwise, we repeat the above algorithm for the set of agents N_1. Recall that if $|N| \leq \theta$, then the grand coalition N forms a stable partition. We therefore define our generalization of the CEA algorithm (Section 3.1) for agent sets with at least $\theta + 1$ agents.

θ-CEA Algorithm

Input: $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}^N_\theta$.

Step 1. Let $N_1 := N$. By applying the θ-CEA set algorithm with agent set N_1 we obtain the first coalition S_1. Set $N_2 := N \setminus S_1$. If $|N_2| \leq \theta$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step k ($k > 1$). Recall from Step $k - 1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > \theta$. We relabel the agents in N_k such that $N_k = \{1', \ldots, n'\}$ and $c_{1'} \leq \ldots \leq c_{n'}$. By applying the θ-CEA set algorithm with agent set N_k, we obtain set S_k. Set $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$.

If $|N_{k+1}| \leq \theta$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \ldots, S_l\}$ for the coalition formation problem with agent set N induced by $((c, E), CEA)$ such that for each $k \in \{1, \ldots, l\}$, $|S_k| \leq \theta$. If $|N|$ is divisible by θ, then partition π is constructed in $l - 1 = \frac{n-\theta}{\theta}$ steps. If $|N|$ is not divisible by θ, then partition π is constructed in $l - 1 = \lfloor \frac{n}{\theta} \rfloor$ steps.

The following result states that the partition obtained by the θ-CEA algorithm is stable.

Theorem 7. *Let $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEA)$. Then, the partition obtained by the θ-CEA algorithm is stable.*

We prove Theorem 7 in Appendix F but we explain the key intuition of the proof here. The θ-CEA set algorithm describes a process of adding agents one by one such that the respective existing coalition weakly prefers the added agent to any other agent that could be added. This process already captures an aspect of coalitional stability but it is *myopic* in nature and we still need to show that the thus obtained coalitions are "*farsightedly*" stable as well. To see this, we assume, by contradiction, that a blocking coalition S exists. As part of the proof, we then show that an agent who received an over-proportional payoff in her coalition in partition π will receive an even larger over-proportional payoff in S. Furthermore, an agent who received an under-proportional payoff in her coalition in partition π will receive an over-proportional payoff in S or transfer less to other agents in S. This, together with some additional proof steps, leads to an imbalance between the transfers that are being made within coalition S between agents with over-proportional and under-proportional payoffs.

4.2 Stable coalitions under the constrained equal losses rule

For the case of the CEL rule, remember first that in Section 3.2, for $\theta = 2$, we have constructed an algorithm to find a stable partition such that in each step two lowest-claims agents are paired off. Therefore, one could suspect that a similar result could arise for any $\theta \in \mathbb{N}$. That is, intuitively, in order to form a stable coalition of size θ, one could suspect that an agent with a very low claim will pair up with $\theta - 1$ low-claim agents. So again, when the CEL rule is used, low-claim agents play a special role.

Let $N = \{1, \dots n\}$ and assume that $c_1 \leq c_2 \leq \cdots \leq c_n$. First, consider a lowest-claim agent, e.g., agent 1. As explained above, agent 1 in a coalition with agents $2, \dots, \theta$ would be a contender to be part of a stable partition with the following possible justification: agent 1 needs to team up with some agents to obtain a positive payoff and agents $2, \dots, \theta$ provide the lowest possible loss to coalition $\{1, 2, \dots, \theta\}$ while requiring a smaller transfer from the proportional payoff compared to other agents. We capture this intuition in an algorithm that determines a stable partition by sequentially pairing off lowest-claims agents. Note that if $|N| \leq \theta$, then the grand coalition N forms a stable partition. We therefore define the algorithm for agent sets with at least $\theta + 1$ agents.

θ-CEL Algorithm

Input: $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$.

Step 1. Let $N_1 := N$, $|N_1| > \theta$. Set $S_1 := \{1, 2, \dots, \theta\}$ and $N_2 := N \setminus S_1$. If $|N_2| \leq \theta$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step $\boldsymbol{k}$ ($\boldsymbol{k > 1}$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > \theta$. Set $S_k := \{\theta k - (\theta - 1), \theta k - (\theta - 2), \dots, \theta k\}$ and $N_{k+1} := N \setminus \cup_{j=1}^{k} S_j$. If $|N_{k+1}| \leq \theta$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, S_2, \dots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

Output: A partition $\pi = \{S_1, \dots, S_l\}$ for the coalition formation problem with agent set N induced by $((c, E), CEL)$. If $|N|$ is divisible by θ, then partition π is constructed in $l - 1 = \frac{n-\theta}{\theta}$ steps. If $|N|$ is not divisible by θ, then partition π is constructed in $l - 1 = \lfloor \frac{n}{\theta} \rfloor$ steps.

The following result states that the partition obtained by the θ-CEL algorithm is stable.

Theorem 8. *Let $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$. Then, the partition obtained by the θ-CEL algorithm is stable.*

We prove Theorem 8 in Appendix G but we explain the key intuition of the proof here. Consider a coalition $S \subsetneq N$ with $E_S = \alpha c^S$ and associated loss equals $(1-\alpha)c^S$. Hence, the coalitional loss decreases (increases) if we switch an agent in S with a lower-claim (higher-claim) agent from $N \setminus S$. Since losses are split as equally as possible (taking zero as lower bound), and, by an iterative reasoning, sequentially matching up θ lowest-claims agents will lead to a stable partition.

5 Conclusion

In this paper, we continue the analysis of Gallo and Inarra's (2018) *coalition formation problems with claims.* They focus on the existence of stable partitions but they do not analyze their exact structure. To make more precise predictions about the possible size and composition of stable partitions, we restrict attention to what we call non-singleton proportional generalized claims problems where singleton coalitions receive zero endowments and all remaining coalitional endowments are a fixed proportion of the sum of the claims of coalition members. Let us briefly summarize our results.

We first characterize the structure of any possible stable partition when the rule applied satisfies continuity, strict resource monotonicity, and consistency. For the weaker notion of resource monotonicity, we demonstrate the existence of a stable pairwise partition with at most one singleton coalition if the set of agents is odd. Furthermore, we provide two algorithms to construct stable pairwise coalitions under CEA and CEL, respectively. For the CEA rule, the obtained stable partition assortatively pairs off either highest-claims agents (assortative coalition) or a highest-claim and a lowest-claim agent (extremal coalition). For the CEL rule, an assortative stable partition is obtained by sequentially pairing off lowest-claims agents.

All these results can be generalized to the case where not only singleton coalitions are disincentivized but also other coalitions of a larger size, say size θ. This generalization can be found in Section 4. For this subclass of generalized claims problems, θ-minimal proportional generalized claims problems, we first characterize the structure of any possible stable partition when the rule applied satisfies continuity, strict resource monotonicity, and consistency. Then, we show that when the rule applied to allocate coalitional endowments satisfies weak resource monotonicity instead of the strict notion, a stable partition formed by the maximal possible number of coalitions of size θ and one coalition of size lower than θ formed by the remaining agents always arises. For this last case, we also provide two algorithms to construct that partition under CEA and CEL, respectively.

Observe that our results are based on the assumption of proportional coalitional endowments. Future research could consider another principle of assigning coalitional endowments than proportionality. More generally, a two-step model in which first the total endowment is split between coalitions (by a claims rule) and second, within each coalition the coalitional endowment is split between its members (by another rule, possibly the same as the first one) could be considered (for a related two-step model in a bankruptcy framework see, for instance, Izquierdo and Timoner, 2019).[12] Therefore, coalition formation will depend both on the rule that divides the total endowment among the different coalitions and on the rule that is used to distribute the coalitional endowments among its members. Observe that our model can be straightforwardly extended to a two-step procedure in which the rule used in the first step is the proportional rule for any coalition of size larger than or equal to two (θ) and the constant zero rule for singleton coalitions (coalitions of size lower than θ), and the rule applied in the second step satisfies continuity, (strict) resource monotonicity, and consistency (or, for some of our results, equals the CEA or CEL rule).

Finally, as already mentioned in the introduction, there are many papers dealing with assortative stable coalitions. We briefly discuss three of them.

Barberà et al. (2015) consider a model in which each agent is endowed with a productivity level and agents can cooperate to perform certain tasks. Each coalition generates an output equal to the sum of its members' productivities. The authors then analyze the formation of coalitions when all agents in a society vote between meritocracy and egalitarianism. They find societies where assortative and non-assortative partitions (in terms of productivity) arise.

In a bargaining framework, Pycia (2012) presents a model in which each agent has a utility function and, for each possible coalition of agents, there is an output to be distributed among its members. He analyzes coalition formation games induced by different bargaining rules and shows that when agents are endowed with productivity levels and "when shares are divided by a stability-inducing sharing rule, agents sort themselves into coalitions in a predictably assortative way". Pycia (2012) deals with many-to-one problems and his notion of assortativeness implies that the most productive agents join the most productive firms.

[12]Two-step procedures have also been analyzed, among others, by Lorenzo-Freire et al. (2010) and Bergantiños et al. (2010) for multi-issue allocation problems. Moreno-Ternero (2011) studies a coalition procedure (two or more steps) for bankruptcy situations.

Finally, Bogomolnaia et al. (2008) study societies where agents are located in an interval and form jurisdictions to consume public projects, which are located in the same interval. Agents share their costs equally and they divide transportation costs to the location of the public project based on its distance to each agent. They analyze both core and Nash stable partitions with a focus on assortative and non-assortative (in terms of location) stable jurisdiction structures.

Appendix

A Proof of Theorem 2

We first introduce some lemmas that will be used to prove Theorem 2. We first show that, given a non-singleton proportional generalized claims problem and a consistent rule, if all agents receive proportional payoffs in a coalition, then all agents receive proportional payoffs in any subcoalition (except singleton subcoalitions) as well.

Lemma 1. *Let* $N \in \mathcal{N}$ *and* $(c, E) \in \widetilde{\mathcal{P}}^N$. *Consider* $F \in \mathcal{F}$ *satisfying consistency. If* $S \subseteq N$, $|S| > 2$, *is such that for each* $i \in S$, $F_i(c_S, E_S) = \alpha c_i$, *then for each* $S' \subsetneq S$ *with* $|S'| \geq 2$ *and each* $j \in S'$, $F_j(c_{S'}, E_{S'}) = \alpha c_j$.

Proof. Let N, (c, E), and F as specified in the lemma. Consider $S \subseteq N$, $|S| > 2$, such that for each $i \in S$, $F_i(c_S, E_S) = \alpha c_i$ and consider $S' \subsetneq S$ with $|S'| \geq 2$ and problem $(c_{S'}, E_{S'}) \in C^{S'}$. By consistency, for each $j \in S'$, $F_j\left(c_{S'}, \sum_{k \in S'} F_k(c_S, E_S)\right) = F_j(c_S, E_S)$ and hence, $F_j\left(c_{S'}, \sum_{k \in S'} F_k(c_S, E_S)\right) = \alpha c_j$. Since $(c, E) \in \widetilde{\mathcal{P}}^N$ we have that $E_{S'} = \alpha c^{S'} = \sum_{k \in S'} F_k(c_S, E_S)$. Thus, for each $j \in S'$, $F_j(c_{S'}, E_{S'}) = \alpha c_j$. □

We next show that, given a non-singleton proportional generalized claims problem and a resource monotonic and consistent rule, if some agent in a coalition of size larger than two does not receive a proportional payoff, then there exists a subcoalition that is strictly preferred by at least one member of the subcoalition.

Lemma 2. *Let* $N \in \mathcal{N}$ *and* $(c, E) \in \widetilde{\mathcal{P}}^N$. *Consider* $F \in \mathcal{F}$ *satisfying resource monotonicity and consistency. If* $S \subseteq N$, $|S| > 2$, *is such that for some* $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$, *then there is* $S' \subsetneq S$, *with* $|S'| = |S| - 1$, *such that for some agent* $l \in S'$, $S' \succ_l^{((c,E),F)} S$.

***Proof*.** Let N, (c, E), and F as specified in the lemma. Consider $S \subseteq N$, $|S| > 2$, such that for some $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$. By consistency, for each $j \in S \setminus \{i\}$,

$$F_j\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) = F_j(c_S, E_S). \tag{1}$$

Since $E_S = \alpha c^S$ and $F_i(c_S, E_S) \neq \alpha c_i$, we can assume, without loss of generality, that agent i receives an over-proportional payoff, i.e., $F_i(c_S, E_S) > \alpha c_i$. Therefore, subcoalition $S \setminus \{i\}$ can achieve a larger joint endowment without agent i compared to what they jointly receive at (c_S, E_S), i.e.,

$$E_{S\setminus\{i\}} = \alpha c^{S\setminus\{i\}} > \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S). \tag{2}$$

Hence, the endowment at problem $(c_{S\setminus\{i\}}, E_{S\setminus\{i\}})$ is larger than at problem $\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right)$ and by resource monotonicity, for each agent $j \in S\setminus\{i\}$,

$$F_j(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}) \geq F_j\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) \overset{(1)}{=} F_j(c_S, E_S),$$

and, by strict inequality (2), for some agent $l \in S \setminus \{i\}$,

$$F_l(c_{S\setminus\{i\}}, E_{S\setminus\{i\}}) > F_l\left(c_{S\setminus\{i\}}, \sum_{k\in S\setminus\{i\}} F_k(c_S, E_S)\right) \overset{(1)}{=} F_l(c_S, E_S).$$

Hence, for subcoalition $S' = S \setminus \{i\}$, there exists an agent $l \in S'$ such that

$$S' \succ_l^{((c,E),F)} S. \qquad \square$$

We next show that, given a non-singleton proportional generalized claims problem and a resource monotonic and consistent rule, there is a coalition of size 2 that is weakly preferred by all its agents to any other coalition of size larger than 2.

Proposition 2. *Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying resource monotonicity and consistency. If $S \subseteq N$, $|S| > 2$, then there is $S' \subsetneq S$, with $|S'| = 2$, such that for each agent $i \in S'$, $S' \succsim_i^{((c,E),F)} S$.*

Proof. Let N, (c, E), and F as specified in the lemma. Consider $S \subseteq N$, $|S| > 2$. We distinguish two cases:

Case 1. $S \subseteq N$ is such that for each $i \in S$, $F_i(c_S, E_S) = \alpha c_i$.

Then, by Lemma 1, for each $S' \subsetneq S$ with $|S'| \geq 2$ and each $j \in S'$, $F_j(c_{S'}, E_{S'}) = \alpha c_j$. In particular, this is true for any $S' \subsetneq S$ with $|S'| = 2$. Thus, for each agent $i \in S'$, $S' \succsim_i^{((c,E),F)} S$ and we are done.

Case 2. $S \subseteq N$ is such that for some $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$.

Then, by Lemma 2, there exists a subcoalition $S' \subsetneq S$ with $|S'| = |S| - 1$, such that for each agent $j \in S'$, $S' \succsim_j^{((c,E),F)} S$. If $|S'| = 2$, then we are done. Otherwise, starting with set S', successively apply Cases 1 and 2 to obtain a coalition $S' \subsetneq S$ with $|S'| = 2$ such that for each $j \in S'$, $S' \succsim_j^{((c,E),F)} S$. □

Now we introduce a property for coalition formation problems that plays an important role in the proof of Theorem 2. Let $\succsim$ be a coalition formation problem with agent set N and $S \subseteq N$. Then, a coalition $S' \subseteq S$ is a **top coalition of S** if for each $i \in S'$ and each $T \subseteq S$ with $i \in T$, we have $S' \succsim_i T$. A coalition formation problem satisfies the **top coalition property** if each non-empty set of agents $S \subseteq N$ has a top coalition. This property (Banerjee et al., 2001) is sufficient to guarantee stability.

We are now ready to prove Theorem 2.

Proof of Theorem 2. Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. By Gallo and Inarra (2018, Lemmas 1, 3, and Theorem 1), $\succsim^{((c,E),F)}$ satisfies the top coalition property; hence, a stable partition exists for the coalition formation problem with agent set N induced by $((c, E), F)$ (Gallo and Inarra, 2018, Theorem 2). If $|N| \leq 2$, we have nothing further to prove (the partition consisting of the grand coalition N is stable). Hence, assume that $|N| > 2$.

First, we iteratively construct a stable partition $\pi \in St(\succsim^{((c,E),F)})$ with coalition sizes of at most two.

Step 1. Let $N_1 := N$, $|N_1| > 2$. Then, there exists a top coalition $S_1' \subseteq N_1$ of N_1, i.e., for each $i \in S_1'$ and each $T \subseteq N_1$ with $i \in T$, we have $S_1' \succsim_i^{((c,E),F)} T$. If $|S_1'| = 2$, then set $S_1 = S_1'$. Otherwise, by Proposition 2, there is a coalition $S_1 \subsetneq S_1'$, such that $|S_1| = 2$ and for each $j \in S_1$, $S_1 \succsim_j^{((c,E),F)} S_1'$. Hence, for each $i \in S_1$ and each $T \subseteq N_1$ with $i \in T$, we have $S_1 \succsim_i^{((c,E),F)} S_1' \succsim_i^{((c,E),F)} T$ and S_1 is a top coalition as well. Hence, agents in S_1 can never be strictly better off in any other coalition $T \subseteq N_1 = N$. Thus, if S_1 is part of a stable partition, no agent in S_1 can block it.

Set $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{i=1}^{k-1} S_i\right)$ and $|N_k| > 2$. Then, there exists a top coalition $S'_k \subseteq N_k$ of N_k, i.e., for each $i \in S'_k$ and each $T \subseteq N_k$ with $i \in T$, we have $S'_k \succsim_i^{((c,E),F)} T$. If $|S'_k| = 2$, then set $S_k = S'_k$. Otherwise, by Proposition 2, there is a coalition $S_k \subsetneq S'_k$, such that $|S_k| = 2$ and for each $j \in S_k$, $S_k \succsim_j^{((c,E),F)} S'_k$. Hence, for each $i \in S_k$ and each $T \subseteq N_k$ with $i \in T$, we have $S_k \succsim_i^{((c,E),F)} S'_k \succsim_i^{((c,E),F)} T$ and S_k is a top coalition as well. Hence, agents in S_k can never be strictly better off in any other coalition $T \subseteq N_k$. In addition, it follows from previous steps that for each $j \in \{1, \ldots, k\}$, agents in S_j can never be strictly better off in any other coalition $T \subseteq N_j$. Thus, if $S_1, \ldots, S_k$ are part of a stable partition, no agent in $\cup_{i=1}^{k} S_i$ can block it.

Set $N_{k+1} := N \setminus \left(\cup_{i=1}^{k} S_i\right)$. If $|N_{k+1}| \leq 2$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

After at most $|N|-2$ steps, we have constructed a stable partition $\pi = \{S_1, \ldots, S_l\}$ of coalitions with size at most two. □

B Proof of Theorem 3

Recall that $N = \{1, \ldots n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. For each $S \subseteq N$, we denote the CEA parameter associated with (c_S, E_S) by λ_{E_S}, i.e., for each $i \in S$, $CEA_i(c_S, E_S) = \min\{c_i, \lambda_{E_S}\}$, where λ_{E_S} is chosen so that $\sum_{j \in S} \min\{c_j, \lambda_{E_S}\} = E_S$.

We first introduce some lemmas that will be used to prove Theorem 3. By the first lemma, we show that each agent $i \in N \setminus \{n\}$ weakly prefers to form a pairwise coalition with highest-claim agent n, instead of with any other agent.

Lemma 3. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c,E), CEA)$. Then, for each $i \in N \setminus \{n\}$ and each $j \in N \setminus \{i, n\}$,*

$$\{i, n\} \succsim_i^{((c,E),CEA)} \{i, j\}.$$

***Proof*.** Let N, (c,E), and $\succsim^{((c,E),CEA)}$ be as specified in the lemma. Let $i \in N \setminus \{n\}$ and $j \in N \setminus \{i,n\}$. We prove $\{i,n\} \succsim_i^{((c,E),CEA)} \{i,j\}$ by showing that

$$CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}}).$$

Given that $(c,E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{i,n\}} = \alpha(c_i + c_n)$ and $E_{\{j,n\}} = \alpha(c_j + c_n)$. If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$, then the above inequality holds automatically. Hence, assume that $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{E_{\{i,n\}}} < c_i$. Since $c_i \leq c_n$, this implies $CEA_n(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{E_{\{i,n\}}} \leq c_n$. Thus, $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \frac{\alpha(c_i+c_n)}{2}$. We distinguish two cases:

Case 1. $CEA_j(c_{\{i,j\}}, E_{\{i,j\}}) = c_j$.[13] Hence, $CEA_i(c_{\{i,j\}}, E_{\{i,j\}}) = \alpha(c_i+c_j) - c_j$. Then,

$$\begin{aligned} & CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) \geq CEA_i(c_{\{i,j\}}, E_{\{i,j\}}) \\ \Leftrightarrow \quad & \frac{\alpha(c_i + c_n)}{2} \geq \alpha(c_i + c_j) - c_j \\ \Leftrightarrow \quad & \alpha c_i + \alpha c_n \geq 2\alpha c_i + 2(\alpha - 1)c_j \\ \Leftrightarrow \quad & \underbrace{\alpha}_{>0} \underbrace{(c_n - c_i)}_{\geq 0} \geq 2\underbrace{(\alpha - 1)}_{<0} \underbrace{c_j}_{\geq 0}. \end{aligned}$$

Case 2. $CEA_j(c_{\{i,j\}}, E_{\{i,j\}}) = \lambda_{E_{\{i,j\}}} = \frac{\alpha(c_i+c_j)}{2}$. Thus,

$$CEA_i(c_{\{i,j\}}, E_{\{i,j\}}) = \frac{\alpha(c_i + c_j)}{2} \leq \frac{\alpha(c_i + c_n)}{2} = CEA_i(c_{\{i,n\}}, E_{\{i,n\}}). \qquad \square$$

We will now focus on agent n and discover with whom she wants to form a pairwise coalition.

Our next lemma captures the intuition that for low values of α that are associated to the situation in which all agents receive an equal split of the endowment, i.e., for each $i \in N \setminus \{n\}$, $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \lambda_{E_{\{i,n\}}} = CEA_n(c_{\{i,n\}}, E_{\{i,n\}})$, agent n weakly prefers coalition $\{n-1, n\}$ to any other coalition of size two.

Lemma 4. *Let $N \in \mathcal{N}$, $(c,E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c,E), CEA)$. If $\alpha \leq \frac{2c_1}{c_1+c_n} \equiv \beta_1$, then for each $i \in N \setminus \{n-1, n\}$,*

$$\{n-1, n\} \succsim_n^{((c,E),CEA)} \{i, n\}.$$

[13]Note that this case only happens when $j < i$.

Proof. Let N, (c,E), $\succsim^{((c,E),CEA)}$, and α be as specified in the lemma. Let $i \in N \setminus \{n-1,n\}$. We prove $\{n-1,n\} \succsim_n^{((c,E),CEA)} \{i,n\}$ by showing that

$$CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}}).$$

First note that

$$\alpha \leq \frac{2c_1}{c_1+c_n}$$
$$\Leftrightarrow \alpha(c_1+c_n) \leq 2c_1$$
$$\Leftrightarrow \frac{\alpha(c_1+c_n)}{2} \leq c_1.$$

Given that $(c,E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{1,n\}} = \alpha(c_1 + c_n)$. Therefore, $CEA_1(c_{\{1,n\}}, E_{\{1,n\}}) = \frac{\alpha(c_1+c_n)}{2} = CEA_n(c_{\{1,n\}}, E_{\{1,n\}})$.

Observe also that

$$\alpha \leq \frac{2c_1}{c_1+c_n}$$
$$\Leftrightarrow \alpha c_n \leq (2-\alpha)c_1.$$

Let $j \in N \setminus \{1,n\}$. Then, $c_1 \leq c_j$ and $(2-\alpha) > 0$,

$$\alpha c_n \leq (2-\alpha)c_j$$

$$\Leftrightarrow \frac{\alpha(c_j+c_n)}{2} \leq c_j.$$

Therefore, $CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = \frac{\alpha(c_j+c_n)}{2} = CEA_n(c_{\{j,n\}}, E_{\{j,n\}})$.

Finally, let $i \in N \setminus \{n-1,n\}$. Given than $c_i \leq c_{n-1} \leq c_n$,

$$CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2} \geq \frac{\alpha(c_i+c_n)}{2} = CEA_n(c_{\{i,n\}}, E_{\{i,n\}}). \quad \square$$

Similarly, the following lemma captures the intuition that for high values of α that are associated to the situation in which all agents receive their claim, i.e., for each $i \in N \setminus \{n\}$, $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$, agent n weakly prefers coalition $\{1,n\}$ to any other coalition of size two.

Lemma 5. *Let* $N \in \mathcal{N}$*,* $(c,E) \in \widetilde{\mathcal{P}}^N$*, and* $\succsim^{((c,E),CEA)}$ *be the coalition formation problem with agent set* N *induced by* $((c,E),CEA)$*. If* $\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \equiv \gamma_1$*, then for each* $i \in N \setminus \{1,n\}$*,*

$$\{1,n\} \succsim_n^{((c,E),CEA)} \{i,n\}.$$

Proof. Let N, (c,E), $\succsim^{((c,E),CEA)}$, and α be as specified in the lemma. Let $i \in N \setminus \{1,n\}$. We prove $\{1,n\} \succsim_n^{((c,E),CEA)} \{i,n\}$ by showing that

$$CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_n(c_{\{i,n\}}, E_{\{i,n\}}).$$

First note that

$$\begin{aligned} &\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \\ \Leftrightarrow\ &\alpha(c_{n-1}+c_n) \geq 2c_{n-1} \\ \Leftrightarrow\ &\frac{\alpha(c_{n-1}+c_n)}{2} \geq c_{n-1}. \end{aligned}$$

Given that $(c,E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{n-1,n\}} = \alpha(c_{n-1}+c_n)$. Therefore, $CEA_{n-1}(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = c_{n-1}$ and $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \alpha c_n - (1-\alpha)c_{n-1}$.

Observe also that

$$\begin{aligned} &\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \\ \Leftrightarrow\ &\alpha c_n \geq (2-\alpha)c_{n-1}. \end{aligned}$$

Let $j \in N \setminus \{n-1,n\}$. Then, since $c_j \leq c_{n-1}$ and $(2-\alpha) > 0$,

$$\begin{aligned} &\alpha c_n \geq (2-\alpha)c_j \\ \Leftrightarrow\ &\frac{\alpha(c_j+c_n)}{2} \geq c_j. \end{aligned}$$

Therefore, $CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = c_j$ and $CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) = \alpha c_n - (1-\alpha)c_j$.

Finally, let $i \in N \setminus \{1,n\}$. Given than $c_1 \leq c_i \leq c_n$ and $(1-\alpha) > 0$,

$$CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) = \alpha c_n - (1-\alpha)c_1 \geq \alpha c_n - (1-\alpha)c_i = CEA_n(c_{\{i,n\}}, E_{\{i,n\}}). \quad \square$$

Lemmas 4 and 5 now imply that agents 1 and $n-1$ are potential "stable partners" for agent n. When α is low ($\alpha \leq \frac{2c_1}{c_1+c_n} \equiv \beta_1$), then $\{n-1,n\}$ is the candidate for a stable pairwise coalition, and when α is large ($\alpha \geq \frac{2c_{n-1}}{c_{n-1}+c_n} \equiv \gamma_1$), then $\{1,n\}$ is the candidate for a stable pairwise coalition. Next, we show that for any other value of α in between β_1 and γ_1, agents 1 and $n-1$ are also potential "stable partners" for agent n. Thus, we next need to determine a threshold value δ_1 for parameter α when $\beta_1 < \alpha \leq \gamma_1$ to see when agent n's partner of choice is $n-1$ (for $\alpha \leq \delta_1$) and when it is 1 (for $\alpha \geq \delta_1$).

We next show that $\delta_1 \equiv \frac{2c_1}{2c_1 - c_{n-1} + c_n}$, the value specified in Step 1 of the CEA algorithm to trigger either Case (i) with coalition $\{n-1, n\}$ or Case (ii) with coalition $\{1, n\}$.

Lemma 6. *Let $N \in \mathcal{N}$, $(c, E) \in \widetilde{\mathcal{P}}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Assume that $\beta_1 \leq \alpha \leq \gamma_1$. Then, for $\delta_1 \equiv \frac{2c_1}{2c_1 - c_{n-1} + c_n}$ we have*

(i) *If $\alpha \leq \delta_1$, then for each $i \in N \setminus \{n-1, n\}, \{n-1, n\} \succsim_n^{((c,E),CEA)} \{i, n\}$.*

(ii) *If $\alpha \geq \delta_1$, then for each $i \in N \setminus \{1, n\}, \{1, n\} \succsim_n^{((c,E),CEA)} \{i, n\}$.*

Proof. Let N, (c, E), and $\succsim^{((c,E),CEA)}$ as specified in the lemma. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{1,n\}} = \alpha(c_1 + c_n)$ and $E_{\{n-1,n\}} = \alpha(c_{n-1} + c_n)$. Assume that $\beta_1 \leq \alpha \leq \gamma_1$, which implies

$$(2-\alpha)c_1 \leq \alpha c_n \leq (2-\alpha)c_{n-1}.$$

This, together with similar arguments to those in the proofs of Lemmas 4 and 5, implies

$$\frac{\alpha(c_1 + c_n)}{2} \geq c_1 \text{ and } \frac{\alpha(c_{n-1} + c_n)}{2} \leq c_{n-1}.$$

Therefore, $CEA_1(c_{\{1,n\}}, E_{\{1,n\}}) = c_1$ and $CEA_{n-1}(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2} \leq c_{n-1}$. Hence, $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) = \alpha(c_1 + c_n) - c_1$ and $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2}$. Next, we check when agent n prefers to form a coalition with either agent 1 or agent $n-1$.

$$CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) = \alpha(c_1 + c_n) - c_1 \gtreqless \frac{\alpha(c_{n-1} + c_n)}{2} = CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}})$$

$$\Leftrightarrow \quad \alpha(2c_1 + 2c_n - c_{n-1} - c_n) \gtreqless 2c_1$$

$$\Leftrightarrow \quad \alpha \gtreqless \frac{2c_1}{(2c_1 - c_{n-1} + c_n)} = \delta_1.$$

It follows that

(a.1) If $\alpha \leq \delta_1$, then $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_n(c_{\{1,n\}}, E_{\{1,n\}})$.

(a.2) If $\alpha > \delta_1$, then $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) > CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}})$.

To prove the general cases (i) and (ii) we first establish two facts that do not depend on α.

Fact 1. When considering two agents $j, k \in N \setminus \{n\}$ who, with agent n, receive an equal share under CEA, agent n prefers to form a coalition with the higher-claim agent.

Let $CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = \frac{\alpha(c_j+c_n)}{2}$, $CEA_k(c_{\{k,n\}}, E_{\{k,n\}}) = \frac{\alpha(c_k+c_n)}{2}$, and $c_j \leq c_k$. Then, $CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) = \frac{\alpha(c_j+c_n)}{2}$ and $CEA_n(c_{\{k,n\}}, E_{\{k,n\}}) = \frac{\alpha(c_k+c_n)}{2}$. Since $\alpha > 0$ and $c_j \leq c_k$,

$$CEA_n(c_{\{k,n\}}, E_{\{k,n\}}) \geq CEA_n(c_{\{j,n\}}, E_{\{j,n\}}).$$

Fact 2. When considering two agents $j, k \in N \setminus \{n\}$ who, with agent n, receive their claim under CEA, agent n prefers to form a coalition with the lower-claim agent.

Let $CEA_j(c_{\{j,n\}}, E_{\{j,n\}}) = c_j$, $CEA_k(c_{\{k,n\}}, E_{\{k,n\}}) = c_k$, and $c_j \leq c_k$. Then, $CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) = \alpha(c_j + c_n) - c_j = \alpha c_n - (1-\alpha)c_j$ and $CEA_n(c_{\{k,n\}}, E_{\{k,n\}}) = \alpha(c_k + c_n) - c_k = \alpha c_n - (1-\alpha)c_k$. Since $(1-\alpha) > 0$ and $c_j \leq c_k$,

$$CEA_n(c_{\{j,n\}}, E_{\{j,n\}}) \geq CEA_n(c_{\{k,n\}}, E_{\{k,n\}}).$$

We are now ready to prove (i) and (ii).

(i) Let $\alpha \leq \delta_1$ and $i \in N \setminus \{n-1, n\}$.

If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \frac{\alpha(c_i+c_n)}{2}$, then $CEA_{n-1}(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2}$ and by Fact 1, $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$.

If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$, then $CEA_1(c_{\{1,n\}}, E_{\{1,n\}}) = c_1$ and either by $i = 1$ or by Fact 2, $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$. Together with (a.1) this implies $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$. Hence,

$$\{n-1, n\} \succsim_n^{((c,E),CEA)} \{i, n\}.$$

(ii) Let $\alpha > \delta_1$ and $i \in N \setminus \{1, n\}$.

If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = c_i$, then $CEA_1(c_{\{1,n\}}, E_{\{1,n\}}) = c_1$ and by Fact 2, $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$.

If $CEA_i(c_{\{i,n\}}, E_{\{i,n\}}) = \frac{\alpha(c_i+c_n)}{2}$, then $CEA_{n-1}(c_{\{n-1,n\}}, E_{\{n-1,n\}}) = \frac{\alpha(c_{n-1}+c_n)}{2}$ and either by $i = n-1$ or by Fact 2, $CEA_n(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$. Together with (a.2) this implies $CEA_n(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_i(c_{\{i,n\}}, E_{\{i,n\}})$. Hence,

$$\{1, n\} \succsim_n^{((c,E),CEA)} \{i, n\}. \qquad \square$$

We now show that parameters β_1, γ_1, and δ_1 as defined in Lemmas 4, 5, and 6 satisfy

$$\beta_1 \leq \delta_1 \leq \gamma_1. \tag{3}$$

$$\begin{aligned} & \beta_1 \leq \delta_1 \\ \Leftrightarrow \quad & \frac{2c_1}{(c_1+c_n)} \leq \frac{2c_1}{(2c_1 - c_{n-1} + c_n)} \\ \Leftrightarrow \quad & \frac{2c_1}{(c_1+c_n)} \leq \frac{2c_1}{(c_1+c_n) - \underbrace{(c_{n-1} - c_1)}_{\geq 0}} \end{aligned}$$

and

$$\begin{aligned} & \delta_1 \leq \gamma_1 \\ & \frac{2c_1}{(2c_1 - c_{n-1} + c_n)} \leq \frac{2c_{n-1}}{(c_{n-1}+c_n)} \\ \Leftrightarrow \quad & 2c_1(c_{n-1}+c_n) \leq 2c_{n-1}(2c_1 - c_{n-1} + c_n) \\ \Leftrightarrow \quad & c_1c_{n-1} + c_1c_n \leq 2c_1c_{n-1} - c_{n-1}c_{n-1} + c_{n-1}c_n \\ \Leftrightarrow \quad & 0 \leq c_1c_{n-1} - c_1c_n - c_{n-1}c_{n-1} + c_{n-1}c_n \\ \Leftrightarrow \quad & 0 \leq c_{n-1}(c_n - c_{n-1}) - c_1(c_n - c_{n-1}) \\ \Leftrightarrow \quad & 0 \leq \underbrace{(c_{n-1} - c_1)}_{\geq 0}\underbrace{(c_n - c_{n-1})}_{\geq 0} \end{aligned}$$

We are now ready to prove Theorem 3.

Proof of Theorem 3. Let $N \in \mathcal{N}$ such that $|N| > 2$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEA)$. We show that the partition $\pi = \{S_1, \ldots, S_l\}$ obtained by the CEA algorithm is stable.

First, we show that the coalitions obtained by the CEA algorithm cannot be blocked by coalitions of size lower than or equal to two.

Step 1. Recall that $N_1 := N$, $|N_1| > 2$, and according to Cases (i) or (ii) in Step 1 of the CEA algorithm, $S_1 \in \{\{n-1, n\}, \{1, n\}\}$. We show that in either case, S_1 is for each of its members at least as desirable as any other coalition of size lower than or equal to two.

Case (i). $\alpha \leq \delta_1$. Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$, we have that for each $j \in \{n-1, n\}$,

$$\{n-1, n\} \succ_j^{((c,E),CEA)} \{j\}.$$

Next, let $j \in \{n-1, n\}$ and $i \in N_1 \setminus \{n-1, n\}$. We prove $\{n-1, n\} \succsim_j^{((c,E),CEA)} \{i, j\}$ by showing that

$$CEA_j(c_{\{n-1,n\}}, E_{\{n-1,n\}}) \geq CEA_j(c_{\{i,j\}}, E_{\{i,j\}}). \tag{4}$$

For $j = n-1$, inequality (4) follows from Lemma 3. For $j = n$, inequality (4) follows from Lemmas 4 and 6 (i).

Case (ii). $\alpha > \delta_1$. Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$, we have that for each $j \in \{1, n\}$,

$$\{1, n\} \succ_j^{((c,E),CEA)} \{j\}.$$

Next, let $j \in \{1, n\}$ and $i \in N_1 \setminus \{1, n\}$. We prove $\{1, n\} \succsim_j^{((c,E),CEA)} \{i, j\}$ by showing that

$$CEA_j(c_{\{1,n\}}, E_{\{1,n\}}) \geq CEA_j(c_{\{i,j\}}, E_{\{i,j\}}). \tag{5}$$

For $j = 1$, inequality (5) follows from Lemma 3. For $j = n$, inequality (4) follows from Lemmas 5 and 6 (ii).

In particular, there exists no coalition $T \subseteq N_1$ such that $|T| \leq 2$, and for each $i \in S_1 \cap T$, $T \succ_i^{((c,E),CEA)} S_1$, i.e., an agent from set S_1 cannot be part of a blocking coalition of size lower than or equal to two with an agent from set N_1.

Step *k* (*k* > 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Furthermore, agents in N_k are relabelled such that $N_k = \{1', \ldots, n'\}$, $c_{1'} \leq \ldots \leq c_{n'}$, and $\delta_k = \frac{2c_{1'}}{2c_{1'} - c_{(n-1)'} + c_{n'}}$. Then, according to Cases (i) or (ii) in Step k of the CEA algorithm, $S_k \in \{\{(n-1)', n'\}, \{1', n'\}\}$. Hence, by a similar reasoning than in Step 1 (with agents $1'$, $(n-1)'$, and n' in the roles of agents 1, $n-1$, and n respectively), it follows that there exists no coalition $T \subseteq N_k$ such that $|T| \leq 2$, and for each $i \in S_k \cap T$, $T \succ_i^{((c,E),CEA)} S_k$, i.e., an agent from set S_k cannot be part of a blocking coalition with an agent from set N_k. By the previous steps, an agent from set $\bigcup_{j\in\{1,\ldots,k-1\}} S_j$ cannot be part of a blocking coalition with an agent from set S_k. Hence, no agent from set S_k can be part of a blocking coalition of size lower than or equal to two.

After $l-1$ steps, we have shown that there is no blocking coalition $T \subsetneq N$ such that $|T| \leq 2$.

Second, assume, by contradiction, that π is not stable. Then, there exists a blocking coalition $T \subseteq N$ such that for each agent $i \in T$, $T \succ_i^{((c,E),CEA)} \pi(i)$. In particular, $|T| > 2$.

Case 1. For each $i \in T$, $CEA_i(c_T, E_T) = \alpha c_i$.

Then, by Lemma 1, there is a coalition $T' \subsetneq T$ with $|T'| = 2$ such that for each $j \in T'$,

$$T' \sim_j^{((c,E),CEA)} T \succ_j^{((c,E),CEA)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to two.

Case 2. For some $i \in T$, $CEA_i(c_T, E_T) \neq \alpha c_i$.

Then, by Proposition 2, there is a coalition $T' \subsetneq T$ with $|T'| = 2$ such that for each $j \in T'$,

$$T' \succsim_j^{((c,E),CEA)} T \succ_j^{((c,E),CEA)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to two.

We have proven that partition π obtained by the CEA algorithm is stable. □

C Proof of Theorem 4

Recall that $N = \{1, \ldots n\}$ and $c_1 \leq c_2 \leq \ldots \leq c_n$. For each $S \subseteq N$, we denote the CEL parameter associated with (c_S, E_S) by λ_{E_S}, i.e., for each $i \in S$, $CEL_i(c_S, E_S) = \max\{0, c_i - \lambda_{E_S}\}$, where λ_{E_S} is chosen so that $\sum_{j\in S} \max\{0, c_j - \lambda_{E_S}\} = E_S$.

Proof of Theorem 4. Let $N \in \mathcal{N}$ such that $|N| > 2$ and $(c, E) \in \widetilde{\mathcal{P}}^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$. We show that the partition $\pi = \{S_1, \ldots, S_l\}$ obtained by the CEL algorithm is stable.

First, we show that the coalitions obtained by the CEL algorithm cannot be blocked by coalitions of size lower than or equal to two.

Step 1. Recall that $N_1 := N$, $|N_1| > 2$, and $S_1 := \{1, 2\}$. We show that S_1 is for each of its members at least as desirable as any other coalition of size lower than or equal to two.

Note that since $(c, E) \in \widetilde{\mathcal{P}}^N$, we have that for each $i \in \{1, 2\}$,

$$\{1, 2\} \succsim_i^{((c,E),CEL)} \{i\}.$$

Next, let $i \in \{1, 2\}$ and $j \in N_1 \setminus \{1, 2\}$. We prove $\{1, 2\} \succsim_i^{((c,E),CEL)} \{i, j\}$ by showing that

$$CEL_i(c_{\{1,2\}}, E_{\{1,2\}}) \geq CEL_i(c_{\{i,j\}}, E_{\{i,j\}}), \text{ or, equivalently, } \lambda_{E_{\{1,2\}}} \leq \lambda_{E_{\{i,j\}}}. \tag{6}$$

Note that $i \leq j$ and hence $c_1 \leq c_i \leq c_j$.

If $CEL_i(c_{\{i,j\}}, E_{\{i,j\}}) = 0$, then inequality (6) holds automatically. Hence, assume that $CEL_i(c_{\{i,j\}}, E_{\{i,j\}}) = c_i - \lambda_{E_{\{i,j\}}} > 0$. Since $c_i \leq c_j$, we have $CEL_j(c_{\{i,j\}}, E_{\{i,j\}}) = c_j - \lambda_{E_{\{i,j\}}} > 0$ and $E_{\{i,j\}} = c_i + c_j - 2\lambda_{E_{\{i,j\}}}$. Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we also have $E_{\{i,j\}} = \alpha(c_i + c_j)$ and corresponding loss $(1 - \alpha)(c_i + c_j)$. Thus,

$$\lambda_{E_{\{i,j\}}} = \frac{(1 - \alpha)(c_i + c_j)}{2} < c_i \leq c_j. \tag{7}$$

Next, let $\{i, k\} = \{1, 2\}$. Note that $c_k \leq c_j$, together with inequality (7), implies

$$\frac{(1 - \alpha)(c_1 + c_2)}{2} = \frac{(1 - \alpha)(c_i + c_k)}{2} \leq \lambda_{E_{\{i,j\}}} < c_i.$$

Given that $(c, E) \in \widetilde{\mathcal{P}}^N$, we have $E_{\{1,2\}} = \alpha(c_1 + c_2)$ with corresponding loss $(1 - \alpha)(c_1 + c_2)$.

If $\frac{(1-\alpha)(c_1+c_2)}{2} \leq c_1$, then $\lambda_{E_{\{1,2\}}} = \frac{(1-\alpha)(c_1+c_2)}{2} \leq \lambda_{E_{\{i,j\}}}$ and inequality (6) holds.

Hence, assume that $\frac{(1-\alpha)(c_1+c_2)}{2} > c_1$. Then, $CEL_1(c_{\{1,2\}}, E_{\{1,2\}}) = 0$ and $CEL_2(c_{\{1,2\}}, E_{\{1,2\}}) = c_2 - \lambda_{E_{\{1,2\}}} > 0$. Hence, $i = 2$ and $E_{\{1,2\}} = c_2 - \lambda_{E_{\{1,2\}}}$. Thus, $\lambda_{E_{\{1,2\}}} = (1 - \alpha)c_2 - \alpha c_1$. Then, inequality (7) together with $c_2 \leq c_j$, implies

$$\lambda_{E_{\{2,j\}}} = \frac{(1 - \alpha)(c_2 + c_j)}{2} \geq \frac{(1 - \alpha)2c_2}{2} = (1 - \alpha)c_2 > (1 - \alpha)c_2 - \alpha c_1 = \lambda_{E_{\{1,2\}}}$$

and inequality (6) holds.

In particular, there exists no coalition $T \subseteq N_1$ such that $|T| \leq 2$, and for each $i \in S_1 \cap T$, $T \succ_i^{((c,E),CEL)} S_1$, i.e., an agent from set S_1 cannot be part of a blocking coalition of size lower than or equal to two with an agent from set N_1.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > 2$. Set $S_k := \{2k-1, 2k\}$. Note that agents $2k-1$ and $2k$ are lowest-claims agents in N_k. Hence, by a similar reasoning than in Step 1 (with agents $2k-1$ and $2k$ in the roles of agents 1 and 2 respectively), it follows that there exists no coalition $T \subseteq N_k$ such that $|T| \leq 2$, and for each $i \in S_k \cap T$, $T \succ_i^{((c,E),CEL)} S_k$, i.e., an agent from set S_k cannot be part of a blocking coalition with an agent from set N_k. By the previous steps, an agent from set $\bigcup_{j \in \{1,\ldots,k-1\}} S_j$ cannot be part of a blocking coalition with an agent from set S_k. Hence, no agent from set S_k can be part of a blocking coalition of size lower than or equal to two.

After $l-1$ steps, we have shown that there is no blocking coalition $T \subsetneq N$ such that $|T| \leq 2$.

Second, assume, by contradiction, that π is not stable. Then, there exists a blocking coalition $T \subseteq N$ such that for each agent $i \in T$, $T \succ_i^{((c,E),CEL)} \pi(i)$. In particular, $|T| > 2$.

Case 1. For each $i \in T$, $CEL_i(c_T, E_T) = \alpha c_i$.

Then, by Lemma 1, there is a coalition $T' \subsetneq T$ with $|T'| = 2$ such that for each $j \in T'$,

$$T' \sim_j^{((c,E),CEL)} T \succ_j^{((c,E),CEL)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to two.

Case 2. For some $i \in T$, $CEL_i(c_T, E_T) \neq \alpha c_i$.

Then, by Proposition 2, there is a coalition $T' \subsetneq T$ with $|T'| = 2$ such that for each $j \in T'$,

$$T' \succsim_j^{((c,E),CEL)} T \succ_j^{((c,E),CEL)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to two.

We have proven that partition π obtained by the CEL algorithm is stable. □

D Proof of Theorem 6

We first introduce some lemmas that will be used to prove Theorem 6. We first show that, given a θ-minimal proportional generalized claims problem and a consistent rule, if all agents receive proportional payoffs in a coalition, then all agents receive proportional payoffs in any subcoalition (except subcoalitions of size lower than θ) as well.

Lemma 7. *Let $N \in \mathcal{N}$ and $(c,E) \in \widetilde{\mathcal{P}}_{\theta}^{N}$. Consider $F \in \mathcal{F}$ satisfying consistency. If $S \subseteq N$, $|S| > \theta$, is such that for each $i \in S$, $F_i(c_S, E_S) = \alpha c_i$, then for each $S' \subsetneq S$ with $|S'| \geq \theta$ and each $j \in S'$, $F_j(c_{S'}, E_{S'}) = \alpha c_j$.*

The proof of Lemma 7 is identical to the proof of Lemma 1 (Appendix A) by simply replacing $\theta = 2$ with $\theta \in \mathbb{N}$.

We next show that, given a θ-minimal proportional generalized claims problem and a resource monotonic and consistent rule, if some agent in a coalition of size larger than θ does not receive a proportional payoff, then there exists a subcoalition that is strictly preferred by at least one member of the subcoalition.

Lemma 8. *Let $N \in \mathcal{N}$ and $(c,E) \in \widetilde{\mathcal{P}}_{\theta}^{N}$. Consider $F \in \mathcal{F}$ satisfying resource monotonicity and consistency. If $S \subseteq N$, $|S| > \theta$, is such that for some $i \in S$, $F_i(c_S, E_S) \neq \alpha c_i$, then there is $S' \subsetneq S$, with $|S'| = |S| - 1$, such that for some agent $l \in S'$, $S' \succ_l^{((c,E),F)} S$.*

The proof of Lemma 8 is identical to the proof of Lemma 2 (Appendix A) by simply replacing $\theta = 2$ with $\theta \in \mathbb{N}$.

We next show that, given a θ-minimal proportional generalized claims problem and a resource monotonic and consistent rule, there is a coalition of size θ that is weakly preferred by all its agents to any other coalition of size larger than θ.

Proposition 3. *Let $N \in \mathcal{N}$ and $(c,E) \in \widetilde{\mathcal{P}}_{\theta}^{N}$. Consider $F \in \mathcal{F}$ satisfying resource monotonicity and consistency. If $S \subseteq N$, $|S| > \theta$, then there is $S' \subsetneq S$, with $|S'| = \theta$, such that for each agent $j \in S'$, $S' \succsim_j^{((c,E),F)} S$.*

The proof of Proposition 3 is identical to the proof of Proposition 2 (Appendix A) by simply replacing $\theta = 2$ with $\theta \in \mathbb{N}$.

We are now ready to prove Theorem 6.

Proof of Theorem 6. Let $N \in \mathcal{N}$ and $(c, E) \in \widetilde{\mathcal{P}}^N_\theta$. Consider $F \in \mathcal{F}$ satisfying continuity, resource monotonicity, and consistency. By Gallo and Inarra (2018, Lemmas 1, 3, and Theorem 1), $\succsim^{((c,E),F)}$ satisfies the top coalition property (as defined in Appendix A); hence, a stable partition exists for the coalition formation problem with agent set N induced by $((c, E), F)$ (Gallo and Inarra, 2018, Theorem 2). If $|N| = \theta$, then the only stable partition consists of the grand coalition N. If $|N| < \theta$, then any partition of agents is stable, in particular the one consisting of the grand coalition N. Hence, assume that $|N| > \theta$.

First, we iteratively construct a stable partition $\pi \in St(\succsim^{((c,E),F)})$ with coalition sizes of at most θ.

Step 1. Let $N_1 := N$, $|N_1| > \theta$. Then, there exists a top coalition $S'_1 \subseteq N_1$ of N_1, i.e., for each $i \in S'_1$ and each $T \subseteq N_1$ with $i \in T$, we have $S'_1 \succsim_i^{((c,E),F)} T$. If $|S'_1| = \theta$, then set $S_1 = S'_1$. Otherwise, by Proposition 3, there is a coalition $S_1 \subsetneq S'_1$, such that $|S_1| = \theta$ and for each $j \in S_1$, $S_1 \succsim_j^{((c,E),F)} S'_1$. Hence, for each $i \in S_1$ and each $T \subseteq N_1$ with $i \in T$, we have $S_1 \succsim_i^{((c,E),F)} S'_1 \succsim_i^{((c,E),F)} T$ and S_1 is a top coalition as well. Hence, agents in S_1 can never be strictly better off in any other coalition $T \subseteq N_1 = N$. Thus, if S_1 is part of a stable partition, no agent in S_1 can block it.

Set $N_2 := N \setminus S_1$. If $|N_2| \leq 2$, then set $S_2 := N_2$, define $\pi := \{S_1, S_2\}$, and stop. Otherwise, go to Step 2.

Step *k* (*k* > 1). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{i=1}^{k-1} S_i\right)$ and $|N_k| > \theta$. Then, there exists a top coalition $S'_k \subseteq N_k$ of N_k, i.e., for each $i \in S'_k$ and each $T \subseteq N_k$ with $i \in T$, we have $S'_k \succsim_i^{((c,E),F)} T$. If $|S'_k| = \theta$, then set $S_k = S'_k$. Otherwise, by Proposition 3, there is a coalition $S_k \subsetneq S'_k$, such that $|S_k| = \theta$ and for each $j \in S_k$, $S_k \succsim_j^{((c,E),F)} S'_k$. Hence, for each $i \in S_k$ and each $T \subseteq N_k$ with $i \in T$, we have $S_k \succsim_i^{((c,E),F)} S'_k \succsim_i^{((c,E),F)} T$ and S_k is a top coalition as well. Hence, agents in S_k can never be strictly better off in any other coalition $T \subseteq N_k$. In addition, it follows from previous steps that for each $j \in \{1, \ldots, k\}$, agents in S_j can never be strictly better off in any other coalition $T \subseteq N_j$. Thus, if $S_1, \ldots, S_k$ are part of a stable partition, no agent in $\cup_{i=1}^k S_i$ can block it.

Set $N_{k+1} := N \setminus \left(\cup_{i=1}^k S_i\right)$. If $|N_{k+1}| \leq \theta$, then set $S_{k+1} := N_{k+1}$, define $\pi := \{S_1, \ldots, S_{k+1}\}$, and stop. Otherwise, go to Step $k+1$.

After at most $|N| - \theta$ steps, we have constructed a stable partition $\pi = \{S_1, \ldots, S_l\}$ of coalitions with size at most θ. □

E Cases and parameters in the θ-CEA algorithm

Recall that $N = \{1, \dots n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. Throughout this appendix, we consider sets of cardinalities $2, \dots, \theta$ and proportional generalized claims problems, i.e., problems in $\mathcal{P}^N$ (without imposing a minimal coalition size of θ).

Note that in the θ-CEA set algorithm, starting with agent n, we construct a set of cardinality θ by adding agents one by one. More precisely, in each Step k, a set of agents S'_{k-1} considers to add either the lowest-label or the highest-label agent of the remaining set of agents. The results in this appendix show that the agent who is added, according to θ-CEA set algorithm Case (i) or (ii), is weakly preferred by the agents in S'_{k-1} to any other agent who could have been added.

Let $S \subsetneq N$, $S \neq \emptyset$, and consider agents $i, j \in N \setminus S$ such that $i < j$, i.e., $c_i \leq c_j$. We first show that if both agents receive their claim in coalitions $S \cup \{i\}$ and $S \cup \{j\}$ respectively, then all agents in S weakly prefer to form a coalition with agent i.

Lemma 9. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Let $S \subsetneq N$, $S \neq \emptyset$, and $j \in N \setminus S$ such that*

$$CEA_j(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) = c_j.$$

Then, for each $i \in N \setminus (S \cup \{j\})$ such that $i < j$,

$$CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = c_i,$$

and for each $k \in S$,

$$S \cup \{i\} \succsim_k^{((c,E),CEA)} S \cup \{j\}. \tag{8}$$

Proof. Let N, (c, E), $\succsim^{((c,E),CEA)}$, $S \subsetneq N$ and $i, j \in N \setminus S$ as specified in the lemma. Hence, $c_i \leq c_j$. Given that $(c, E) \in \mathcal{P}^N$, we have $E_{(S\cup\{i\})} = \alpha c^{(S\cup\{i\})}$ and $E_{(S\cup\{j\})} = \alpha c^{(S\cup\{j\})}$.

Starting from problem $(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) \in \mathcal{C}^{(S\cup\{j\})}$, we replace agent j (with claim c_j) with agent i (with claim c_i) but without changing the CEA payoffs of agents in S in the resulting problem in $\mathcal{C}^{(S\cup\{i\})}$. That is, we consider the auxiliary problem $(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})}) \in \mathcal{C}^{(S\cup\{i\})}$ such that for each $k \in S$, $CEA_k(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})}) =$

$CEA_k(c_{(S\cup\{j\})}, E_{(S\cup\{j\})})$, and $CEA_i(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})}) = c_i$. Hence,

$$\bar{E}_{(S\cup\{i\})} = \sum_{k\in S} CEA_k(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) + c_i = E_{(S\cup\{j\})} - c_j + c_i.$$

Note that $(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})})$ is such that $\bar{E}_{(S\cup\{i\})} \neq \alpha c^{(S\cup\{i\})}$. Furthermore, by construction, $\lambda_{\bar{E}_{(S\cup\{i\})}} = \lambda_{E_{(S\cup\{j\})}}$.

Finally, consider problem $(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) \in \mathcal{C}^{(S\cup\{i\})}$. We have that $E_{(S\cup\{i\})} = E_{(S\cup\{j\})} - \alpha c_j + \alpha c_i$. Therefore, since $c_j - c_i \geq 0$ and $\alpha \in (0,1)$,

$$E_{(S\cup\{i\})} = E_{(S\cup\{j\})} - \alpha(c_j - c_i) \geq E_{(S\cup\{j\})} - (c_j - c_i) = \bar{E}_{(S\cup\{i\})}.$$

Then, by definition of the CEA rule, $\lambda_{E_{(S\cup\{i\})}} \geq \lambda_{\bar{E}_{(S\cup\{i\})}} = \lambda_{E_{(S\cup\{j\})}}$. Thus, we have that

$$CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = c_i = CEA_i(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})}),$$

and for each $k \in S$,

$$CEA_k(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) \geq CEA_k(c_{(S\cup\{i\})}, \bar{E}_{(S\cup\{i\})}) = CEA_k(c_{S\cup\{j\}}, E_{(S\cup\{j\})}),$$

or equivalently,

$$S \cup \{i\} \succsim_k^{((c,E),CEA)} S \cup \{j\}. \qquad \square$$

Lemma 9 implies that, given any coalition $S \subsetneq N$, among all agents receiving their claim when added, coalition S weakly prefers to add a lowest-claim agent. In particular, if there are several such agents, we add the agent with the lowest label.

Let $S \subsetneq N$, $S \neq \emptyset$, and consider agents $i, j \in N \setminus S$ such that $i < j$, i.e., $c_i \leq c_j$. We second show that if both agents do not receive their claim in coalitions $S \cup \{i\}$ and $S \cup \{j\}$ respectively, then all agents in S weakly prefer to form a coalition with agent j.

Lemma 10. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Let $S \subsetneq N$, $S \neq \emptyset$, and $i \in N \setminus S$ such that*

$$CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = \lambda_{E_{(S\cup\{i\})}} < c_i.$$

Then, for each $j \in N \setminus (S \cup \{i\})$ such that $i < j$,

$$CEA_j(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) = \lambda_{E_{(S\cup\{j\})}} < c_j,$$

and for each $k \in S$,

$$S \cup \{j\} \succsim_k^{((c,E),CEA)} S \cup \{i\}. \tag{9}$$

Proof. Let N, (c, E), $\succsim^{((c,E),CEA)}$, $S \subsetneq N$ and $i, j \in N \setminus S$ as specified in the lemma. Hence, $c_i \leq c_j$. Given that $(c, E) \in \mathcal{P}^N$, we have $E_{(S\cup\{i\})} = \alpha c^{(S\cup\{i\})}$ and $E_{(S\cup\{j\})} = \alpha c^{(S\cup\{j\})}$. Suppose that $CEA_j(c_{(S\cup\{i\})}, E_{(S\cup\{j\})}) = c_j$. Then, by Lemma 9, $CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = c_i$, contradicting our assumption that $CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = \lambda_{E_{(S\cup\{i\})}} < c_i$. Hence,

$$CEA_j(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) = \lambda_{E_{(S\cup\{j\})}} < c_j.$$

Starting from problem $(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) \in \mathcal{C}^(S \cup \{i\})$, we replace agent i (with claim c_i) with agent j (with claim c_j) without changing the endowment. That is, we consider the auxiliary problem $(c_{(S\cup\{j\})}, E_{(S\cup\{i\})}) \in \mathcal{C}^{(S\cup\{j\})}$. Note that $(c_{(S\cup\{j\})}, E_{(S\cup\{i\})})$ is such that $E_{(S\cup\{i\})} \neq \alpha c^{(S\cup\{j\})}$. Since $CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = \lambda_{E_{(S\cup\{i\})}} < c_i$ and $c_i \leq c_j$,

$$CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = \lambda_{E_{(S\cup\{i\})}} = \lambda_{E_{(S\cup\{j\})}} = CEA_j(c_{(S\cup\{j\})}, E_{(S\cup\{i\})})$$

and for each agent $k \in S$,

$$CEA_k(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = CEA_k(c_{(S\cup\{j\})}, E_{(S\cup\{i\})}).$$

Finally, consider problem $(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) \in \mathcal{P}^{(S\cup\{j\})}$. Since $E_{(S\cup\{j\})} = \alpha c^{(S\cup\{j\})} \geq \alpha c^{(S\cup\{i\})} = E_{(S\cup\{i\})}$, by the definition of the CEA rule, we have that $\lambda_{E_{(S\cup\{j\})}} \geq \lambda_{E_{(S\cup\{i\})}}$. Thus, for each $k \in S$,

$$CEA_k(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) \geq CEA_k(c_{S\cup\{i\}}, E_{(S\cup\{j\})}),$$

or equivalently,

$$S \cup \{j\} \succsim_k^{((c,E),CEA)} S \cup \{i\}.$$ □

Lemma 10 implies that, given any coalition $S \subsetneq N$, among all the agents who do not receive their claim when added, coalition S weakly prefers to add a highest-claim agent. In particular, if there are several such agents, we add the agent with the highest label.

To summarize, by Lemmas 9 and 10, a coalition $S \subsetneq N$ weakly prefers to either add the lowest-label agent or the highest-label agent to their coalition. Finally, if applicable, we determine which of these two agents the coalition weakly prefers to add.

Let $S \subsetneq N$, $S \neq \emptyset$, and consider agents $i, j \in N \setminus S$ such that $i < j$, i.e., $c_i \leq c_j$. We next show which of the agents i or j all agents in S weakly prefer to form a coalition with if agent i receives her claim in coalition $S \cup \{i\}$ and agent j does not receive her claim in coalition $S \cup \{j\}$ respectively.

Lemma 11. *Let $N \in \mathcal{N}$, $(c, E) \in \mathcal{P}^N$, and $\succsim^{((c,E),CEA)}$ be the coalition formation problem with agent set N induced by $((c, E), CEA)$. Let $S \subsetneq N$, $S \neq \emptyset$, and $i, j \in N \setminus S$ be such that $i < j$ and*

$$CEA_i(c_{(S\cup\{i\})}, E_{(S\cup\{i\})}) = c_i,$$

and

$$CEA_j(c_{(S\cup\{j\})}, E_{(S\cup\{j\})}) = \lambda_{E_{(S\cup\{j\})}} < c_j.$$

(i) *If $\lambda_{E_{(S\cup\{j\})}} \leq (1-\alpha)c_i + \alpha c_j$, then for each $k \in S$, $S \cup \{j\} \succsim_k^{((c,E),CEA)} S \cup \{i\}$.*

(ii) *If $\lambda_{E_{(S\cup\{j\})}} > (1-\alpha)c_i + \alpha c_j$, then for each $k \in S$, $S \cup \{i\} \succsim_k^{((c,E),CEA)} S \cup \{j\}$.*

Proof. Let N, (c, E), $\succsim^{((c,E),CEA)}$, $S \subsetneq N$ and $i, j \in N \setminus S$ as specified in the lemma. Given that $(c, E) \in \mathcal{P}^N$, we have $E_{(S\cup\{i\})} = \alpha c^{(S\cup\{i\})}$ and $E_{(S\cup\{j\})} = \alpha c^{(S\cup\{j\})}$. Since agent i in coalition $S\cup\{i\}$ receives c_i but only contributes αc_i, she receives a subsidy of $(1-\alpha)c_i > 0$. Agent j in coalition $S \cup \{j\}$ receives $\lambda_{E_{(S\cup\{j\})}} < c_j$ and contributes αc_j. Thus, if $\lambda_{E_{(S\cup\{j\})}} \leq \alpha c_j$,[14] agent j weakly transfers $\alpha c_j - \lambda_{E_{(S\cup\{j\})}} \geq 0$, which implies that each agent in S weakly prefers $S \cup \{j\}$ to $S \cup \{i\}$. Otherwise, if $\lambda_{E_{(S\cup\{j\})}} > \alpha c_j$, agent j receives a subsidy of $\lambda_{E_{(S\cup\{j\})}} - \alpha c_j > 0$. Hence, we distinguish two cases:

(i) Agent j weakly transfers $\alpha c_j - \lambda_{E_{(S\cup\{j\})}} \geq 0$ or receives a weakly smaller subsidy than agent i.

[14]Hence, $\lambda_{E_{(S\cup\{j\})}} < (1-\alpha)c_i + \alpha c_j$ and the premise of Case (i) is satisfied.

Then, $\alpha c_j - \lambda_{E_{(S\cup\{j\})}} \leq (1-\alpha)c_i$, which is equivalent to $\lambda_{E_{(S\cup\{j\})}} \leq (1-\alpha)c_i + \alpha c_j$. In this case, for each $k \in S$, $S \cup \{j\} \succsim_k^{((c,E),CEA)} S \cup \{i\}$.

(ii) Agent i receives a smaller subsidy than agent j.

Then, $(1-\alpha)c_i < \alpha c_j - \lambda_{E_{(S\cup\{j\})}}$, which is equivalent to $\lambda_{E_{(S\cup\{j\})}} > (1-\alpha)c_i + \alpha c_j$. In this case, for each $k \in S$, $S \cup \{i\} \succsim_k^{((c,E),CEA)} S \cup \{j\}$. □

Let $S \subsetneq N$. Define $L \equiv \{i' \in N \setminus S : CEA_{i'}(c_{(S\cup\{i'\})}, E_{(S\cup\{i'\})}) = c_{i'}\}$ and $R \equiv \{j' \in N \setminus S : CEA_{j'}(c_{(S\cup\{j'\})}, E_{(S\cup\{j'\})}) = \lambda_{E_{(S\cup\{j'\})}} \neq c_{j'}\}$. Consider $i \equiv \min L$ and $j \equiv \max R$. Lemmas 9 and 10 together imply that agents i and j are potential "stable partners" for coalition S. If $R = \emptyset$, then Lemma 9 implies that agent i is added to coalition S. Similarly, if $L = \emptyset$, then Lemma 10 implies that agent j is added to coalition S. Finally, let $L \neq \emptyset$ and $R \neq \emptyset$. Then, according to Lemma 11, depending on Cases (i) or (ii), together with Lemmas 9 and 10, either agent i or agent j is added to coalition S.

F Proof of Theorem 7

Recall that $N = \{1, \ldots n\}$ and $c_1 \leq c_2 \leq \cdots \leq c_n$. For each $S \subseteq N$, we denote the CEA parameter associated with (c_S, E_S) by λ_{E_S}, i.e., for each $i \in S$, $CEA_i(c_S, E_S) = \min\{c_i, \lambda_{E_S}\}$, where λ_{E_S} is chosen so that $\sum_{j\in S} \min\{c_j, \lambda_{E_S}\} = E_S$.

We introduce some notation before the proof. For $S \subseteq N$ and $CEA(c_S, E_S)$, we denote the set of agents receiving an over-proportional or proportional payoff in coalition S, also called the *set of* ***r****ecipients*, by $R_S \equiv \{i \in S : CEA_i(c_S, E_S) \geq \alpha c_i\}$. Moreover, for each agent $i \in R_S$, we denote the subsidy she receives in coalition S as $r_i^S = CEA_i(c_S, E_S) - \alpha c_i \geq 0$. Similarly, we denote the set of agents receiving an under-proportional payoff in coalition S, also called the *set of* ***t****ransfer agents*, by $T_S \equiv \{i \in S : CEA_i(c_S, E_S) < \alpha c_i\}$. Furthermore, for each agent $i \in T_S$, the transfer she makes to coalition S is $t_i^S = \alpha c_i - CEA_i(c_S, E_S) > 0$.

Note that for any coalition $S \subseteq N$, $S \neq \emptyset$, by the definition of the CEA rule, $R_S \neq \emptyset$ and $[T_S = \emptyset$ if and only if for all $i, j \in S$, $c_i = c_j]$.

Proof of Theorem 7. Let $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEA)$. We show that the partition $\pi = \{S_1, \ldots, S_l\}$ obtained by the θ-CEA algorithm is stable.

First, we show that the coalitions obtained by the θ-CEA algorithm cannot be blocked by coalitions of size lower than or equal to θ.

Step 1. Recall that $N_1 := N$, $|N_1| > \theta$. We show that S_1 cannot be blocked by a coalition of size lower than or equal to θ. Note that since $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$, we have that for each $S \subsetneq N$ with $|S| < \theta$ and each $k \in S_1 \cap S$,

$$S_1 \succsim_k^{((c,E),CEA)} S.$$

Next, assume, by contradiction, that S_1 can be blocked by a coalition of size θ, i.e., there exists a set $S \subsetneq N$ with $|S| = \theta$ such that for each $k \in S \cap S_1$,

$$S \succ_k^{((c,E),CEA)} S_1.$$

Note that no agent receiving her claim in S_1 can be in S. Hence,

$$\lambda_{E_S} > \lambda_{E_{S_1}}.$$

Let i^* be the agent with the highest label in S_1 receiving c_i, i.e., $i^* \equiv \max\{i \in S_1 : CEA_i(c_{S_1}, E_{S_1}) = c_i\}$ and j^* be the agent with the lowest label in S_1 receiving $\lambda_{E_{S_1}} \geq \alpha c_{j^*}$, $\lambda_{E_{S_1}} \neq c_{j^*}$, i.e., $j^* \equiv \min\{j \in S_1 : CEA_j(c_{S_1}, E_{S_1}) = \lambda_{E_{S_1}} \geq \alpha c_j$ and $\lambda_{E_{S_1}} \neq c_j\}$. Since $R_{S_1} \neq \emptyset$, $\{i^*, j^*\} \neq \emptyset$, i.e., at least one of these agents (or both) exist. Note that if both agents i^* and j^* exist, then there is no agent in S_1 whose label is in between that of these two agents, i.e., there exists no $l \in S_1$ such that $i^* < l < j^*$.

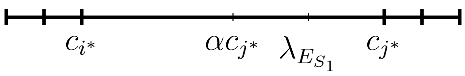

Figure 1: Both agents, i^* and j^*, exist.

We divide the rest of the proof into four parts.

Part 1. *We show that all agents in $S \setminus S_1$ are recipients. That is, for each $l \in S \setminus S_1$, $l \in R_S$. We distinguish two cases.*

Case 1. Agent j^* exists, i.e., j^* is the agent with the lowest label in S_1 receiving $\lambda_{E_{S_1}} \geq \alpha c_{j^*}$ ($\lambda_{E_{S_1}} \neq c_{j^*}$).

Let $l \in S \setminus S_1$. Note first that $l < j^*$ because, by construction of coalition S_1 by the algorithm, each agent with a higher label than j^* is necessarily in S_1.[15] Therefore, $c_l \leq c_{j^*}$. Recall that $\lambda_{E_S} > \lambda_{E_{S_1}} \geq \alpha c_{j^*}$. Hence,

$$\lambda_{E_S} > \lambda_{E_{S_1}} \geq \alpha c_{j^*} \geq \alpha c_l,$$

which implies $\lambda_{E_S} > \alpha c_l$. Thus, $l \in R_S$.

Case 2. Agent j^* does not exist and i^* is the agent with the highest label in S_1 receiving c_{i^*}.

Thus, $T_{S_1} \neq \emptyset$. Denote by j^t the transfer agent in S_1 with the lowest label, i.e., $j^t \equiv \min\{j \in T_{S_1} : \lambda_{E_{S_1}} < \alpha c_j\}$. Since agent j^* does not exist, agent i^* is now the agent with the next lowest label in S_1, i.e., there exists no $l \in S_1$ such that $i^* < l < j^t$.

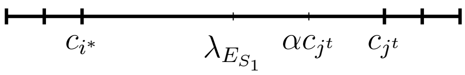

Figure 2: Agent j^* does not exist and agent j^t is a transfer agent with the lowest label.

First, by construction of coalition S_1 by the algorithm, in particular, by Step $\theta - 1$ (ii) of the θ-CEA set algorithm, agent i^* has been chosen against agent $j^t - 1$. Therefore,

$$(1-\alpha)c_{i^*} < \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}} - \alpha c_{j^t-1},$$

which means that if agent i^* is added to coalition $(S_1 \setminus \{i^*\})$, even if receiving her claim, she would receive a lower subsidy than if agent $j^t - 1$ is added to coalition $(S_1 \setminus \{i^*\})$. This, together with the definition of the CEA rule, implies

$$\lambda_{E_{S_1}} \geq \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}}.$$

Furthermore, since $(1-\alpha)c_{i^*} > 0$, we have $\lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}} - \alpha c_{j^t-1} > 0$. Thus,

$$\lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}} > \alpha c_{j^t-1}.$$

[15]Note that coalition S_1 constructed by the algorithm is such that each agent with a higher label than j^* is either a transfer agent or a recipient with a lower subsidy, which means that they are chosen before j^* in the algorithm.

Let $l \in S \setminus S_1$. Note first that $l < j^t - 1$ because, by construction of coalition S_1 by the algorithm, each agent with a higher label than j^t is necessarily in S_1. Therefore, $\alpha c_l \leq \alpha c_{j^t-1}$. Recall that $\lambda_{E_S} > \lambda_{E_{S_1}} \geq \lambda_{E_{(S_1 \setminus \{i^*\}) \cup \{j^t-1\}}} > \alpha c_{j^t-1}$. Hence,

$$\lambda_{E_S} > \lambda_{E_{S_1}} \geq \lambda_{E_{(S_1 \setminus \{i^*\}) \cup \{j^t-1\}}} > \alpha c_{j^t-1} \geq \alpha c_l,$$

which implies $\lambda_{E_S} > \alpha c_l$. Thus, $l \in R_S$.

As a result of Part 1, all new agents in coalition S are recipients, i.e., $S \setminus S_1 \subseteq R_S$.

Part 2. *We show that each agent in $S \cap S_1$ who is a recipient in S_1 is also a recipient in S. Furthermore, the subsidy a recipient in $S \cap S_1$ receives in S is higher than the subsidy she receives in S_1. That is, for each $l \in S \cap R_{S_1}$, we have $l \in R_S$ and $r_l^S > r_l^{S_1}$.*

Let $l \in S \cap R_{S_1}$. Since $l \in R_{S_1}$, we have $\lambda_{E_{S_1}} \geq \alpha c_l$. Hence, $\lambda_{E_S} > \lambda_{E_{S_1}}$ implies $\lambda_{E_S} > \alpha c_l \geq 0$ and $l \in R_S$. Recall that since $l \in S \cap S_1$, $CEA_l(c_{S_1}, E_{S_1}) = \lambda_{E_{S_1}} \neq c_l$. Now, we distinguish two cases.

(i) If $CEA_l(c_S, E_S) = c_l$, then $r_l^S = CEA_l(c_S, E_S) - \alpha c_l = (1-\alpha)c_l > \lambda_{E_{S_1}} - \alpha c_l = CEA_l(c_{S_1}, E_{S_1}) - \alpha c_l = r_l^{S_1}$.

(ii) If $CEA_l(c_S, E_S) = \lambda_{E_S}$, then $r_l^S = CEA_l(c_S, E_S) - \alpha c_l = \lambda_{E_S} - \alpha c_l > \lambda_{E_{S_1}} - \alpha c_l = CEA_l(c_{S_1}, E_{S_1}) - \alpha c_l = r_l^{S_1}$.

Hence, for each $l \in S \cap R_{S_1}$,

$$r_l^S > r_l^{S_1}. \tag{10}$$

As a result of Part 2, all recipients of S_1 who are in S are recipients in S, i.e., $S \cap R_{S_1} \subseteq R_S$. Moreover, by Part 1, we know that $S \setminus S_1 \subseteq R_S$. Thus, since $|S| = |S_1|$, we conclude that

$$|R_S| \geq |R_{S_1}|. \tag{11}$$

Part 3. *We show that each agent in $S \cap S_1$ who is a transfer agent in S_1 is either a transfer agent in S transferring less in S than in S_1 or a recipient in S. That is, for each $l \in S \cap T_{S_1}$, we have either $[l \in T_S$ and $t_l^S < t_l^{S_1}]$ or $l \in R_S$.*

Let $l \in S \cap T_{S_1}$. Since $l \in T_{S_1}$, we have $\lambda_{E_{S_1}} < \alpha c_l$. Hence, given that $\lambda_{E_S} > \lambda_{E_{S_1}}$, we distinguish two cases:

(i) If $\alpha c_l > \lambda_{E_S}$, then agent l continues being a transfer agent, i.e., $l \in T_S$. Recall that since $l \in T_{S_1}$, $CEA_l(c_{S_1}, E_{S_1}) = \lambda_{E_{S_1}} \neq c_l$. Therefore,

$$t_l^S = \alpha c_l - CEA_l(c_S, E_S) = \alpha c_l - \lambda_{E_S} < \alpha c_l - \lambda_{E_{S_1}} = \alpha c_l - CEA_l(c_{S_1}, E_{S_1}) = t_l^{S_1}. \quad (12)$$

(ii) If $\alpha c_l \leq \lambda_{E_S}$, then agent l becomes a recipient, i.e., $l \in R_S$.

Equations (11) and (12) together imply

$$\sum_{l \in T_S} t_l^S \leq \sum_{l' \in T_{S_1}} t_{l'}^{S_1}. \quad (13)$$

Part 4. *We show that for each agent in $S \setminus S_1$, the subsidy she receives in S is higher than or equal to the maximal subsidy any agent in S_1 receives. That is, for each $l \in S \setminus S_1$, $r_l^S \geq \max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}}$.*

Note first that, by construction of coalition S_1 by the algorithm, the maximal subsidy any recipient in S_1 receives is either $(1-\alpha)c_{i^*}$ or $\lambda_{E_{S_1}} - \alpha c_{j^*}$.

Consider now the agent in $S \setminus S_1$ with the lowest subsidy, i.e., let $l^* \in S \setminus S_1$ be such that for each $l \in S \setminus S_1$, $r_{l^*}^S \leq r_l^S$. We show that $r_{l^*}^S$ is higher than or equal to $\max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}}$. Recall that no agent receiving her claim in S_1 can be in S. In particular, neither agent i^* nor any agent with a lower label than i^* can be in S. Furthermore, by construction of coalition S_1 by the algorithm, any agent with a higher label than j^* is in S_1. Thus, $i^* < l^* < j^*$ and $c_{i^*} \leq c_{l^*} \leq c_{j^*}$.

We distinguish three cases.

Case 1. Agent j^* does not exist.

Then, agent i^* has been chosen in the last step of the algorithm against agent $j^t - 1$ (recall that j^t is the transfer agent in S_1 with the lowest label and $c_{i^*} \leq c_{j^t}$.) That is, $\max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}} = (1-\alpha)c_{i^*}$ and by Step $\theta - 1$ (ii) of the θ-CEA set algorithm,

$$(1-\alpha)c_{i^*} < \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t - 1\})}} - \alpha c_{j^t - 1}. \quad (14)$$

By construction of coalition S_1 by the algorithm, we have $\lambda_{E_{S_1}} \geq \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t - 1\})}}$ (this follows from Lemma 11 in Appendix E). We distinguish two cases.

(i) If $CEA_{l^*}(c_S, E_S) = c_{l^*}$, then, since $c_{l^*} \geq c_{i^*}$ and $(1-\alpha) > 0$,

$$r_{l^*}^S = (1-\alpha)c_{l^*} \geq (1-\alpha)c_{i^*} = \max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}}.$$

(ii) If $CEA_{l^*}(c_S, E_S) = \lambda_{E_S} < c_{l^*}$, then, since $\lambda_{E_S} > \lambda_{E_{S_1}} \geq \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t - 1\})}}$ and $c_{l^*} \leq c_{j^t-1}$,

$$\lambda_{E_S} - \alpha c_{l^*} \geq \lambda_{E_S} - \alpha c_{j^t-1} > \lambda_{E_{S_1}} - \alpha c_{j^t-1} \geq \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}} - \alpha c_{j^t-1}.$$

This, together with inequality (14), implies

$$r^S_{l^*} = \lambda_{E_S} - \alpha c_{l^*} > \lambda_{E_{((S_1 \setminus \{i^*\}) \cup \{j^t-1\})}} - \alpha c_{j^t-1} > (1 - \alpha) c_{i^*} = \max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}}.$$

Case 2. Agent i^* does not exist.

Then, agent j^* has been chosen in the last step of the algorithm against agent 1. That is, $\max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}} = \lambda_{E_{S_1}} - \alpha c_{j^*}$ and by Step $\theta - 1$ (i) of the θ-CEA set algorithm,

$$\lambda_{E_{S_1}} - \alpha c_{j^*} \leq (1 - \alpha) c_1.$$

We distinguish two cases.

(i) If $CEA_{l^*}(c_S, E_S) = c_{l^*}$, then, since $\lambda_{E_{S_1}} - \alpha c_{j^*} \leq (1 - \alpha) c_1$, and $c_1 \leq c_{l^*}$,

$$r^S_{l^*} = (1 - \alpha) c_{l^*} \geq (1 - \alpha) c_1 \geq \lambda_{E_{S_1}} - \alpha c_{j^*} = \max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}}.$$

(ii) If $CEA_{l^*}(c_S, E_S) = \lambda_{E_S} < c_{l^*}$, then, since $c_{l^*} \leq c_{j^*}$ and $\lambda_{E_S} > \lambda_{E_{S_1}}$,

$$r^S_{l^*} = \lambda_{E_S} - \alpha c_{l^*} \geq \lambda_{E_S} - \alpha c_{j^*} > \lambda_{E_{S_1}} - \alpha c_{j^*} = \max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}}.$$

Case 3. Both agents i^* and j^* exist.

Then, $\max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}} = \max\{(1 - \alpha) c_{i^*}, \lambda_{E_{S_1}} - \alpha c_{j^*}\}$.

Note that in Cases 1 and 2 it is clear who the last agent added to coalition S_1 is and therefore, what the highest subsidy in set S_1 is. However, when both agents i^* and j^* exist, we need to distinguish cases depending on the last agent added to coalition S_1, who after all is the agent with the highest subsidy in S_1. We distinguish the following cases.

(i) $CEA_{l^*}(c_S, E_S) = c_{l^*}$.

(i.1) If agent i^* is the last agent added to coalition S_1, then $\max\{r^{S_1}_{l'}\}_{l' \in R_{S_1}} = (1 - \alpha) c_{i^*}$ and we apply Case 1 (i).

(i.2) If agent j^* is the last agent added to coalition S_1, then $\max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}} = \lambda_{E_{S_1}} - \alpha c_{j^*}$ and we apply Case 2 (i) by replacing agent 1 with agent $i^* + 1$.

(ii) $CEA_{l^*}(c_S, E_S) = \lambda_{E_S} < c_{l^*}$.

(ii.1) If agent i^* is the last agent added to coalition S_1, then $\max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}} = (1-\alpha)c_{i^*}$ and we apply Case 1 (ii) by replacing agent $j^t - 1$ with agent $j^* - 1$.

(ii.2) If agent j^* is the last agent added to coalition S_1, then $\max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}} = \lambda_{E_{S_1}} - \alpha c_{j^*}$ and we apply Case 2 (ii).

Hence, we can conclude that $r_{l^*}^S \geq \max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}}$. Furthermore, since for each $l \in S \setminus S_1$, $r_{l^*}^S \leq r_l^S$, we have that for each $l \in S \setminus S_1$,

$$r_l^S \geq \max\{r_{l'}^{S_1}\}_{l' \in R_{S_1}}.$$

Part 4, together with inequality (10), implies that the total subsidy in coalition S is higher than or equal to the total subsidy in coalition S_1. That is,

$$\sum_{l \in R_S} r_l^S \geq \sum_{l' \in R_{S_1}} r_{l'}^{S_1}. \tag{15}$$

Finally, inequalities (13) and (15) together imply that the total transfer in coalition S is lower than the total subsidy in coalition S. That is,

$$\sum_{k \in T_S} t_k^S < \sum_{k' \in R_S} r_{k'}^S.$$

However, since the total transfer has to be equal to the total subsidy, a contradiction is reached.

In particular, there exists no coalition $T \subseteq N_1$ such that $|T| \leq \theta$, and for each $i \in S_1 \cap T$, $T \succ_i^{((c,E),CEA)} S_1$, i.e., an agent from set S_1 cannot be part of a blocking coalition of size lower than or equal to θ with agents from set N_1.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j\right)$ and $|N_k| > \theta$. Furthermore, agents in N_k are relabelled such that $N_k = \{1', \ldots, n'\}$, $c_{1'} \leq \ldots \leq c_{n'}$. Hence, by a similar reasoning than in Step 1 (with the new set of agents N_k), it follows that there exists no coalition $T \subseteq N_k$ such that $|T| \leq \theta$, and for each $i \in S_k \cap T$, $T \succ_i^{((c,E),CEA)} S_k$, i.e., an agent from set S_k cannot be part of blocking coalition of size lower than or equal to θ with agents from set N_k. By the previous steps, an agent from set $\bigcup_{j\in\{1,\ldots,k-1\}} S_j$ cannot be part of a blocking coalition with agents from set S_k. Hence, no agent from set S_k can be part of a blocking coalition of size lower than or equal to θ.

After $l-1$ steps, we have shown that there is no blocking coalition $T \subsetneq N$ such that $|T| \leq \theta$.

Finally, assume, by contradiction, that π is not stable. Then, there exists a blocking coalition $T \subseteq N$ such that for each agent $i \in T$, $T \succ_i^{((c,E),CEA)} \pi(i)$. In particular, $|T| > \theta$.

Case 1. For each $i \in T$, $CEA_i(c_T, E_T) = \alpha c_i$.

Then, by Lemma 7, there is a coalition $T' \subsetneq T$ with $|T'| = \theta$ such that for each $j \in T'$,

$$T' \sim_j^{((c,E),CEA)} T \succ_j^{((c,E),CEA)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to θ.

Case 2. For some $i \in T$, $CEA_i(c_T, E_T) \neq \alpha c_i$.

Then, by Proposition 3, there is a coalition $T' \subsetneq T$ with $|T'| = \theta$ such that for each $j \in T'$,

$$T' \succsim_j^{((c,E),CEA)} T \succ_j^{((c,E),CEA)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to θ.

We have proven that partition π obtained by the θ-CEA algorithm is stable. □

G Proof of Theorem 8

Recall that $N = \{1, \ldots n\}$ and $c_1 \leq c_2 \leq \ldots \leq c_n$. For each $S \subseteq N$, we denote the CEL parameter associated with (c_S, E_S) by λ_{E_S}, i.e., for each $i \in S$, $CEL_i(c_S, E_S) = \max\{0, c_i - \lambda_{E_S}\}$, where λ_{E_S} is chosen so that $\sum_{j\in S} \max\{0, c_j - \lambda_{E_S}\} = E_S$.

Proof of Theorem 8. Let $N \in \mathcal{N}$ such that $|N| > \theta$ and $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$. Consider the coalition formation problem with agent set N induced by $((c, E), CEL)$. We show that the partition $\pi = \{S_1, \ldots, S_l\}$ obtained by the θ-CEL algorithm is stable.

First, we show that the coalitions obtained by the θ-CEL algorithm cannot be blocked by coalitions of size lower than or equal to θ.

Step 1. Recall that $N_1 := N$, $|N_1| > \theta$, and $S_1 := \{1, \ldots, \theta\}$. We show that S_1 is for each of its members at least as desirable as any other coalition of size lower than or equal to θ.

Note that since $(c, E) \in \widetilde{\mathcal{P}}_\theta^N$, we have that for each $S \subsetneq N$ with $|S| < \theta$ and each $i \in S \cap S_1$,

$$S_1 \succsim_i^{((c,E),CEL)} S.$$

Next, let $j \in S_1$ and $k \in N \setminus S_1$. Then, $k > \theta$ and $c_k \geq c_\theta$. We consider coalition $T = (S_1 \setminus \{j\}) \cup \{k\}$ and prove for each $i \in S_1 \setminus \{j\}$, $S_1 \succsim_i^{((c,E),CEL)} T$ by showing that

$$CEL_i(c_{S_1}, E_{S_1}) \geq CEL_i(c_T, E_T).$$

If $c_k = c_j$, then $CEL_i(c_{S_1}, E_{S_1}) = CEL_i(c_T, E_T)$ follows immediately. Hence, assume that $c_k > c_j$. We consider two cases:

Case 1. $j \in S_1$ is such that $CEL_j(c_{S_1}, E_{S_1}) \neq 0$.

Then, at the problem $(c_{S_1}, E_{S_1}) \in \mathcal{C}^{S_1}$, since agent j contributes αc_j to and requests c_j from coalition S_1, she contributes a loss of $(1-\alpha)c_j$ to coalition S_1.

Next, consider the auxiliary problem $(c_T, \bar{E}_T) \in \mathcal{C}^T$ at which agent k also contributes a loss of $(1-\alpha)c_j$ to coalition T. Then, agent k contributes $c_k - (1-\alpha)c_j$ to coalition T and $\bar{E}_T = E_{S_1} - \alpha c_j + (c_k - c_j + \alpha c_j) = E_{S_1} + (c_k - c_j)$. Note that the auxiliary problem $(c_T, \bar{E}_T)$ is such that $\bar{E}_T \neq \alpha c^T$. Since $c_k > c_j$, we have, by the definition of the CEL rule, that for each agent $i \in S_1 \setminus \{j\} = T \setminus \{k\}$,

$$CEL_i(c_{S_1}, E_{S_1}) = CEL_i(c_T, \bar{E}_T).$$

Now, consider the problem $(c_T, E_T) \in \mathcal{C}^T$ and note that agent k contributes a loss of $(1-\alpha)c_k$ to coalition T. Then, agent k contributes αc_k to coalition T and $E_T = E_{S_1} + \alpha(c_k - c_j)$.

Since $\alpha \in (0,1)$ and $c_k > c_j$,

$$\bar{E}_T = E_{S_1} + (c_k - c_j) > E_{S_1} + \alpha(c_k - c_j) = E_T.$$

Then, by resource monotonicity of the CEL rule, for each agent $i \in S_1 \setminus \{j\}$,

$$CEL_i(c_T, \bar{E}_T) \geq CEL_i(c_T, E_T)$$

and hence,

$$CEL_i(c_{S_1}, E_{S_1}) \geq CEL_i(c_T, E_T).$$

Case 2. $j \in S_1$ is such that $CEL_j(c_{S_1}, E_{S_1}) = 0$.

Starting from problem $(c_{S_1}, E_{S_1}) \in \mathcal{C}^{S_1}$, we consider the auxiliary problem that is obtained when agent j leaves with her payoff $CEL_j(c_{S_1}, E_{S_1}) = 0$. Thus, the total endowment is now divided between agents in coalition $S_1 \setminus \{j\}$ and the auxiliary problem $(c_{S_1\setminus\{j\}}, E_{S_1}) \in \mathcal{C}^{S_1\setminus\{j\}}$ is such that $E_{S_1} \neq \alpha c^{S_1\setminus\{j\}}$. By consistency of the CEL rule, for each $i \in S_1 \setminus \{j\}$,

$$CEL_i(c_{S_1\setminus\{j\}}, E_{S_1}) = CEL_i(c_{S_1}, E_{S_1}). \tag{16}$$

Next, let $T = (S_1 \setminus \{j\}) \cup k$ and consider problem $(c_T, E_T) \in \mathcal{C}^T$. Since $E_T > 0$ and $c_k \geq c_\theta$, by the definition of the CEL rule, $CEL_k(c_T, E_T) = c_k - \lambda_{E_T} > 0$. Then, the amount of the endowment that is allocated to agents in $T \setminus \{k\} = S_1 \setminus \{j\}$ is

$$\bar{E}_{T\setminus\{k\}} = E_T - (c_k - \lambda_{E_T}) = \alpha c^{T\setminus\{k\}} + \alpha c_k - (c_k - \lambda_{E_T}) = \alpha c^{T\setminus\{k\}} + (\alpha - 1)c_k + \lambda_{E_T}. \tag{17}$$

By the definition of the CEL rule, if an agent receives a zero payoff, then agents with the highest claims receive an over-proportional payoff. Hence, $c_k - \lambda_{E_T} \geq \alpha c_k$, or equivalently,

$$\lambda_{E_T} \leq (1 - \alpha)c_k. \tag{18}$$

Starting with equality (17) and using inequality (18), we obtain

$$\bar{E}_{T\setminus\{k\}} = \alpha c^{T\setminus\{k\}} + (\alpha - 1)c_k + \lambda_{E_T} \leq \alpha c^{T\setminus\{k\}} + (\alpha - 1)c_k + (1 - \alpha)c_k = \alpha c^{T\setminus\{k\}}. \tag{19}$$

Now, starting from problem $(c_T, E_T) \in \mathcal{C}^T$, we consider the auxiliary problem that is obtained when agent k leaves with her payoff $CEL_k(c_T, E_T)$. The auxiliary problem $(c_{T\setminus\{k\}}, \bar{E}_{T\setminus\{k\}}) \in \mathcal{C}^{(T\setminus\{k\})}$ is such that $\bar{E}_{T\setminus\{k\}} \neq \alpha c^{T\setminus\{k\}}$. By consistency of the CEL rule, for each $i \in T \setminus \{k\} = S_1 \setminus \{j\}$,

$$CEL_i(c_{T\setminus\{k\}}, \bar{E}_{T\setminus\{k\}}) = CEL_i(c_T, E_T). \tag{20}$$

We will now put the above arguments together. First, $E_{S_1} = \alpha c^{S_1} \geq \alpha c^{T \setminus \{k\}} \geq \bar{E}_{S_1 \setminus \{k\}}$, where the first inequality follows from $[S_1 \setminus \{j\} = T \setminus \{k\}$ and $\alpha c_j \geq 0]$ and the second inequality follows from $[S_1 \setminus \{j\} = T \setminus \{k\}$ and inequality (19)]. Then, inequalities (16) and (20), together with resource monotonicity of the CEL rule, imply that for each $i \in S_1 \setminus \{j\} = T \setminus \{k\}$,

$$CEL_i(c_{S_1}, E_{S_1}) \overset{(16)}{=} CEL_i(c_{S_1 \setminus \{j\}}, E_{S_1}) \geq CEL_i(c_{T \setminus \{k\}}, \bar{E}_{T \setminus \{k\}}) \overset{(20)}{=} CEL_i(c_T, E_T).$$

Note that so far we have only shown that any agent $i \in S_1$ weakly prefers coalition S_1 to a coalition T where another agent was replaced by an agent outside of S_1. However, this proof step can be repeated by replacing other agents one by one and hence, for each agent $i \in S_1$ and each coalition $S \subsetneq N_1$ such that $[i \in S$ and $|S| = \theta]$,

$$S_1 \succsim_i^{((c,E),CEL)} S.$$

In particular, there exists no coalition $T \subseteq N_1$ such that $|T| \leq \theta$, and for each $i \in S_1 \cap T$, $T \succ_i^{((c,E),CEL)} S_1$, i.e., an agent from set S_1 cannot be part of a blocking coalition of size lower than or equal to θ with agents from set N_1.

Step k ($k > 1$). Recall from Step $k-1$ that $N_k := N \setminus \left(\cup_{j=1}^{k-1} S_j \right)$ and $|N_k| > \theta$. Set $S_k := \{\theta k - (\theta - 1), \theta k - (\theta - 2), \ldots, \theta k\}$. Note that agents $\theta k - (\theta - 1), \ldots, \theta k$ are lowest-claims agents in N_k. Hence, by a similar reasoning than in Step 1 (with agents $\theta k - (\theta - 1), \ldots, \theta k$ in the roles of $1, \ldots, \theta$ respectively), it follows that there exists no coalition $T \subseteq N_k$ such that $|T| \leq \theta$, and for each $i \in S_k \cap T$, $T \succ_i^{((c,E),CEL)} S_k$, i.e., an agent from set S_k cannot be part of a blocking coalition of size lower than or equal to θ with agents from set N_k. By the previous steps, an agent from set $\bigcup_{j \in \{1,\ldots,k-1\}} S_j$ cannot be part of a blocking coalition with an agent from set S_k. Hence, no agent from set S_k can be part of a blocking coalition of size lower than or equal to θ.

After $l-1$ steps, we have shown that there is no blocking coalition $T \subsetneq N$ such that $|T| \leq \theta$.

Finally, assume, by contradiction, that π is not stable. Then, there exists a blocking coalition $T \subseteq N$ such that for each agent $i \in T$, $T \succ_i^{((c,E),CEL)} \pi(i)$. In particular, $|T| > \theta$.

Case 1. For each $i \in T$, $CEL_i(c_T, E_T) = \alpha c_i$.

Then, by Lemma 7, there is a coalition $T' \subsetneq T$ with $|T'| = \theta$ such that for each $j \in T'$,

$$T' \sim_j^{((c,E),CEL)} T \succ_j^{((c,E),CEL)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to θ.

Case 2. For some $i \in T$, $CEL_i(c_T, E_T) \neq \alpha c_i$.

Then, by Proposition 3, there is a coalition $T' \subsetneq T$ with $|T'| = \theta$ such that for each $j \in T'$,

$$T' \succsim_j^{((c,E),CEL)} T \succ_j^{((c,E),CEL)} \pi(j),$$

which contradicts the fact that there is no blocking coalition of size lower than or equal to θ.

We have proven that the partition π obtained by the θ-CEL algorithm is stable. □

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