Partial Verifiability Induced Contests

Octavian Strîmbu *

April 2022

Abstract

In a simple common agency (a first-price menu auction with two bidders), the interest groups compete for the political prize awarded by an official. I show that if in the complete information setting one drops the perfect verifiability of the outcome assumption, the ensuing game turns out to be a contest. As the conflicting interest groups observe but may only partially verify the policy chosen by the official, they will play mixed strategies. I characterize these strategies and the interest groups' betting behaviour relative to their verification technologies. The probability of getting an inefficient policy as outcome of the influence game is derived from the equilibrium strategies.

Keywords : Contests, Common agency, Partial verifiability

JEL Codes : D44, D72, D80

---

*Faculty of Business and Economics (HEC Lausanne), University of Lausanne, Switzerland and Geneva School of Economics and Management (GSEM), University of Geneva, Switzerland *(octavian.strimbu@unil.ch)*.

The author wants to thank Salvatore Di Falco and all those who welcomed him as research fellow at GSEM.

1 Introduction

A sound democracy needs executive and legislative decision processes that ensure social efficiency. The influence of interest groups on policy-making has been the subject of intense study in economics and political science. From Olson (1965) to Fukuyama (2014), free-riding or the costs of organisation within large interest groups have been put forward as causes of inefficiency.

A great accomplishment of political economy has been to recognise and formalise the competitive nature of the policy-making process. Two of the workhorse models in this field have been the common agency (a first-price auction with multiple prizes) and the all-pay auction (a type of contest). In time, the range of applications to political behaviour has significantly increased for both these models. Their inner-workings are quite different, though.

In a common agency setting, the interest groups *perfectly condition* their transfers on the policy eventually chosen by an official. Not surprisingly, the outcome of the complete information common agency is efficient. In consequence, it has extensively been employed as benchmark against which policy-making under asymmetric information has been characterised. In an all-pay auction, on the other hand, all the interest groups trying to influence an official announce a bid and then, whether or not they obtain the prize, all bids have to be paid. Rent-seeking scenarios where the interest groups sunk a certain effort in order to influence the official's choice have been formalized as all-pay auctions, for example.

In this paper I argue that there is a whole range of policy-making settings in-between the perfect conditioning specific to common agency and the lack of conditioning on winning from the all-pay auctions. These settings are induced by the imperfect verifiability of the official's choice and should be analysed as a specific form of contest. To make my point, I consider the simplest type of common agency, i.e. two interest groups and two policies under complete information, but I assume that the interest groups may only partially verify the official's policy choice. To avoid triviality, a second assumption is that the interest groups have conflicting interests as each strictly prefers a different policy.

Partial verifiability means that, once they observe the implemented policy, the interest groups may gather evidence about the official's choice only with a certain probability. With a better verification technology, this probability may increase but is always strictly less than one. A consequence of this assumption is that the official engages in opportunistic behaviour and an interest group may have to pay a positive transfer even if he doesn't get the political prize. This doesn't occur by design like in the case of standard contests, but because the interest groups cannot always prove that the official has made a false promise.

Evidently, there are numerous actual public policies for which the outcome is only partially verifiable by one or more interest groups. In fact, any public policy that is a rather complex matter may fall into this category. Consider for example an infrastructure, like a bridge or a school, that may be more or less earthquake-resistant. For obvious safety reasons, in an earthquake-prone area the public is likely to prefer an earthquake-proof construction. However, even if the local government promises an earthquake-proof bridge or school, it might be prohibitively expensive, or simply too late, for the citizens to prove the contrary *ex post*.[1] Consider also a tax hike for the rich. The interest group consisting of relatively poor citizens should support this policy and try to influence the officials to choose it. Again, even if the officials promise to implement the tax hike, it would be quite difficult for the poor citizens to produce hard evidence that the rich, eventually, do not pay higher taxes.

The first finding in this paper is that under partial verifiability there is no pure strategy equilibrium. The following step is to characterize the equilibrium mixed strategies and the betting behaviour of the interest groups relative to their respective verification technologies. Surprisingly, in some

[1] Unfortunately, this kind of situation occurred during the 2008 Sichuan earthquake, when a number of schools reportedly collapsed too quickly.

cases an interest group may bid less aggressively even if he gets a better verification technology. I argue that this behavior has to do with the flexibility of the incentive schemes at their disposal.

Because in equilibrium the interest groups randomize over a set of bids, the official will choose an inefficient policy with a probability strictly higher than zero. As a last result, I calculate this probability and I discuss its simple link with the expected total welfare and, especially, its more complex relation with the interest groups' access to hard evidence. A part of the discussion focuses on the official's incentive constraint which is in itself important because, as far as I know, it is new in the literature.

Related literature

The beginnings of contest literature coincide with the Virginia School's attempt to understand the functioning of state and the political behaviour using tools specific to economics. Tullock (1980) proposes a first contest where an interest group wins a favour from an official with a certain probability. This probability is a simple, exogenous and proportional function of the *non-refundable* contributions made by the interest groups and it made the "Tullock contest" highly popular. A drawback of the initial model is, though, the fact that all the interest groups have the same valuation of the political prize.

Two capital additions to the contest literature are Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1993). They generalise the Tullock contest in the sense that interest groups have different valuations of the political prize and the probability of winning is derived from the bidders' mixed equilibria strategies. The new game is dubbed all-pay auction. Besides analysing the canonical all pay-auction, Baye, Kovenock, and de Vries (1993) propose a relevant application: the "wining-and-dining" lobbying as practised in Washington DC. Subsequently, the all-pay auction has proved to be an useful tool in determining the effect of caps on political contributions, as in Che and Gale (1998), or in comparing different forms of lobbying, as in

Cotton (2009).[2]

But what if the bidders may actually condition their contributions on the official's choice? The answer has essentially been given with the help of the common agency model. The seminal articles in this field are Bernheim and Whinston (1986a,b). The authors show that by letting the interest groups condition their bids on each potential prize and by restricting the equilibria to a set called *truthful*, one gets the official to choose the socially efficient outcome. Among the most influential applications of the model, one should mention Grossman and Helpman (1994) in the political economy of tariffs, or Martimort and Semenov (2007) and Le Breton and Salanié (2003) in lobbying under asymmetric information. Together with the efficiency result, the fact that common agency is tractable in the presence of private information or contract imperfections has surely added to its success.

In a notable generalization effort, Siegel (2010) proposes a "simple contest"where each player pays his entire bid if winning, but only a fraction α if he looses. There are two main differences with the game that I analyse here. First, partial verifiability offers a relevant foundation for the imperfect conditioning on winning: in the contest described below, the interest groups do not sunk a part of their bids by assumption, but will have to pay this part because the official might be pretending and they cannot prove otherwise. Then, technically, the fraction paid upon loosing in Siegel (2010) is the same across contestants. My model assumes heterogeneity among bidders and it will turn out to be an important determinant of their behaviour.

2 The model

The benchmark model is a common agency (first-price menu auction) with two interest groups (bidders) denoted by $i \in \{1, 2\}$ and an official (auctioneer of a political prize). The official chooses between two public policies

[2]For a comprehensive discussion on contests, see Konrad (2009).

$\{a, b\}$.[3] The public policies could be two similar infrastructures which have one different, and potentially difficult to verify, attribute. The first interest group prefers policy a and the second interest group prefers policy b. Let $u, v \in \mathbb{R}_{++}$ denote these preferences; that is, the first interest group gets an utility of u if a is chosen and zero otherwise, while the second interest group gets an utility of v if b is chosen and zero otherwise. I assume that

$$u > v$$

so a is the efficient policy.

To influence the official, each interest group offers a conditional transfer (bid) $t_i : \{a, b\} \to \mathbb{R}_+$.[4] It is straightforward that no interest group is willing to pay anything to the official unless he chooses his respective preferred policy. Hence, without further proof, I will suppose throughout that $t_1(b) = t_2(a) = 0$ and I resume the conditional transfers by denoting $t_1(a) = t_1$ and $t_2(b) = t_2$.

By awarding the political prize, the official maximizes his rent, that is, he chooses a if

$$t_1 \geq t_2.$$

The standard "truthful" solution[5] of this game is that the first bidder offers $t_1 = v$, the second also offers $t_2 = v$ and the official chooses the efficient project a.

[3] A common agency with two bidders and two choices for public policy may seem rather specific. However, if one considers the "truthful" Nash Equilibrium refinement, the benchmark model turns out to cover quite a large spectrum of scenarios. See the first corollary of Theorem 2 in Bernheim and Whinston (1986b).

[4] For some political systems, the bids may be considered political contributions. For others, t_i may be a bribe. In a more general interpretation, the bids are any type of costly political support.

[5] See Bernheim and Whinston (1986b).

2.1 The partial verification model

I begin by presenting an extreme version of the partial verification model: the first interest group observes but may never verify (produce evidence about) the official's choice. The second interest group may, as before, always produce evidence and back out from paying t_2 if the second policy is not implemented. The previous strategy profile is no longer an equilibrium since the official would pretend to the first interest group that he plays a and would actually play b to gather $2v$. In fact, the first interest group finds himself powerless and can't compete with the second interest group to influence the official's choice. The interest groups offer $t_1 = 0$, $t_2 = 0$, and in equilibrium, the official plays the inefficient b.

I now analyze the general partial verification model. The timing of the game is:

1. The interest groups bid t_i. For each interest group, the verifiability level of the official's choice is common knowledge and denoted $p \in [0, 1]$ and $q \in [0, 1]$, respectively.

2. The official chooses a policy and announces his choice to each interest group, separately. According to the official's announcement, the interest groups pay their bids.

3. The interest groups observe the policy actually implemented and get their utility. If the interest group i observes his less-preferred policy being implemented despite the official's announcement promising to favour him, with probability p and respectively q he may produce evidence and back out from paying t_i.[6]

The interest groups basically make full upfront payments (they pay before observing the implemented policy). Therefore, the official may display

[6] In contract theory terms, there is a third-party, e.g. a court, that enforces the contracts t_i with probability p and respectively q. For an in-depth theoretical analysis of partial verifiability in the more general context of bilateral opportunism see for example Kvaløy and Olsen (2009).

an opportunistic behaviour: he may announce to an interest group that he chooses his preferred policy even if he doesn't. As argued in the introduction, this kind of unilateral opportunism should be relevant for policy-making settings.

By assumption, the "punishment" of the official for lying is also quite mild. When caught, upon presentation of evidence, he only has to comply with the initial rules of the auction but there is no incurred disutility.

In any Nash equilibrium, the official plays a if

$$t_1 + (1-q)t_2 > (1-p)t_1 + t_2 \tag{1}$$

which is equivalent to

$$pt_1 > qt_2. \tag{2}$$

He plays b if the inequality is reversed, and he is indifferent between the projects if they both bring him the same expected payoff. To resume, in a Nash equilibrium, the official plays a with probability r

$$r \in \begin{cases} \{1\} \text{ if } pt_1 > qt_2 \\ [0,1] \text{ if } pt_1 = qt_2 \\ \{0\} \text{ if } pt_1 < qt_2 \end{cases} \tag{3}$$

Proposition 1. *Suppose that* $0 < p < 1$ *and* $0 < q < 1$. *Then there is no Nash equilibrium where the official plays a with probability* 1*, nor one where he plays a with probability* 0*. There is no Nash equilibrium where an interest group plays a pure strategy.*

Proof. **The official doesn't always play the same move:**

Consider a strategy profile such that the official plays a with probability 1. I will show that it is not a Nash equilibrium. Suppose it is; then (3) implies

$$\Pr(pt_1 > qt_2) = 1.$$

The second interest group's payoff is $-(1-q)t_2$. Because $q < 1$, this can only be a best response payoff if $t_2 = 0$. The first interest group's payoff is $u - t_1$. Because $t_2 = 0$, the first interest group could decrease t_1 without inducing a move change from the official. Hence, for $u - t_1$ to be a best response payoff, it must be that $t_1 = 0$. But since $q > 0$, the second interest group could induce the official into playing b by bidding $0 < t_2' < v$; doing so, he would increase his payoff to $v - t_2' > 0$. So, a strategy profile where the official plays a with probability 1 is not a Nash equilibrium. Similarly, one can show that because $0 < p < 1$, a strategy profile where the official plays a with probability 0 is not a Nash equilibrium either.

The interest groups don't play pure strategies:

Suppose there is a Nash equilibrium where the second interest group plays a pure strategy t_2 and the official plays a with probability r when $pt_1 = qt_2$. By playing t_1, the first interest group gets

$$\begin{cases} u - t_1 \text{ if } pt_1 > qt_2 \\ ru - (1-p+rp)t_1 \text{ if } pt_1 = qt_2 \\ -(1-p)t_1 \text{ if } pt_1 < qt_2 \end{cases}$$

If the first interest group has no best response, this can't be a Nash equilibrium. If he has one, it is either $t_1 = \frac{q}{p}t_2$, a strategy that yields

$$ru - (1-p+rp)\frac{q}{p}t_2 \tag{4}$$

or $t_1 = 0$, a strategy that yields 0.

If (4) is not zero, then the first interest group's best response is also a pure strategy so that $\Pr(pt_1 \geq qt_2)$ will either be 0 or 1, and I have shown above that it can't happen in a Nash equilibrium.

If (4) is zero, then

$$u - t_1 = u - \frac{q}{p}t_2 = \frac{(1+p)(1-r)u}{1-p+pr} > 0.$$

If $r \neq 1$, t_1 is not a best response since by marginally raising it, the first interest group could induce the official into playing a to get a positive payoff. So $r = 1$ and since (4) is zero, $t_1 = u$. But then $\Pr(pt_1 \geq qt_2)$ will again be either 0 or 1.

I encounter similar contradictions if I assume that the first interest group plays a pure strategy t_1.

$\square$

I will thus be looking for a Nash equilibrium where both interest groups play mixed strategies.

Proposition 2. *Define*

$$\theta_1 = \min\{\frac{q}{p}v, u\}$$
$$\theta_2 = \min\{\frac{p}{q}u, v\}$$

and let $R : \mathbb{R} \to \mathbb{R}_+$ be the ramp function.

The following profile is a Nash equilibrium:

- *The official plays strategy* (3) *for some r.*
- *The first interest group randomizes on $[0, \theta_1]$ with the distribution function*
  $$F(z) = \frac{1}{q}\frac{qv - p\theta_1 + (1-q)pz}{v - pz}.$$
- *The second interest group randomizes on $[0, \theta_2]$ with the distribution function*
  $$G(z) = \frac{1}{p}\frac{pu - q\theta_2 + (1-p)qz}{u - qz}.$$

Proof. **The interest groups play a best response:**

The first interest group's expected payoff is

$$\Pr(pt_1 > qt_2 \mid t_1)(u - t_1) - \Pr(pt_1 \leq qt_2 \mid t_1)(1-p)t_1 =$$
$$= \Pr(t_2 \leq \frac{p}{q}t_1 \mid t_1)(u - pt_1) - (1-p)t_1 =$$
$$= G(\frac{p}{q}t_1)(u - pt_1) - (1-p)t_1 = u - \frac{q}{p}\theta_2.$$

Hence, the first interest group is playing a best response since his payoff is independent of t_1 over $[0, \theta_1]$.

The model is symmetric, so one can establish that the second interest group is playing a best response in a similar fashion.

□

To fully understand the interest groups' bidding behaviour, in the next section I characterize the change in their equilibrium strategies when the verification technologies improve.

2.2 The interest groups' bidding behavior

In this section I derive the interest groups' bidding behavior from their equilibrium mixed strategies. A legitimate question would be: are the interest groups going to bid more or less aggressively as they get a better verification technology? The next proposition shows that the mathematical answer is quite straightforward, but in the following discussion I also show that the intuition behind it is less obvious.

Proposition 3. *When $pu > qv$, $F(z)$ and $G(z)$ are strictly increasing in p and strictly decreasing in q. When $pu \leq qv$, $F(z)$ and $G(z)$ are strictly decreasing in p and strictly increasing in q.*

Proof. **The derivative of equilibrium mixed strategy distributions with respect to verification technologies:**

$$\frac{dF(z)}{dp} = \begin{cases} \frac{1}{q}\frac{d}{dp}\left(\frac{(1-q)\,pz}{v - pz}\right) & \text{if } pu > qv \\ \frac{1}{q}\frac{d}{dp}\left(\frac{qv - pu + (1-q)pz}{v - pz}\right) & \text{if } pu \leq qv \end{cases}$$

or

$$\frac{dF(z)}{dp} = \begin{cases} \frac{1}{q}\frac{(1-q)vz}{(v-pz)^2} \text{ if } pu > qv \\ -\frac{1}{q}\frac{(u-z)v}{(v-pz)^2} \text{ if } pu \leq qv \end{cases}$$

and since $u, v \in \mathbb{R}_{++}$, $p, q \in (0,1)$, and $z \in [0, u]$ in the second case, it implies that $\frac{dF(z)}{dp} > 0$ if $pu > qv$ and $\frac{dF(z)}{dp} < 0$ if $pu \leq qv$.

Moreover,

$$\frac{dF(z)}{dq} = \begin{cases} \frac{pz}{v-pz}\frac{d}{dq}\left(\frac{1-q}{q}\right) \text{ if } pu > qv \\ \frac{1}{v-pz}\frac{d}{dq}\left(\frac{qv-pu+(1-q)pz}{q}\right) \text{ if } pu \leq qv \end{cases}$$

or

$$\frac{dF(z)}{dq} = \begin{cases} -\frac{1}{q^2}\frac{pz}{v-pz} \text{ if } pu > qv \\ \frac{1}{q^2}\frac{p(u-z)}{v-pz} \text{ if } pu \leq qv. \end{cases}$$

In the first case $z \in [0, \frac{q}{p}v]$ so, $v - pz > 0$ as $v \in \mathbb{R}_{++}$ and $q \in (0,1)$. Also, $z \in [0, u]$ in the second case, and since $u, v \in \mathbb{R}_{++}$ and $p \in (0,1)$, it implies that $v - pz \geq v - pv > 0$ again. Hence, $\frac{dF(z)}{dq} < 0$ if $pu > qv$ and $\frac{dF(z)}{dq} > 0$ if $pu \leq qv$.

The calculus is symmetric for $G(z)$.

□

The mechanics behind third proposition are standard. If for example the first interest group' verification technology marginally improves from p to p' when $pu > qv$, $F'(z)$ is first-order stochastically dominated by $F(z)$, which means that the first interest group is betting less aggressively.

Still, the interpretation of the interest groups behavior is not obvious. When having access to a better verification technology, why an interest group would bid less? The third proposition helps to understand that the tone of the influence game is set by the flexibility of the interest groups' incentive schemes. An interest group has a more flexible incentive scheme if, following a change in his verification technology, he will offer new bids or will renounce some of the initial bids (and he will not only put more or less weight on the existing ones). For example, when $pu > qv$, $F(z)$'s support is $\left[0, \frac{q}{p}v\right]$ and $G(z)$'s support is $[0, v]$. If p increases, because the first interest group has a more flexible incentive scheme and he doesn't have to do as much in order to "beat" the second interest group, he will bid less aggressively by decreasing the upper bound of $F(z)$'s support. The second interest group does respond in equilibrium, but only puts less weight on the higher bids.

Now that the bidding behavior of the interest groups is settled, one should ask who is winning the influence game more often. It is important to calculate the probability that the official chooses the efficient/inefficient policy because it is a measure of the expected welfare.

3 The probability of choosing the inefficient project

In the above model, two interest groups try to get a rent-maximizing official to choose their respective preferred public policies. The first interest group bids up in order to impose the socially efficient policy, while the second one wants to impose the inefficient policy. There is one caveat: the interest groups have to rely on imperfect verification technologies.

As I have proved, typical of a contest, in equilibrium the principals play mixed strategies, so each principal may win the influence game with a certain positive probability. In this setting, because the utility is transferable, if the agent chooses the efficient project more often, the expected social welfare raises. Hence, calculating the probability that the official chooses the efficient/inefficient policy is a central result.

Proposition 4. *The probability of choosing the inefficient project is:*

$$\Pr(pt_1 \leq qt_2) = \frac{(u - q\theta_2)(v - p\theta_1)}{pq(u-v)^2} \ln\left(\frac{v}{u}\frac{u - q\theta_2}{v - q\theta_2}\right) - \frac{[(1-q)u + qv - p\theta_1]\theta_2}{p(u-v)u}.$$

This probability is strictly decreasing in p and strictly increasing in q if $pu \leq qv$.

If $pu > qv$, the probability of choosing the inefficient project is non-monotonic in q: it is strictly increasing when q is small enough, but strictly decreasing when q is big enough.

Proof. See the Appendix.

□

Probably the most important feature of my model is the official's incentive constraint. Up to this point in the contest literature, since the outcome is perfectly verifiable by the bidders, the auctioneer awards the prize function only of the actual bids. In my partial verifiability model, as one can notice from (1) and (2), the interest groups' verification technologies are also part of the official's decision making. This feature has consequences on the interest groups's betting behaviour, but even more importantly, on the probability of getting as outcome a socially inefficient policy.

The last proposition shows how the interest groups' access to a better verification technology influences the probability of choosing the inefficient policy. For the first interest group the matter is simple: a better verification technology will lower this probability, so will improve the total expected welfare.

The access to a better verification technology for the second interest group has a more complex effect, though. If the first interest group has a relatively low access to hard evidence about the official's choice (p is relatively small, so $pu \leq qv$, for example) letting the second interest group have a better verification technology is a bad idea. But if both interest groups may gather hard evidence often enough (p and q approach 1), it turns out to be a good idea to improve the second interest group's verification technology.

4 Conclusion

In a capital contribution to the contract theory literature, Maskin and Tirole (1999) mentions that the incomplete contract paradigm is particularly relevant for the governments and legislators. In the same time, the political economy literature has extensively employed the first-price auctions in order to formalise the competitive nature of the political decision-making process.

The question that I try to answer in this article is: what new insights a very simple form of contract incompleteness would bring within a first-price auction? A very interesting result is that the new game is a contest. The bidders compete in mixed strategies, so with a positive probability the outcome of the game is inefficient. From the contest theory point of view, the innovative part of the above game is the official's incentive constraint. The rent-maximising official keeps having an opportunistic behaviour towards the interest groups because they don't have the verification technology to always prove him wrong.

References

Baye, M. R., D. Kovenock, and C. G. de Vries (1993): "Rigging the Lobbying Process: An Application of the All-Pay Auction," *The American Economic Review*, 83(1), 289–294.

Bernheim, B. D., and M. D. Whinston (1986a): "Common Agency," *Econometrica*, 54(4), 923–942.

——— (1986b): "Menu Auctions, Resource Allocation, and Economic Influence," *The Quarterly Journal of Economics*, 101(1), 1–32.

Che, Y.-K., and I. L. Gale (1998): "Caps on Political Lobbying," *The American Economic Review*, 88(3), 643–651.

Cotton, C. (2009): "Should We Tax or Cap Political Contributions? A Lobbying Model with Policy Favors and Access," *Journal of Public Economics*, 93, 831–842.

Fukuyama, F. (2014): *Political order and political decay.* Farrar, Straus and Giroux.

Grossman, G. M., and E. Helpman (1994): "Protection for Sale," *The American Economic Review*, 84(4), 833–850.

Hillman, A. L., and J. G. Riley (1989): "Politically Contestable Rents and Transfers," *Economics & Politics*, 1(1), 17–39.

Konrad, K. A. (2009): *Strategy and Dynamics in Contests.* Oxford University Press.

Kvaløy, O., and T. E. Olsen (2009): "Endogenous Verifiability and Relational Contracting," *The American Economic Review*, 99(5), 2193–2208.

Le Breton, M., and F. Salanié (2003): "Lobbying under political uncertainty," *Journal of Public Economics*, 87, 2589–2610.

Martimort, D., and A. Semenov (2007): "Political Biases in Lobbying under Asymmetric Information," *Journal of the European Economic Association*, 5(2/3), 614–623.

Maskin, E., and J. Tirole (1999): "Unforeseen Contingencies and Incomplete Contracts," *Review of Economic Studies*, 66(1), 83–114.

Olson, M. (1965): *The logic of collective action: public goods and the theory of groups*. Harvard University Press.

Siegel, R. (2010): "Asymmetric Contests with Conditional Investments," *The American Economic Review*, 100(5), 2230–2260.

Tullock, G. (1980): "Efficient Rent Seeking," in *Towards a theory of the rent-seeking society*, ed. by J. M. Buchanan, R. D. Tollison, and G. Tullock, pp. 97–112. Texas A&M University Press.

Appendix

Proof. **Proof of Proposition 4**

G is absolutely continuous with density

$$g(z) = \frac{q}{p}\frac{u - q\theta_2}{(u - qz)^2} \text{ for } z \in [0, \theta_2]$$

The probability that the agent chooses the inefficient project:

$$\Pr(pt_1 \leq qt_2) = \int_0^{\theta_2} F(\frac{q}{p}z)g(z)dz =$$
$$\frac{u - q\theta_2}{p}\int_0^{\theta_2} \frac{qv - p\theta_1 + (1-q)qz}{(v - qz)(u - qz)^2} =$$
$$\frac{u - q\theta_2}{pq(u - v)}\left[\frac{v - p\theta_1}{u - v}\ln\left(\frac{u - qz}{v - qz}\right) - \frac{(1-q)u + qv - p\theta_1}{u - qz}\right]_0^{\theta_2} =$$
$$\frac{(u - q\theta_2)(v - p\theta_1)}{pq(u - v)^2}\ln\left(\frac{v}{u}\frac{u - q\theta_2}{v - q\theta_2}\right) - \frac{[(1-q)u + qv - p\theta_1]\theta_2}{p(u - v)u}.$$

Now, consider the following inequalities:

$$-\ln(1 - x) < \frac{x}{1 - x} \text{ if } x \in (-\infty, 0) \text{ and}$$
$$\frac{x}{1 + \frac{1}{2}x} < \ln(1 + x) \text{ if } x \in (0, \infty).$$

$$\frac{d\Pr(pt_1 \leq qt_2)}{dp}$$
$$= \begin{cases} \frac{d}{dp}\left(\frac{(1-q)}{pq}\frac{(u - qv)v}{(u - v)^2}\ln\left(\frac{u - qv}{(1-q)u}\right) - \frac{(1-q)v}{p(u - v)}\right) & \text{if } pu > qv \\ \frac{d}{dp}\left(\frac{(1-p)}{pq}\frac{(v - pu)u}{(u - v)^2}\ln\left(\frac{(1-p)v}{v - pu}\right) + 1 - \frac{(1-p)u}{q(u - v)}\right) & \text{if } pu \leq qv \end{cases}$$

or

$$\frac{d\Pr(pt_1 \leq qt_2)}{dp}$$

$$= \begin{cases} \dfrac{(1-q)}{p^2}\dfrac{v}{(u-v)}\left(-\dfrac{(u-qv)}{q\,(u-v)}\ln\left(\dfrac{u-qv}{(1-q)\,u}\right)+1\right) & \text{if } pu > qv \\ \left(-\dfrac{u\,(v-p^2u)}{p^2q\,(u-v)^2}\ln\left(\dfrac{(1-p)\,v}{v-pu}\right)+\dfrac{(1+p)\,u}{pq\,(u-v)}\right) & \text{if } pu \le qv \end{cases}$$

If $pu > qv$, since

$$-\frac{(u-qv)}{q\,(u-v)}\ln\left(\frac{u-qv}{(1-q)\,u}\right) < -1 \text{ is equivalent to}$$

$$-\ln\left(1-\frac{q\,(v-u)}{(1-q)\,u}\right) < \frac{\dfrac{q\,(v-u)}{(1-q)\,u}}{1-\dfrac{q\,(v-u)}{(1-q)\,u}}$$

and $\dfrac{q\,(v-u)}{(1-q)\,u} \in (-\infty, 0)$, it implies that $\dfrac{d\Pr(pt_1 \le qt_2)}{dp} < 0$.

If $pu \le qv$, I will show that

$$\frac{(1+p)\,u}{pq\,(u-v)} < \frac{u\,(v-p^2u)}{p^2q\,(u-v)^2}\ln\left(\frac{(1-p)\,v}{v-pu}\right) \text{ or}$$

$$\frac{(1+p)p\,(u-v)}{(v-p^2u)} < \ln\left(1+\frac{p(u-v)}{v-pu}\right).$$

Since $\dfrac{p(u-v)}{v-pu} \in (0, \infty)$, the last step is to show that

$$\frac{(1+p)p\,(u-v)}{(v-p^2u)} < \frac{\dfrac{p(u-v)}{v-pu}}{1+\dfrac{1}{2}\dfrac{p(u-v)}{v-pu}} \text{ or}$$

$$\frac{1+p}{v-p^2u} < \frac{2}{2v-p(u+v)} \text{ which is equivalent to}$$

$$p < 1.$$

Furthermore, consider the following inequalities:

$$x < -\ln(1-x) \text{ if } x \in (-\infty, 0) \text{ and}$$

$$\ln(1+x) < x\frac{1+\dfrac{x}{2}}{1+x} \text{ if } x \in (0, \infty).$$

$$\frac{d\Pr(pt_1 \le qt_2)}{dq}$$

$$= \begin{cases} \frac{d}{dq}\left(\frac{(1-q)}{pq}\frac{(u-qv)\,v}{(u-v)^2}\ln\left(\frac{u-qv}{(1-q)\,u}\right) - \frac{(1-q)\,v}{p\,(u-v)}\right) & \text{if } pu > qv \\ \frac{d}{dq}\left(\frac{(1-p)}{pq}\frac{(v-pu)\,u}{(u-v)^2}\ln\left(\frac{(1-p)\,v}{v-pu}\right) + 1 - \frac{(1-p)\,u}{q\,(u-v)}\right) & \text{if } pu \le qv \end{cases}$$

or

$$\frac{d\Pr(pt_1 \le qt_2)}{dq}$$

$$= \begin{cases} \left(-\frac{v\,(u-q^2v)}{pq^2\,(u-v)^2}\ln\left(\frac{u-qv}{(1-q)\,u}\right) + \frac{(1+q)\,v}{pq\,(u-v)}\right) & \text{if } pu > qv \\ \frac{(1-p)}{q^2}\frac{u}{(u-v)}\left(-\frac{(v-pu)}{p\,(u-v)}\ln\left(\frac{(1-p)\,v}{v-pu}\right) + 1\right) & \text{if } pu \le qv \end{cases}$$

If $pu > qv$ I will show that for $q \le \frac{1}{2}$, $\frac{d\Pr(pt_1 \le qt_2)}{dq} > 0$ but for $q \to 1$, $\frac{d\Pr(pt_1 \le qt_2)}{dq} < 0$.

The values of q where $\frac{d\Pr(pt_1 \le qt_2)}{dq} > 0$ are the values where

$$\ln\left(\frac{u-qv}{(1-q)\,u}\right) < \frac{q\,(1+q)\,(u-v)}{u-q^2v} \text{ or}$$
$$\ln\left(1 + \frac{q\,(u-v)}{(1-q)\,u}\right) < \frac{q\,(1+q)\,(u-v)}{u-q^2v}.$$

Because $\frac{q\,(u-v)}{(1-q)\,u} \in (0,\infty)$, for the values that satisfy

$$\frac{q\,(u-v)}{(1-q)\,u}\frac{1+\frac{1}{2}\frac{q\,(u-v)}{(1-q)\,u}}{1+\frac{q\,(u-v)}{(1-q)\,u}} < \frac{q\,(1+q)\,(u-v)}{u-q^2v}, \text{ which is equivalent to}$$
$$0 < u - 2qu + q^2v,$$

it must be that $\frac{d\Pr(pt_1 \le qt_2)}{dq} > 0$. So, for $q \le \frac{1}{2}$, $\frac{d\Pr(pt_1 \le qt_2)}{dq} > 0$.

On the other hand, because

$$\lim_{q\to 1}\frac{q\,(1+q)\,(u-v)}{u-q^2v}=2$$

and because

$$\lim_{q\to 1}\ln\left(\frac{u-qv}{(1-q)\,u}\right)>2$$

, for $q\to 1$, $\dfrac{d\Pr(pt_1\le qt_2)}{dq}<0$.

If $pu\le qv$, since

$$-1<-\frac{(v-pu)}{p\,(u-v)}\ln\left(\frac{(1-p)\,v}{v-pu}\right)\text{ is equivalent to}$$
$$\frac{p\,(v-u)}{v-pu}<-\ln\left(1-\frac{p\,(v-u)}{v-pu}\right)$$

and $\dfrac{p\,(v-u)}{v-pu}\in(-\infty,0)$, it implies that $\dfrac{d\Pr(pt_1\le qt_2)}{dq}>0$ for any $q\in(0,1)$.

□