A Characterization of the Coordinate-Wise Top-Trading-Cycles Mechanism for Multiple-Type Housing Markets*

Di Feng[†] Bettina Klaus[‡] Flip Klijn[§]

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Abstract

We consider the generalization of the classical Shapley and Scarf housing market model of trading indivisible objects (houses) (Shapley and Scarf, 1974) to so-called multiple-type housing markets (Moulin, 1995). When preferences are separable, the prominent solution for these markets is the coordinate-wise top-trading-cycles (cTTC) mechanism.

We first show that on the subdomain of lexicographic preferences, a mechanism is *unanimous* (or *onto*), *individually rational*, *strategy-proof*, and *non-bossy* if and only if it is the cTTC mechanism (Theorem 1). Second, using Theorem 1, we obtain a corresponding characterization on the domain of separable preferences (Theorem 2). Finally, we show that on the domain of strict preferences, there is no mechanism satisfying *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness* (Theorem 3).

Our characterization of the cTTC mechanism constitutes the first characterization of an extension of the prominent top-trading-cycles (TTC) mechanism to multiple-type housing markets.

Keywords: multiple-type housing markets; *strategy-proofness*; *non-bossiness*; top-trading-cycles (TTC) mechanism; market design.

JEL codes: C78; D47.

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[†]Faculty of Business and Economics, University of Lausanne, 1015 Lausanne, Switzerland; *e-mail*: di.feng@unil.ch.

[‡]*Corresponding author*. Faculty of Business and Economics, University of Lausanne, 1015 Lausanne, Switzerland; *e-mail*: bettina.klaus@unil.ch.

[§]Institute for Economic Analysis (CSIC) and Barcelona School of Economics, Campus UAB, 08193 Bellaterra (Barcelona), Spain; e-mail: flip.klijn@iae.csic.es.

1 Introduction

In many applied matching problems, indivisible goods that are in unit demand have to be assigned without monetary transfers. One of the most prominent such problems is modeled by classical Shapley-Scarf housing markets (Shapley and Scarf, 1974). Shapley and Scarf (1974) consider an exchange economy in which each agent owns an indivisible object (say, a house); each agent has preferences over houses and wishes to consume exactly one house. The objective of the market designer then is to reallocate houses among agents. When preferences are strict, Shapley and Scarf (1974) show that the strict core (defined by a weak blocking notion) has remarkable features: it is non-empty,[1] and can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Moreover, the TTC mechanism that assigns the unique strict core allocation satisfies important incentive properties, *strategy-proofness* (Roth, 1982) and *group strategy-proofness* (Bird, 1984). Furthermore, Ma (1994) and Svensson (1999) show that the TTC mechanism is the unique mechanism satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness*.

However, more general problems of exchanging indivisible objects that are in multi-unit demand are known to be very difficult. In this paper, we consider an extension of the classical Shapley-Scarf housing markets by allowing multi-unit demand: multiple-type housing markets, to use the language of Moulin (1995).[2] In this model, objects are of different types (say, houses, cars, etc.) and agents initially own and exactly wish to consume one object of each type.[3] This model is firstly studied by Konishi et al. (2001). Their results are mainly negative: they show that even if we further restrict preferences to be additively separable, the strict core may still be empty. Moreover, there exists no mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof*.

Despite their negative results, for (strictly) separable preferences, Wako (2005) suggests an alternative solution concept to the strict core by first decomposing a multiple-type housing market into coordinate-wise sub-markets and second, determining the strict core in each sub-market. Wako (2005) calls this unique outcome the commodity-wise competitive allocation and shows that it is implementable in (self-enforcing) coalition-proof Nash equilibria but not in strong Nash equilibria.[4]

Based on Wako's result, we investigate the mechanism that always assigns the commodity-wise competitive allocation; since this allocation can be obtained by using the TTC algorithm for

[1] Roth and Postlewaite (1977) show that the strict core is single-valued.

[2] There are many other resource allocation models with multi-unit demand, such as Pápai (2001, 2007) and Manjunath and Westkamp (2021).

[3] A partial list of papers that study real world applications of multiple-type housing markets is Bag et al. (2019), Han et al. (2019), Huh et al. (2013), Klaus (2008), Mackin and Xia (2016), and Peng et al. (2015).

[4] However, (1) the commodity-wise competitive allocation may be *Pareto inefficient*; and (2) the mechanism that always assigns this allocation, is *not group strategy-proof* (see Wako, 2005, Section 6, for details).

each object type, we refer to it as the *coordinate-wise TTC (cTTC) mechanism*. Although the cTTC mechanism is not *Pareto efficient*, it does have many desirable properties: it is *individually rational*, *strategy-proof*, and *second-best incentive compatible*, i.e., it is *not Pareto dominated* by any other *strategy-proof* mechanism (Klaus, 2008). In view of these positive results, one may wonder whether the cTTC mechanism can be characterized by weakening *Pareto efficiency* and strengthening *strategy-proofness*.

For Shapley-Scarf housing markets, a characterization along these lines is provided by Takamiya (2001): he shows that the TTC mechanism is the only mechanism satisfying *unanimity*, *individual rationality*, and *group strategy-proofness*.[5] Based on Takamiya's result, one could now conjecture that this characterization of the TTC mechanism for Shapley-Scarf housing markets can be carried over to the cTTC mechanism for multiple-type housing markets. That conjecture is almost true; however, we need to weaken *group strategy-proofness* to [*strategy-proofness* and *non-bossiness*].[6] In other words, inspired by Takamiya's result for Shapley-Scarf housing markets, we show that, remarkably, the cTTC mechanism is the only mechanism satisfying *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness* (see Theorems 1 and 2, for lexicographic and separable preferences, respectively).

Our characterization of the cTTC mechanism constitutes the first characterization of an extension of the prominent top-trading-cycles (TTC) mechanism to multiple-type housing markets. Furthermore, our result suggests that when preferences are separable, the cTTC mechanism is outstanding, first, because some efficiency in the form of *unanimity* is preserved (even if full *Pareto efficiency* cannot be reached), and second, because of its incentive robustness in the form of *strategy-proofness* and *non-bossiness* (even if full *group strategy-proofness* cannot be reached). Moreover, we also provide an impossibility result (Theorem 3) for a more general preference domain: we show that when preferences are strict but otherwise unrestricted, there is no mechanism satisfying *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness*.

The rest of the paper is organized as follows. In the following section, Section 2, we introduce multiple-type housing markets, mechanisms and their properties, and the cTTC mechanism. We state our results in Section 3. In Subsection 3.1, we first show that on the subdomain of lexicographic preferences, a mechanism is *unanimous* (or *onto*), *individually rational*, *strategy-proof*, and *non-bossy* if and only if it is the cTTC mechanism (Theorem 1). In Subsection 3.2, using Theorem 1, we obtain a corresponding characterization on the domain of separable preferences (Theorem 2). We would like to emphasize that the proof strategy to use the preference domain

[5] In fact, Takamiya's characterization is based on *ontoness*, a weakening of *unanimity*. However, in the presence of *group strategy-proofness*, *ontoness* coincides with *unanimity*.

[6] When preferences are strict but otherwise unrestricted, the combination of *strategy-proofness* and *non-bossiness* is equivalent to *group strategy-proofness*. However, this is not true for separable preferences; see Example 1 for details.

of lexicographic preferences as a "stepping stone" to obtain a corresponding result for separable preferences is, to the best of our knowledge, new. In Subsection 3.3, we finally show that on the domain of strict preference, there is no mechanism satisfying *unanimity* (or *ontoness*), *individual rationality*, *strategy-proofness*, and *non-bossiness* (Theorem 3). Section 4 concludes with a discussion of our results and how they relate to the literature. Appendix A contains the proofs of our results that are not included in the main text.

2 The model

Multiple-type housing markets

We consider a barter economy without monetary transfers formed by n agents and $n \times m$ indivisible objects. Let $N = \{1, \dots, n\}$ be a finite *set of agents*. A nonempty subset of agents $S \subseteq N$ is a *coalition*. We assume that there exist $m \geq 1$ *(distinct) types of indivisible objects* and n *(distinct) indivisible objects of each type*. We denote the *set of types* by $T = \{1, ..., m\}$. Note that for $m = 1$ our model equals the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974).

Each agent $i \in N$ is endowed with exactly one object of each type $t \in T$, denoted by o_i^t. Hence, each *agent i's endowment* is a list $o_i = (o_i^1, \dots, o_i^m)$. The *set of type-t objects* is $O^t = \{o_1^t, ..., o_n^t\}$, and the *set of all objects* is $O = \{o_1^1, o_1^2, \dots, o_n^1, o_n^2, \dots, o_n^m\}$. In particular, $|O| = n \times m$.

For each agent i, an *allotment* x_i assigns one object of each type to agent i, i.e., x_i is a list $x_i = (x_i^1, \dots, x_i^m) \in \Pi_{t \in T} O^t$, where $x_i^t \in O^t$ is *agent i's type-t allotment*. We assume that each agent i has *complete*, *antisymmetric*, and *transitive preferences* R_i over all possible allotments, i.e., R_i is a linear order over $\Pi_{t \in T} O^t$.[7] For two allotments x_i and y_i, x_i is *weakly preferred to* y_i if $x_i \, R_i \, y_i$, and x_i is *strictly preferred to* y_i if $[x_i \, R_i \, y_i$ and not $y_i \, R_i \, x_i]$, denoted $x_i \, P_i \, y_i$. Finally, since preferences over allotments are strict, agent i is indifferent between x_i and y_i only if $x_i = y_i$. We denote preferences as ordered lists, e.g., $R_i : x_i, \; y_i, \; z_i$ instead of $x_i \, P_i \, y_i \, P_i \, z_i$. The *set of all preferences* is denoted by $\mathcal{R}$, which we will also refer to as the *strict preference domain*.

A *preference profile* specifies preferences for all agents and is denoted by a list $R = (R_1, \dots, R_n) \in \mathcal{R}^N$. We use the standard notation $R_{-i} = (R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n)$ to denote the list of all agents' preferences, except for agent i's preferences. Furthermore, for each coalition S we define $R_S = (R_i)_{i \in S}$ and $R_{-S} = (R_i)_{i \in N \setminus S}$ to be the lists of preferences of the members of coalitions S and $N \setminus S$, respectively.

[7]Preferences R_i are *complete* if for any two allotments x_i, y_i, $x_i \, R_i \, y_i$ or $y_i \, R_i \, x_i$; they are *antisymmetric* if $x_i \, R_i \, y_i$ and $y_i \, R_i \, x_i$ imply $x_i = y_i$; and they are *transitive* if for any three allotments x_i, y_i, z_i, $x_i \, R_i \, y_i$ and $y_i \, R_i \, z_i$ imply $x_i \, R_i \, z_i$.

In addition to the domain of strict preferences, we consider two preference subdomains based on agents' "marginal preferences": assume that for each $i \in N$ and for each type $t \in T$, agent i has complete, antisymmetric, and transitive preferences R_i^t over the set of type-t objects O^t. We refer to R_i^t as *agent i's type-t marginal preferences*, and denote by $\mathcal{R}^t$ the *set of all type-t marginal preferences*. Then, we can define the following two preference domains.

(Strictly) Separable preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *separable* if for each $t \in T$ there exist type-t marginal preferences $R_i^t \in \mathcal{R}^t$ such that for any two allotments x_i and y_i,

$$\text{if for all } t \in T,\ x_i^t\, R_i^t\, y_i^t,\ \text{then } x_i\, R_i\, y_i.$$

$\mathcal{R}_s$ denotes the *domain of separable preferences.*

Before introducing our next preference domain, we introduce some notation. We use a bijective function $\pi_i : T \to T$ to order types according to agent i's "(subjective) importance", with $\pi_i(1)$ being the most important and $\pi_i(m)$ being the least important object type. We denote π_i as an ordered list of types, e.g., by $\pi_i = (2, 3, 1)$, we mean that $\pi_i(1) = 2$, $\pi_i(2) = 3$, and $\pi_i(3) = 1$. For each agent $i \in N$ and each allotment $x_i = (x_i^1, \ldots, x_i^m)$, $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \ldots, x_i^{\pi_i(m)})$ denotes the allotment after rearranging it with respect to the *object-type importance order* π_i.

(Separably) Lexicographic preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *(separably) lexicographical* if they are separable with type-t marginal preferences $(R_i^t)_{t\in T}$ and there exists an object-type importance order $\pi_i : T \to T$ such that for any two allotments x_i and y_i,

$$\begin{aligned}
&\text{if } x_i^{\pi_i(1)} P_i^{\pi_i(1)} y_i^{\pi_i(1)} \text{ or}\\
&\text{if there exists a positive integer } k \leq m-1 \text{ such that}\\
&x_i^{\pi_i(1)} = y_i^{\pi_i(1)},\ \ldots,\ x_i^{\pi_i(k)} = y_i^{\pi_i(k)},\ \text{and } x_i^{\pi_i(k+1)} P_i^{\pi_i(k+1)} y_i^{\pi_i(k+1)},\\
&\text{then } x_i\, P_i\, y_i.
\end{aligned}$$

$\mathcal{R}_l$ denotes the *domain of lexicographic preferences.*

Note that $R_i \in \mathcal{R}_l$ can be restated as a $m+1$-tuple $R_i = (R_i^1, \ldots, R_i^m, \pi_i) = ((R_i^t)_{t\in T}, \pi_i)$, or a strict ordering of all objects,[8] i.e., R_i lists first all $\pi(1)$ objects (according to $R_i^{\pi(1)}$), then all $\pi(2)$ objects (according to $R_i^{\pi(2)}$), and so on. We provide a simple illustration in Example 1.

Note that

$$\mathcal{R}_l \subsetneq \mathcal{R}_s \subsetneq \mathcal{R}.$$

An *allocation* x partitions the set of all objects O into agents' allotments, i.e., $x = \{x_1, \ldots, x_n\}$ is such that for each $t \in T$, $\cup_{i\in N} x_i^t = O^t$ and for each pair $i \neq j$, $x_i^t \neq x_j^t$. For simplicity, sometimes we will restate an allocation as a list $x = (x_1, \ldots, x_n)$. The *set of all allocations* is denoted by X, and the *endowment allocation* is denoted by $e = (o_1, \ldots, o_n)$.

[8]See Feng and Klaus (2022, Remark 1) for details.

We assume that when facing an allocation x, there are no consumption externalities and each agent $i \in N$ only cares about his own allotment x_i. Hence, each agent i's preferences over allocations X are essentially equivalent to his preferences over allotments $\Pi_{t \in T} O^t$. With some abuse of notation, we use notation R_i to denote an agent i's preferences over allotments as well as his preferences over allocations, i.e., for each agent $i \in N$ and for any two allocations $x, y \in X$, $x \, R_i \, y$ if and only if $x_i \, R_i \, y_i$.[9]

A *(multiple-type housing) market* is a triple (N, e, R); as the set of agents N and the endowment allocation e remain fixed throughout, we will simply denote market (N, e, R) by R. Thus, the strict preference profile domain $\mathcal{R}^N$ also denotes the *set of all markets*.

Mechanisms and properties

Note that all following definitions for the domain of strict preferences $\mathcal{R}$ can be formulated for the domain of separable preferences $\mathcal{R}_s$ or the domain of lexicographic preferences $\mathcal{R}_l$.

A mechanism is a function $f : \mathcal{R}^N \to X$ that assigns to each market $R \in \mathcal{R}^N$ an allocation $f(R) \in X$, and

- $f_i(R)$ is agent i's allotment
- $f_i^t(R)$ is agent i's type-t allotment

under mechanism f at R.

We next introduce and discuss some well-known properties for allocations and mechanisms. Let $R \in \mathcal{R}$.

First we consider a voluntary participation condition for an allocation x to be implementable without causing agents any harm: no agent will be worse off than at his endowment.

Definition 1 (**Individual rationality**).
An allocation $x \in X$ is *individually rational* if for each agent $i \in N$, $x_i \, R_i \, o_i$. A mechanism is *individually rational* if for each market, it assigns an individually rational allocation.

Next, we consider two well-known efficiency criteria.

Definition 2 (**Pareto efficiency**).
An allocation $y \in X$ *Pareto dominates* allocation $x \in X$ if for each agent $i \in N$, $y_i \, R_i \, x_i$, and for at least one agent $j \in N$, $y_j \, P_j \, x_j$. An allocation $x \in X$ is *Pareto efficient* if there is no allocation $y \in X$ that Pareto dominates it. A mechanism is *Pareto efficient* if for each market, it assigns a Pareto efficient allocation.

[9]Note that when extending strict preferences over allotments to preferences over allocations without consumption externalities, strictness is lost because an agent is indifferent between any two allocations where he gets the same allotment.

Definition 3 (**Unanimity**).
An allocation $x \in X$ is *unanimously best* if for each agent $i \in N$ and each allocation $y \in X$, we have $x \; R_i \; y$.[10] A mechanism is *unanimous* if for each market, it assigns only the unanimously best allocation if it exists.

Since a unanimously best allocation is the only Pareto efficient allocation, *Pareto efficiency* implies *unanimity*.

Next, we introduce a weaker condition than *unanimity* that guarantees that no allocation is a priori excluded.

Definition 4 (**Ontoness**).
A mechanism is *onto* if each allocation is assigned to some markets. In other words, a mechanism is *onto* if it is an onto function.

It is immediate that *unanimity* implies *ontoness* (see also Lemma 1).

The next two properties are incentive properties that model that no agent / coalition can benefit from misrepresenting his / their preferences.

Definition 5 (**Strategy-proofness**).
A mechanism f is *strategy-proof* if for each $R \in \mathcal{R}^N$, each agent $i \in N$, and each preference relation $R'_i \in \mathcal{R}$, $f_i(R_i, R_{-i}) \; R_i \; f_i(R'_i, R_{-i})$.

Definition 6 (**Group strategy-proofness**).
A mechanism f is *group strategy-proof* if for each $R \in \mathcal{R}^N$, there is no coalition $S \subseteq N$ and no preference list $R'_S = (R'_i)_{i \in S} \in \mathcal{R}^S$ such that for each $i \in S$, $f_i(R'_S, R_{-S}) \; R_i \; f_i(R)$, and for some $j \in S$, $f_j(R'_S, R_{-S}) \; P_j \; f_j(R)$.

Finally, we consider a property for mechanisms that restricts each agent's influence: no agent can change other agents' allotments without changing his own allotment.[11]

Definition 7 (**Non-bossiness**).
A mechanism f is *non-bossy* if for each $R \in \mathcal{R}^N$, each agent $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R_i, R_{-i}) = f_i(R'_i, R_{-i})$, then $f(R_i, R_{-i}) = f(R'_i, R_{-i})$.

We already mentioned that *unanimity* implies *ontoness*. We next show that, in the presence of *strategy-proofness* and *non-bossiness*, *ontoness* implies *unanimity*.

Lemma 1. *On the domain of strict preferences $\mathcal{R}$ ($\mathcal{R}_s$ / $\mathcal{R}_l$, respectively), if a mechanism is unanimous, then it is onto. If a mechanism is strategy-proof, non-bossy, and onto, then it is unanimous.*

[10] Since all preferences are strict, the set of unanimously best allocations can only be empty or single-valued.

[11] Alva (2017, Proposition 1) shows the equivalence of (a) *effective pairwise strategy-proofness* and (b) the combination of *strategy-proofness* and *non-bossiness*. Thus, his study provides an intuition why not only *strategy-proofness* but also *non-bossiness* can be considered to be an incentive property.

***Proof*.** Let f be a *unanimous* mechanism. Fix any allocation $x \in X$. Let $R \in \mathcal{R}^N$ be a preference profile such that x is *unanimously* best under R. Then, by *unanimity*, $f(R) = x$. Hence, f is an *onto* function.

Let f be *strategy-proof*, *non-bossy*, and *onto*. Let $x \in X$ and $R \in \mathcal{R}^N$ be a preference profile such that x is unanimously best under R. By *ontoness* of f, there exists a preference profile $R' \in \mathcal{R}$ such that $f(R') = x$. Let $i \in N$ and $y = f(R_i, R'_{-i})$. By *strategy-proofness* of f, we have $y_i \, R_i \, x_i$. Since x_i is agent i's most preferred allotment, we have $y_i = x_i$. Then, by *non-bossiness* of f, we have $f(R_i, R'_{-i}) = y = x = f(R')$. By applying this argument repeatedly for all agents in $N \setminus \{i\}$, we find that $f(R) = x = f(R')$. So, f is *unanimous*. □

Recall that the above definitions and result are valid for preference domains $\mathcal{R}$, $\mathcal{R}_s$, and $\mathcal{R}_l$.

We next focus on the domain of separable preferences $\mathcal{R}_s$ (the domain of lexicographic preferences $\mathcal{R}_l$, respectively) and extend Gale's famous top trading cycles (TTC) algorithm to multiple-type housing markets.

Definition 8 (The type-*t* top trading cycles (TCC) algorithm).
Consider a market (N, e, R) such that $R \in \mathcal{R}_s$. For each type $t \in T$, let $(N, e^t, R^t) = (N, (o^t_1, \ldots, o^t_n), (R^t_1, \ldots, R^t_n))$ be its *associated type-t sub-market*.

For each type t, we define the top trading cycles (TTC) allocation for the type-t sub-market as follows.

Input. A type-t sub-market (N, e^t, R^t).

Step 1. Let $N_1 := N$ and $O^t_1 := O^t$. We construct a directed graph with the set of nodes $N_1 \cup O^t_1$. For each agent $i \in N_1$, there is an edge from the agent to his most preferred type-t object in O^t_1 according to R^t_i. For each edge (i, o) we say that agent i points to type-t object o. For each type-t object $o \in O^t_1$, there is an edge from the object to its owner.

A *trading cycle* is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and type-t objects. If there are some agents (and hence objects) left, we continue with the next step. Otherwise we stop.

Step *k*. Let N_k be the set of agents that remain after Step $k-1$ and O^t_k be the set of type-t objects that remain after Step $k-1$. We construct a directed graph with the set of nodes $N_k \cup O^t_k$. For each agent $i \in N_k$, there is an edge from the agent to his most preferred type-t object in O^t_k according to R^t_i. For each type-t object $o \in O^t_k$, there is an edge from the object to its owner. At least one trading cycle exists and we assign to each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and objects. If there are some agents (and hence objects) left, we continue with the next step. Otherwise we stop.

Output. The type-t TTC algorithm terminates when each agent in N is assigned an object in O^t, which takes at most n steps. We denote the object in O^t that agent $i \in N$ obtains in the type-t TTC algorithm by $TTC_i^t(R^t)$ and the final type-t allocation by $TTC^t(R^t)$.

Definition 9 (cTTC allocations and the cTTC mechanism).
The *coordinate-wise top trading cycles (cTTC) allocation*, $cTTC(R)$, is the collection of all type-t TTC allocations, i.e., for each $R \in \mathcal{R}_s^N$,

$$cTTC(R) = \left(\left(TTC_1^1(R^1), \ldots, TTC_1^m(R^m)\right), \ldots, \left(TTC_n^1(R^1), \ldots, TTC_n^m(R^m)\right) \right).$$

The *cTTC mechanism* (introduced by Wako, 2005) assigns to each market $R \in \mathcal{R}_s^N$ its cTTC allocation.

Shapley-Scarf housing market results

As mentioned before, for $m = 1$ our model equals the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974) and the cTTC mechanism reduces to the standard TTC mechanism. The Shapley-Scarf housing market (with strict preferences) results that are pertinent for our analysis of multiple-type housing markets are the following.

Result 1 (Bird (1984))**.**
The TTC mechanism is *group strategy-proof*.

Note that *group strategy-proofness* implies *strategy-proofness* and *non-bossiness*. Thus, Result 1 also implies that the TTC mechanism is *non-bossy* (Miyagawa, 2002, explicitly shows this). Also note that when preferences are strict and unrestricted, the combination of *strategy-proofness* and *non-bossiness* coincides with *group strategy-proofness*. Recently, Alva (2017) identifies a list of properties of domains for which this equivalence holds.

Result 2 (Pápai (2000); Takamiya (2001); Alva (2017))**.**
A mechanism is *strategy-proof* and *non-bossy* if and only if it is *group strategy-proof*.

Result 3 (Ma (1994); Svensson (1999))**.**
A mechanism is *Pareto efficient*, *individually rational*, and *strategy-proof* if and only if it is the TTC mechanism.

Result 4 (Takamiya (2001))**.**
A mechanism is *onto*, *individually rational*, *strategy-proof*, and *non-bossy* if and only if it is the TTC mechanism.

Extension of existing Shapley-Scarf housing market results to multiple-type housing markets

The results in the previous subsection imply that for Shapley-Scarf housing markets, the TTC mechanism satisfies

- *Pareto efficiency*, *unanimity*, and *ontoness*;
- *individual rationality*; and
- *strategy-proofness*, *non-bossiness*, and *group strategy-proofness*.

The cTTC mechanism inherits most of these properties, except for *Pareto efficiency* and *group strategy-proofness*. Hence, TTC Results 1, 2, and 3 do not extend to the cTTC mechanism when more than one object types is allocated.

Proposition 1. *On the domain of separable preferences $\mathcal{R}_s$ ($\mathcal{R}_l$, respectively), the cTTC mechanism satisfies unanimity, ontoness, individual rationality, strategy-proofness, and non-bossiness. The cTTC mechanism satisfies neither Pareto efficiency nor group strategy-proofness.*

Proof. It is straightforward to check that the cTTC mechanism is *individually rational* and *unanimous* (and hence *onto*).

We next show that the cTTC mechanism inherits *strategy-proofness* from the TTC mechanism. Let $R \in \mathcal{R}_s^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_s$ with marginal preferences $(\hat{R}_i^1, \ldots, \hat{R}_i^m)$. By the definition and *strategy-proofness* of the TTC mechanism, for each $t \in T$, $cTTC_i^t(R) = TTC_i^t(R^t)\ R_i^t\ TTC_i^t(\hat{R}_i^t, R_{-i}^t) = cTTC_i^t(\hat{R}_i, R_{-i})$. Then, by the separability of preferences, we have $cTTC_i(R)\ R_i\ cTTC_i(\hat{R}_i, R_{-i})$ and the cTTC mechanism is *strategy-proof*.

Finally, to show that the cTTC mechanism is *non-bossy*, let $R \in \mathcal{R}_s^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_s$, with marginal preferences $(\hat{R}_i^1, \ldots, \hat{R}_i^m)$, be such that $cTTC_i(R) = cTTC_i(\hat{R}_i, R_{-i})$. Thus, for each $t \in T$, $cTTC_i^t(R) = cTTC_i^t(\hat{R}_i, R_{-i})$. Moreover, by definition of the cTTC mechanism, we have for each $t \in T$, $cTTC_i^t(R) = TTC_i(R^t)$ and $cTTC_i^t(\hat{R}_i, R_{-i}) = TTC_i(\hat{R}_i^t, R_{-i}^t)$. Thus, for each $t \in T$, $TTC_i(R^t) = TTC_i(\hat{R}_i^t, R_{-i}^t)$, and since the TTC mechanism is *non-bossy*, we have that for each $t \in T$, $TTC(R^t) = TTC(\hat{R}_i^t, R_{-i}^t)$. Then, for each $t \in T$, $cTTC^t(R) = cTTC^t(\hat{R}_i, R_{-i})$. Thus, $cTTC(R) = cTTC(\hat{R}_i, R_{-i})$ and the cTTC mechanism is *non-bossy*.

Example 1 below shows that the cTTC mechanism is neither *Pareto efficient* nor *group strategy-proof*. □

Example 1 (*cTTC* is neither *Pareto efficient* nor *group strategy-proof*).
Consider the market with $N = \{1, 2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, and where each agent i's endowment is (H_i, C_i). The preference profile $R \in \mathcal{R}_l^N$ is as follows:

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, \boldsymbol{C_1}, C_2,$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, \boldsymbol{H_2}, H_1.$$

Thus, agent 1, who primarily cares for houses, would like to trade houses but not cars and agent 2, who primarily cares about cars, would like to trade cars but not houses. One easily verifies that $cTTC(R) = \big((H_1, C_1), (H_2, C_2)\big)$, the no trade outcome. However, note that since preferences are lexicographic, both agents would be strictly better off if they traded cars and houses. Thus, allocation $\big((H_2, C_2), (H_1, C_1)\big)$ Pareto dominates $cTTC(R)$. Hence, $cTTC$ is not *Pareto efficient*. Furthermore, assume that both agents (mis)report their preferences as follows:

$$\boldsymbol{R'_1} : H_2, \boldsymbol{H_1}, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R'_2} : C_1, \boldsymbol{C_2}, H_1, \boldsymbol{H_2}.$$

Then, $cTTC(R') = \big((H_2, C_2), (H_1, C_1)\big)$, making both agents better off compared to $cTTC(R)$. Hence, $cTTC$ is not *group strategy-proof*. ◇

Example 1 shows that the cTTC mechanism does not satisfy the three properties that were used in Result 3. Is there another mechanism that does satisfy the three properties? The following result gives an answer in the negative: there is no mechanism that satisfies *Pareto efficiency*, *individual rationality*, and *strategy-proofness*, neither on the domain of separable preferences nor on the domain of lexicographic preferences.

Result 5 (Impossible trinity)**.**

(a) On the domain of separable preferences $\mathcal{R}_s$, there is no mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof* (Konishi et al., 2001, Proposition 4.1).

(b) On the domain of lexicographic preferences $\mathcal{R}_l$, there is no mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof* (Sikdar et al., 2017, Theorem 2).

Result 5 implies that there is no other mechanism that does better than the cTTC mechanism by satisfying the three properties on either the domain of separable preferences or the domain of lexicographic preferences. However, the cTTC mechanism does satisfy all the properties used in Result 4. In the next section we answer the question if Takamiya's characterization of the TTC mechanism for Shapley-Scarf housing markets can be extended to characterize the cTTC mechanism for multiple-type housing markets.

Finally, Proposition 1 also demonstrates that the equivalence of *strategy-proofness* and *non-bossiness* with *group strategy-proofness* (Result 2) does not extend to multiple-type housing markets with separable or lexicographic preferences (because *strategy-proofness* and *non-bossiness* do not imply *group strategy-proofness*).

3 Characterizing the cTTC mechanism

From now on, we focus on the multiple-type extension of the Shapley-Scarf housing market model as introduced by Moulin (1995), where $|N| = n \geq 2$ and $|T| = m \geq 2$.[12]

3.1 Characterizing the cTTC mechanism for lexicographic preferences

We first show that Takamiya's result (Takamiya, 2001, Corollary 4.16) can indeed be extended to characterize the cTTC mechanism for lexicographic preferences.

Theorem 1. *On the domain of lexicographic preferences $\mathcal{R}_l$, a mechanism is*

(a) *onto, individually rational, strategy-proof, and non-bossy*

(b) *unanimous, individually rational, strategy-proof, and non-bossy*

if and only if it is the cTTC mechanism.

We prove Theorem 1 in Appendix A.2.

Note that even if one does not consider the domain of lexicographic preferences as an interesting or relevant preference domain for multiple-type housing markets, Theorem 1 serves as an important stepping stone to establish the corresponding characterization of the cTTC mechanism on the domain of separable preferences, see Subsection 3.2.

The following examples establish the logical independence of the properties in Theorem 1. We label the examples by the property/properties that is/are not satisfied.

Example 2 (***Ontoness*** **and** ***unanimity***)**.**
The no trade mechanism that always assigns the endowment allocation to each market is *individually rational*, *(group) strategy-proof*, and *non-bossy*, but neither *onto* nor *unanimous*. ◇

Example 3 (***Individual rationality***)**.**
By ignoring property rights that are established via the endowments, we can easily adjust the well-known mechanism of serial dictatorship to our setting: based on an ordering of agents, we let agents sequentially choose their allotments. Serial dictatorship mechanisms have been shown in various resource allocation models to satisfy *Pareto efficiency* (and hence *ontoness* and *unanimity*), *strategy-proofness*, and *non-bossiness*; since property rights are ignored, they violate *individual rationality* (e.g., see Monte and Tumennasan, 2015, Theorem 1). ◇

[12] One agent multiple-type housing market problems are rather trivial since no trade occurs and for just one object type, we are back to the Shapley-Scarf housing market model.

Example 4 (*Strategy-proofness*).

We adapt so-called Multiple-Serial-IR mechanisms introduced by Biró et al. (2021) for their circulation model to our multiple-type housing markets model. A Multiple-Serial-IR mechanism is determined by a fixed order of the agents. At any preference profile and following the order, the rule lets each agent pick her most preferred allotment from the available objects such that this choice together with previous agents' choices is compatible with an *individually rational* allocation. Formally,

Input. An order $\delta = (i_1, \ldots, i_n)$ of the agents and a multiple-type housing market $R \in \mathcal{R}_l^N$.

Step 0. Let $Y(0)$ be the set of individually rational allocations in X.

Step 1. Let Y_1 be the set of agent i_1's allotments that are compatible with some allocation in $Y(0)$, i.e., Y_1 consists of all $y_{i_1} \in \Pi_{t \in T} O^t$ for which there exists an allocation $x \in Y(0)$ such that $x_{i_1} = y_{i_1}$.

Let $y^*_{i_1}$ be agent i_1's most preferred allotment in Y_1, i.e., for each $y_{i_1} \in Y_1$, $y^*_{i_1} \; R_i \; y_{i_1}$.

Let $Y(1) \subseteq Y(0)$ be the set of allocations in $Y(0)$ that are compatible with $y^*_{i_1}$, i.e., $Y(1)$ consists of all $x \in Y(0)$ with $x_{i_1} = y^*_{i_1}$.

Step $\boldsymbol{k = 2, \ldots, n}$. Let Y_k be the set of agent i_k's allotments that are compatible with some allocation in $Y(k-1)$.

Let $y^*_{i_k}$ be agent i_k's most preferred allotment in Y_k.

Let $Y(k) \subseteq Y(k-1)$ be the set of allocations in $Y(k-1)$ that are compatible with $y^*_{i_k}$.

Output. The allocation of the Multiple-Serial-IR mechanism associated with δ at R is $MSIR(\delta, R) \equiv (y^*_1, y^*_2, \ldots, y^*_n)$.

Given an order δ, the associated Multiple-Serial-IR mechanism Δ assigns to each market R the allocation $\Delta(R) \equiv MSIR(\delta, R)$.

Biró et al. (2021) show that Multiple-Serial-IR mechanisms are *individually rational* and *Pareto efficient*.

Next, we show that Multiple-Serial-IR mechanisms are *non-bossy*. Let $\delta = (i_1, \ldots, i_n)$ be an order of the agents and let Δ denote the associated Multiple-Serial-IR mechanism.

Let $R \in \mathcal{R}_l^N$, $i \in N$, and $R'_i \in \mathcal{R}_l$. Let $R' \equiv (R'_i, R_{-i})$, $x \equiv \Delta(R)$, and $y \equiv \Delta(R')$. Assume $y_i = x_i$. We show that $y = x$.

Let $i_k \equiv i$. Since $y_i = x_i$ and for each $\ell = 2, \ldots, k-1, k+1, \ldots, n$, $R'_{i_\ell} = R_{i_\ell}$, agent i_1's choice at Step 1 under R' is restricted in the same way as agent i_1's choice at Step 1 under R. Thus, since $R'_{i_1} = R_{i_1}$, we have $y_{i_1} = x_{i_1}$. Similar arguments show that for each $\ell = 2, \ldots, k-1, k+1, \ldots, n$, $y_{i_\ell} = x_{i_\ell}$. Hence, Δ is non-bossy.

In the context of their circulation model, Biró et al. (2021, Example 5) show that Multiple-Serial-IR mechanisms are not *strategy-proof*. In the context of multiple-type housing markets, Konishi et al. (2001) show that there is no mechanism that is *Pareto efficient*, *individually* *rational*, and *strategy-proof*. Since Multiple-Serial-IR mechanisms are *Pareto efficient* and *individually rational*, they are not *strategy-proof*. We include a simple illustrative example for $n = 2$ agents and $m = 2$ types for completeness.

Let $N = \{1, 2\}$ and $T = \{H(ouse), C(ar)\}$. For each $i \in N$, let (H_i, C_i) be agent i's endowment. Let $R \in \mathcal{R}_l^N$ be given by

$$\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : H_1, \boldsymbol{H_2}, \boldsymbol{C_2}, C_1.$$

Consider the Multiple-Serial-IR mechanism Δ induced by $\delta = (1, 2)$, i.e., agent 1 moves first (note that since there are only two agents, when agent 1 picks his allotment, the final allocation is completely determined). Since allocation $x \equiv ((H_2, C_2), (H_1, C_1))$ is *individually rational* at R and $x_1 = (H_2, C_2)$ is agent 1's most preferred allotment, $\Delta(R) = x$.

Next, consider $\boldsymbol{R'_2} : \boldsymbol{C_2}, C_1, H_1, \boldsymbol{H_2}$. Note that at (R_1, R'_2), only $y \equiv ((H_2, C_1), (H_1, C_2))$ and e are *individually rational*. Thus, agent 1 can only pick y_1 or o_1. Since $y_1 \; R_1 \; o_1$, agent 1 picks y_1 and hence $\Delta(R_1, R'_2) = y$. Finally, we see that $y_2 \; R_2 \; x_2$, which implies that agent 2 has an incentive to misreport R'_2 at R. Hence, the Multiple-Serial-IR mechanism induced by $\delta = (1, 2)$ is not *strategy-proof*. $\diamond$

Note that if $n = 2$, then any mechanism is *non-bossy*. Thus, for our last independence example, we assume $n > 2$.

Example 5 (*Non-bossiness*).
We first provide an example of a mechanism for $n = 3$ and $m = 1$. Let $N = \{1, 2, 3\}$ and $T = \{H(ouse)\}$. Let $R \in \mathcal{R}^N$. We say that agents 1 and 3 are *in conflict* if H_2 is the most preferred object for both R_1 and R_3. Similarly, we say that agents 1 and 2 are *in conflict* if H_3 is the most preferred object for both R_1 and R_2. Let mechanism f be defined as follows: for each $R \in \mathcal{R}^N$,

- **(a)** if agents 1 and 2 are in conflict, then (i) transform R_2 to $\bar{R}_2$ by dropping H_3 to the bottom, i.e., $\bar{R}_2 : \ldots, H_3$, while keeping the relative order of H_1 and H_2, and (ii) set $f(R) \equiv TTC(R_1, \bar{R}_2, R_3)$;
- **(b)** if agents 1 and 3 are in conflict, then (i) transform R_3 to $\bar{R}_3$ by dropping H_2 to the bottom, i.e., $\bar{R}_3 : \ldots, H_2$, while keeping the relative order of H_1 and H_3, and (ii) set $f(R) \equiv TTC(R_1, R_2, \bar{R}_3)$;
- **(c)** if agent 1 is not in conflict with either agent 2 or agent 3, then $f(R) \equiv TTC(R)$.

It is easy to verify that f is *individually rational* and *unanimous*. We prove that f is *strategy-proof* in Appendix A.3. To see that f is *bossy*, let R be such that

$$\boldsymbol{R_1} : H_3, \boldsymbol{H_1}, H_2,$$

$$\boldsymbol{R_2} : H_3, \boldsymbol{H_2}, H_1,$$

$$\boldsymbol{R_3} : H_2, \boldsymbol{H_3}, H_1.$$

Then, since agents 1 and 2 are in conflict, we have $\boldsymbol{\bar{R}_2} : \boldsymbol{H_2}, H_1, H_3$ and $f(R) = TTC(\bar{R}_2, R_{-2})$. In particular, for $i = 1, 2, 3$, $f_i(R) = H_i$. Next consider $\boldsymbol{R'_1} : \boldsymbol{H_1}, \ldots$. Then, $f(R'_1, R_{-1}) = TTC(R'_1, R_{-1})$. In particular, $f_1(R'_1, R_{-1}) = H_1$, $f_2(R'_1, R_{-1}) = H_3$, and $f_3(R'_1, R_{-1}) = H_2$. Therefore, $f_1(R'_1, R_{-1}) = H_1 = f_1(R)$, $f_2(R'_1, R_{-1}) = H_3 \neq H_2 = f_2(R)$, and $f_3(R'_1, R_{-1}) = H_2 \neq H_3 = f_3(R)$. Hence, f is *bossy* (and not *Pareto efficient*).

Next, we extend mechanism f from $n = 3$ to any $n > 3$. Let $n > 3$. Let mechanism g be defined as follows: for each $R \in \mathcal{R}^N$,

Case (A) if some agent $i \in \{4, \ldots, n\}$ finds some object different from her endowment acceptable, then set $g(R) \equiv TTC(R)$;

Case (B) if each agent $i \in \{4, \ldots, n\}$ only finds her own endowment acceptable, then

- let $N' \equiv \{1, 2, 3\}$ and for each $i \in N'$, let $g_i(R) \equiv f_i(R_{|N'})$ where $R_{|N'}$ denotes the preferences of agents in N' restricted to $\{H_1, H_2, H_3\}$;
- for each agent $i \in \{4, \ldots, n\}$, $g_i(R) \equiv H_i$.

Since f and TTC are *individually rational* and *unanimous*, g is *individually rational* and *unanimous*. Since f is *bossy*, g is *bossy* as well.

Next, we show that g is *strategy-proof*. First, we verify that no agent $i \in \{4, \ldots, n\}$ can profitably misreport her preferences. If R is in case (A), then a misreport by agent i that creates another profile in case (A) does not lead to a more preferred allotment because TTC is *strategy-proof*; a misreport that creates a profile in case (B) assigns endowment H_i to agent i. In either case, the misreport does not yield a more preferred allotment for agent i. If R is in case (B), then each agent $i \in \{4, \ldots, n\}$ obtains her most preferred object (her own endowment) and hence cannot gain by misreporting her preferences.

Second, no agent in $\{1, 2, 3\}$ can "move" R from case (A) to case (B). If R is in case (A), no agent in $\{1, 2, 3\}$ can profitably misreport her preferences because TTC is *strategy-proof*. If R is in case (B), no agent in $\{1, 2, 3\}$ can profitably misreport her preferences because f is *strategy-proof*. Hence, g is *strategy-proof*.

Finally, we extend rule g from Shapley-Scarf housing markets to multiple-type housing markets with lexicographic (or separable) preferences by applying it coordinate-wise to all object types. Let h be the mechanism that assigns the objects of each type t according to g. Then, h is *unanimous* (and hence *onto*), *individually rational*, and *strategy-proof*, but *bossy*. ◇

3.2 Characterizing the cTTC mechanism for separable preferences

Note that for lexicographic preferences, under the cTTC mechanism, the importance order of types plays no role because the allocation of each type only depends on the agents' marginal preferences of each type, i.e., for each market R and type t, $cTTC^t(R) = TTC(R_1^t, \ldots, R_n^t)$. Thus, one could conjecture that Theorem 1 also holds on the domain of separable preferences. This conjecture is correct.

Theorem 2. *On the domain of separable preferences $\mathcal{R}_s$, a mechanism is*

(a) *onto, individually rational, strategy-proof, and non-bossy*

(b) *unanimous, individually rational, strategy-proof, and non-bossy*

if and only if it is the cTTC mechanism.

We prove Theorem 2 in Appendix A.4. Examples 2, 3, 4, and 5 are well defined on the domain of separable preferences and establish the logical independence of the properties in Theorem 2.

3.3 An impossibility for strict preferences

Note that the cTTC mechanism is not well-defined on the domain of strict preferences since for non-separable preferences, marginal type preferences cannot be derived. Then, a natural question is if there exists an extension of the cTTC mechanism to the strict preference domain that satisfies our properties. First, observe that the impossibility trinity result (Result 5) implies that on the domain of strict preferences, no mechanism satisfies *Pareto efficiency*, *individual rationality*, *strategy-proofness*, and *non-bossiness*. Our next result shows that weakening *Pareto efficiency* to *unanimity* (or *ontoness*) cannot resolve this impossibility, i.e., there is no extension of the cTTC mechanism to the strict preference domain that satisfies our properties.

Theorem 3. *On the domain of strict preferences $\mathcal{R}$, there is no mechanism satisfying*

(a) *ontoness, individual rationality, strategy-proofness, and non-bossiness;*

(b) *unanimity, individual rationality, strategy-proofness, and non-bossiness.*

Proof. **(a)** Let $f : \mathcal{R} \to X$ be *onto*, *individually rational*, *strategy-proof*, and *non-bossy*. First, since the domain of strict preferences is "rich," *strategy-proofness* and *non-bossiness* together imply *group strategy-proofness* (see Alva, 2017, Theorem 1, for details). Thus, f is *group strategy-proof*. Second, *group strategy-proofness* and *ontoness* together imply *Pareto efficiency*. Thus, f is also *Pareto efficient*.[13] Therefore, by the impossible trinity (Result 5 for strict preferences), f is not *individually rational*. **(b)** follows from **(a)** and Lemma 1. □

[13] See Takamiya (2001, Lemma 3.5) and Pápai (2001, Lemma 2) for details.

4 Discussion

Shapley-Scarf housing markets

Our results (Theorem 1 and Theorem 2) can be compared to Takamiya (2001, Corollary 4.16) for Shapley-Scarf housing markets. In the proof of Theorem 1 we make explicit use of the steps used by the TTC algorithm to compute the TTC allocation. In contrast, Takamiya's original proof is based on *strict core-stability*.[14] His proof is not based on the TTC algorithm but the absence of weak blocking coalitions and profitable coalitional deviations: essentially his proof consists of two steps: (1) *strict core-stability* implies *group strategy-proofness* and (2) *group strategy-proofness* and *ontoness* together imply *Pareto efficiency*. Since the cTTC mechanism neither satisfies *Pareto efficiency* nor *group strategy-proofness*, our results and proof strategy are logically independent. Moreover, Takamiya's proof strategy cannot be extended to multiple-type housing markets because weak blocking coalitions and profitable coalitional deviations need not coincide (see Feng and Klaus, 2022, for details).

Furthermore, comparing the classical TTC characterization by Ma (1994) with that of Takamiya (2001) yields the following result. For Shapley-Scarf housing markets, an *individually rational* and *strategy-proof* mechanism is *Pareto efficient* if and only if it is *unanimous* and *non-bossy*. However, this result does not extend to multiple-type housing markets, as illustrated in Example 1 which shows that $cTTC$ is not Pareto efficient (recall that there the no trade allocation $cTTC(R) = ((H_1, C_1), (H_2, C_2))$ is Pareto dominated by the full trade allocation $((H_2, C_2), (H_1, C_1)))$.

Object allocation problems with multi-demand and without ownership

Our results can be compared to Monte and Tumennasan (2015) and Pápai (2001) for object allocation problems with multi-demand and without ownership, i.e., agents can consume more than one object, and the set of objects is a social endowment.

While Monte and Tumennasan (2015) still assume that objects are of different types and agents can only consume one object of each type, Pápai (2001) imposes no consumption restriction.[15] Although both models are slightly different, their characterization results are similar: the only mechanisms satisfying *strategy-proofness*, *non-bossiness*, and *Pareto efficiency* are sequential dictatorships. Clearly, if agents, like in our model, have property rights, sequential dictatorship mechanisms will not satisfy *individual rationality*. Thus, their characterization results imply an impossibility result for our model, in line with our Theorem 3; however, note that our efficiency notion in Theorem 3, *unanimity*, is weaker than *Pareto efficiency*.

[14] A mechanism is *strictly core-stable* if it always assigns a strict core allocation.

[15] In Pápai (2001), agents can consume any set of objects, and their preferences are linear orders over all sets of objects.

Object allocation problems with multi-demand and with ownership

Finally, we compare our results (Theorem 1 and Theorem 2) to Pápai (2003).

Similarly to Pápai (2001), Pápai (2003) considers a more general model of allocating objects to the set of agents who can consume any set of objects. In contrast to Pápai (2001), each object now is owned by an agent and each agent has strict preferences over all objects, and his preferences over sets of objects are monotonically responsive to these "objects-preferences."[16] In our model, we impose more structure by assuming that (i) the set of objects is partitioned into sets of exogenously given types and (ii) each agent owns and wishes to consume one object of each type.

Pápai (2003) considers *strategy-proofness*, *non-bossiness* (as we do) and she introduces two additional (non-standard) properties: *trade sovereignty* and *strong individual rationality*. *Trade sovereignty* requires that every feasible allocation that consists of "admissible transactions" should be realized at some preference profile; it allows for trade restrictions and some objects never being traded and is hence weaker than *ontoness* (for details see Pápai, 2003). *Strong individual rationality* requires that for each agent and all preference relations with the same objects-preferences as the agent has, *individual rationality* holds (for details see Pápai, 2003). Note that *strong individual rationality* is stronger than *individual rationality*. For instance, if agent 1's endowment is (H_1, C_1), and his objects-preferences are $\boldsymbol{R_1} : H_2, \boldsymbol{H_1}, \boldsymbol{C_1}, C_2$, then allotment (H_2, C_2) is not *strongly individually rational*.[17]

Pápai (2003) shows that the set of mechanisms satisfying *trade sovereignty*, *strong individual rationality*, *strategy-proofness*, and *non-bossiness* coincides with the set of segmented trading cycle mechanisms. In this class of mechanisms, all objects are (endogenously) decomposed into different segments that can be expressed as the components of a trading possibility graph (which can express trading restrictions that can even mean that certain objects cannot be traded). Agents can own at most one object per segment and the TTC algorithm is then executed separately for each segment. The set of segmented trading cycle mechanisms is large and, for our model, would include the cTTC mechanism, the no trade rule, and many segmented trading cycles mechanisms with restricted trades.

[16]Formally, let O be a finite set of objects. A preference relation $\succeq$ over all non-empty sets of objects is *monotonically responsive* if (i) it is monotonic, i.e., for any two non-empty subsets of objects, $A, B \subseteq O$, $A \subseteq B$ implies that $B \succeq A$; and (ii) *responsive*, i.e., there exists a strict "objects-preference relation" over all objects, R, such that for any two distinct objects $o, o' \in O$, and a subset of objects $A \subseteq O \setminus \{o, o'\}$, $o \, P \, o'$ implies that $\{o\} \cup A \succ \{o'\} \cup A$. In our model, that since agents' allotments have a fixed number of objects, monotonicity of preferences over sets of objects plays no role. Furthermore, lexicographic preferences are responsive, but not all separable preferences may induce objects-preferences, i.e., separable preferences need not be responsive. Vice versa, Pápai's responsiveness condition implies separability.

[17]Let $\widetilde{\succ}_1 : (H_2, C_1), (H_1, C_1), (H_2, C_2), (H_1, C_2)$ and $\widehat{\succ}_1 : (H_2, C_1), (H_2, C_2), (H_1, C_1), (H_1, C_2)$. Note that both preferences are responsive to R_1. We see that $(H_2, C_2) \, \widehat{\succ}_1 \, (H_1, C_1)$ but $(H_1, C_1) \, \widetilde{\succ}_1 \, (H_2, C_2)$. Thus, (H_2, C_2) is *individually rational* at $\widehat{\succ}_1$ but *not individually rational* at $\widetilde{\succ}_1$.

The cTTC mechanism is a specific segmented trading cycle mechanism in the sense that all segments are a priori determined by object types. Thus, our characterization result of the cTTC mechanism can be seen as characterizing a specific segmented trading cycle mechanism while Pápai characterizes the whole class of segmented trading cycle mechanisms. On the one hand, we weaken *strong individual rationality* to *individual rationality* but strengthen *trade sovereignty* to *ontoness*. On the other hand, one of our preference domains, the domain of separable preferences, could be considered as a larger preference domain than Pápai's domain of monotonically responsive preferences (see Footnote 16). Therefore, while there is a close connection between our models and results, there is no direct logical relation between Pápai (2003)'s result and ours (Theorems 1 and 2).

A Appendix: proofs

A.1 Auxiliary properties and results

We introduce the well-known property of *(Maskin) monotonicity*, which requires that if an allocation is chosen, then that allocation will still be chosen if each agent shifts it up in his preferences. We formulate *monotonicity* as well as our first auxiliary result for the domain of strict preferences $\mathcal{R}$; however, we could use preference domains $\mathcal{R}_s$ and $\mathcal{R}_l$ instead.

Let $i \in N$. Given preferences $R_i \in \mathcal{R}$ and an allotment x_i, let $L(x_i, R_i) = \{y_i \in \Pi_{t \in T} O^t \mid x_i R_i y_i\}$ be the *lower contour set of R_i at x_i*. Preference relation R'_i is a *monotonic transformation of R_i at x_i* if $L(x_i, R_i) \subseteq L(x_i, R'_i)$. Similarly, given a preference profile $R \in \mathcal{R}^N$ and an allocation x, a preference profile $R' \in \mathcal{R}^N$ is a *monotonic transformation of R at x* if for each $i \in N$, R'_i is a monotonic transformation of R_i at x_i.

Definition 10 (**Monotonicity**).
A mechanism f is *monotonic* if for each $R \in \mathcal{R}^N$ and for each monotonic transformation $R' \in \mathcal{R}^N$ of R at $f(R)$, we have $f(R') = f(R)$.

We show that *strategy-proofness* and *non-bossiness* imply *monotonicity*.

Lemma 2. *On the domain of strict preferences $\mathcal{R}$ ($\mathcal{R}_s$ / $\mathcal{R}_l$, respectively), if a mechanism is strategy-proof and non-bossy, then it is monotonic.*

Proof. The proof is a straightforward extension of Takamiya (2001, Theorem 4.12) and Pápai (2001, Lemma 1). Let $R \in \mathcal{R}^N$ and let $x = f(R)$. Consider an agent $i \in N$ with a monotonic transformation $R'_i \in \mathcal{R}$ of R_i at x_i. Let $y = f(R'_i, R_{-i})$. By *strategy-proofness* of f, we have $x_i \, R_i \, y_i$, which implies that $y_i \in L(x_i, R_i) \subseteq L(x_i, R'_i)$. However, by *strategy-proofness* of f, we also have $y_i \, R'_i \, x_i$. Thus, since $y_i \in L(x_i, R'_i)$, $x_i = y_i$. Then, by *non-bossiness* of f, we have $x = y$. By applying this argument sequentially for all agents in $N \setminus \{i\}$ and any $R' \in \mathcal{R}^N$ that is a monotonic transformation of R at x, we find that $f(R) = x = f(R')$. □

The converse of Lemma 2 is in our model only true for the domain of strict preferences $\mathcal{R}$; the domain of lexicographic/separable preferences is not rich enough to satisfy Alva's preference domain richness condition (Alva, 2017, two-point connectedness).

Next, for preference domain $\mathcal{R}_l$, we introduce a "marginal version" of monotonic preference transformations; we could use preference domain $\mathcal{R}_s$ instead. Let $i \in N$. Given preferences $R_i \in \mathcal{R}_l$ and an allotment x_i, for each type t, consider the associated marginal preferences R_i^t and marginal allotment x_i^t. Let $L(x_i^t, R_i^t) = \{y_i^t \in O^t \mid x_i^t \; R_i^t \; y_i^t\}$ be the lower contour set of R_i^t at x_i^t. Marginal preference relation $\hat{R}_i^t$ is a monotonic transformation of R_i^t at x_i^t if $L(x_i^t, R_i^t) \subseteq L(x_i^t, \hat{R}_i^t)$.

Fact 1. Let x_i be an allotment. Let $R_i, \hat{R}_i$ be lexicographic preferences such that (1) $\pi_i = \hat{\pi}_i$ and (2) for each $t \in T$, $\hat{R}_i^t$ is a monotonic transformation of R_i^t at x_i^t. Then, $\hat{R}_i$ is a monotonic transformation of R_i at x_i.

***Proof*.** We show that $L(x_i, R_i) \subseteq L(x_i, \hat{R}_i)$. Let $y_i \in L(x_i, R_i)$ with $y_i \neq x_i$. Then, $x_i \; P_i \; y_i$. Restate y_i and x_i as $y_i^{\pi_i} = (y_i^{\pi_i(1)}, \ldots, y_i^{\pi_i(m)})$ and $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \ldots, x_i^{\pi_i(m)})$, respectively. Let k be the first type for which x_i and y_i assign different objects, i.e., for all $l < k$, $y_i^{\pi_i(l)} = x_i^{\pi_i(l)}$ and $y_i^{\pi_i(k)} \neq x_i^{\pi_i(k)}$. Since $x_i \; P_i \; y_i$ and preferences are lexicographic, we have $x_i^{\pi_i(k)} \; P_i^{\pi_i(k)} \; y_i^{\pi_i(k)}$. Thus, $y_i^{\pi_i(k)} \in L(x_i^{\pi_i(k)}, R_i^{\pi_i(k)}) \subseteq L(x_i^{\pi_i(k)}, \hat{R}_i^{\pi_i(k)})$, which implies that $x_i^{\pi_i(k)} \; \hat{P}_i^{\pi_i(k)} \; y_i^{\pi_i(k)}$. Then, since $\pi_i = \hat{\pi}_i$, $x_i \; \hat{P}_i \; y_i$, i.e., $y_i \in L(x_i, \hat{R}_i)$. □

Therefore, by *monotonicity*, if an agent receives an allotment and shifts each of its objects up in the marginal preferences (without changing his importance order), he still receives that allotment and the allocation of the other agents also does not change.

Next, for preference domain $\mathcal{R}_l$, we introduce a new property, *marginal individual rationality*, which is a stronger property than *individual rationality*; we could use preference domain $\mathcal{R}_s$ instead.

Definition 11 (**Marginal individual rationality**).
A mechanism f is *marginally individually rational* if for each $R \in \mathcal{R}_l^N$, each $i \in N$, and each $t \in T$, $f_i^t(R) \; R_i^t \; o_i^t$.

Note that while *marginal individual rationality* can also be defined on the domain of separable preferences $\mathcal{R}_s$, the following lemma and proof are established only for lexicographic preferences.

Lemma 3. *On the domain of lexicographic preferences $\mathcal{R}_l$, if a mechanism is unanimous, individually rational, strategy-proof, and non-bossy, then it is marginally individually rational.*

***Proof*.** Suppose mechanism f is *unanimous*, *individually rational*, *strategy-proof*, *non-bossy*, and not *marginally individually rational*, i.e., there is a preference profile $R \in \mathcal{R}_l^N$, an agent $i \in N$, and a type $t \in T$ such that $o_i^t \; P_i^t \; f_i^t(R)$. Then, by *individual rationality* of f, we know that $t \neq \pi_i(1)$.

Let $x \equiv f(R)$. Consider a preference profile $\hat{R} \in \mathcal{R}_l^N$ such that

for agent i,

- $\hat{R}_i^t : o_i^t, x_i^t, \ldots,$

for each $\tau \in T \setminus \{t\}$,

- $\hat{R}_i^\tau : x_i^\tau, \ldots,$ and
- $\hat{\pi}_i = \pi;$

and for each agent $j \in N \setminus \{i\}$,

for each $t \in T$,

- $\hat{R}_j^t : x_j^t, \ldots,$ and
- $\hat{\pi}_j = \pi_j.$

Note that, by Fact 1, $\hat{R}$ is a monotonic transformation of R at x. By Lemma 2, f is *monotonic*. Thus, $f(\hat{R}) = x$.

Next, consider a preference profile $(\bar{R}_i, \hat{R}_{-i}) \in \mathcal{R}_l^N$, where $\bar{R}_i$ is such that

for each $t \in T$,

- $\bar{R}_i^t = \hat{R}_i^t$, and
- $\bar{\pi}_i(1) = t.$

Note that $\bar{R}_i$ can be interpreted as a linear order over all objects such that $\bar{R}_i : o_i^t, \ldots$, i.e., object o_i^t is the most preferred object.

Let $y \equiv f(\bar{R}_i, \hat{R}_{-i})$. By *individual rationality* of f, $y_i^t = o_i^t$. Thus, $y_i \neq x_i$. By *strategy-proofness* of f, $x_i = f(\hat{R}_i, \hat{R}_{-i}) \ \hat{P}_i \ f(\bar{R}_i, \hat{R}_{-i}) = y_i$. Since agent i gains in type t by misreporting (i.e., $y_i^t \ \hat{P}_i^t \ x_i^t$), he must lose in other more important types according to $\hat{\pi}_i$. That is, there is a type t' such that (1) $\hat{\pi}_i^{-1}(t') < \hat{\pi}_i^{-1}(t)$ and (2) $x_i^{t'} \ \hat{P}_i^{t'} \ y_i^{t'}$. In words, there exists a type t' that is more important than t according to $\hat{\pi}_i$; and $y_i^{t'}$, a type-t' object received when i misreports $\bar{R}_i$, is worse than $x_i^{t'}$ (according to $\hat{R}_i^{t'}$), which is received by reporting $\hat{R}_i$. In particular, $x_i^{t'} \neq y_i^{t'}$.

Next, consider a preference profile $\bar{R} \equiv (\bar{R}_i, \bar{R}_{-i})$ such that

for each agent $j \in N \setminus \{i\}$,

- $\bar{R}_j^t : y_j^t, \ldots,$

for each $\tau \in T \setminus \{t\}$,

- $\bar{R}_j^\tau = \hat{R}_i^\tau$, and

- $\bar{\pi}_j = \hat{\pi}_j$.

Note that the only relevant difference between $\bar{R}$ and $(\bar{R}_i, \hat{R}_{-i})$ is that under $\bar{R}$, each agent $j \neq i$ positions y_j^t as his most preferred type-t object. Thus, $\bar{R}$ is a monotonic transformation of $(\bar{R}_i, \hat{R}_{-i})$ at y. Therefore, by *monotonicity* of f, $f(\bar{R}) = y$.

However, under $\bar{R}$, for each agent $k \in N$, his most preferred allotment is $z_k = (x_k^1, \ldots, x_k^{t-1}, y_k^t, x_k^{t+1}, \ldots, x_k^m)$. Note that $z = (z_k)_{k \in N} \in X$ is an allocation because z is a mixture of y (for type t) and x (for other types). Thus, by *unanimity* of f, $f(\bar{R}) = z$. So, $y = z$. However, for type t', $z_i^{t'} = x_i^{t'} \neq y_i^{t'}$, a contradiction. □

A.2 Proof of Theorem 1

We are ready to prove our first characterization result (Theorem 1): on the domain of lexicographic preferences, a mechanism is *unanimous* (*onto*, respectively), *individually rational*, *strategy-proof*, and *non-bossy* if and only if it is the *cTTC* mechanism.

Proof. It is straightforward to check that the *cTTC* mechanism is *unanimous* (and hence *onto*) and *individually rational*.

To show that the *cTTC* mechanism is *non-bossy*, let $R \in \mathcal{R}_l^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_l$ with $\hat{R}_i \equiv (\hat{R}_i^1, \ldots, \hat{R}_i^m, \hat{\pi}_i)$ be such that $cTTC_i(R) = cTTC_i(\hat{R}_i, R_{-i})$. So, for each $t \in T$, $cTTC_i^t(R) = cTTC_i^t(\hat{R}_i, R_{-i})$. Moreover, by the definition of the *cTTC* mechanism, for each $t \in T$, $cTTC_i^t(R) = TTC_i(R^t)$ and $cTTC_i^t(\hat{R}_i, R_{-i}) = TTC_i(\hat{R}_i^t, R_{-i}^t)$. Since Miyagawa (2002) shows that for Shapley-Scarf housing markets the TTC mechanism is *non-bossy*, we have that for each $t \in T$, $TTC(R^t) = TTC(\hat{R}_i^t, R_{-i}^t)$. So, for each $t \in T$, $cTTC^t(R) = cTTC^t(\hat{R}_i, R_{-i})$. Thus, $cTTC(R) = cTTC(\hat{R}_i, R_{-i})$.

Finally, we show that the *cTTC* mechanism inherits *strategy-proofness* from the TTC mechanism (for Shapley-Scarf housing markets) as well. Let $R \in \mathcal{R}_l^N$, $i \in N$, and $\hat{R}_i \in \mathcal{R}_l$. Since for each $t \in T$, $cTTC_i^t(R) = TTC_i^t(R^t)\ R_i^t\ TTC_i^t(\hat{R}_i^t, R_{-i}^t) = cTTC_i^t(\hat{R}_i, R_{-i})$, we have $cTTC_i(R)\ R_i\ cTTC_i(\hat{R}_i, R_{-i})$. Therefore, *cTTC* satisfies all properties listed in Theorem 1.

Next, suppose that there is a mechanism $f : \mathcal{R}_l^N \to X$, different from the *cTTC* mechanism, that satisfies the properties listed in Theorem 1 (by Lemma 1, *ontoness* and *unanimity* can be used interchangeably). Then, there is a market R such that $y \equiv f(R) \neq cTTC(R) \equiv x$. In particular, there is a type t such that $(y_1^t, \ldots, y_n^t) \neq (x_1^t, \ldots, x_n^t)$.

By Lemma 2, both mechanisms, f and *cTTC*, are *monotonic*. By Lemma 3, both mechanisms, f and *cTTC*, are *marginally individually rational*. Since both mechanisms are *marginally individually rational*, for each $i \in N$ and each $\tau \in T$, $y_i^\tau R_i^\tau o_i^\tau$ and $x_i^\tau R_i^\tau o_i^\tau$. So, we can define a preference profile $\hat{R} \in \mathcal{R}_l^N$ such that for each agent $i \in N$,

- $\hat{R}_i^t : \begin{cases} x_i^t, y_i^t, o_i^t, \ldots \text{ if } & x_i^t \, R_i^t \, y_i^t \\ y_i^t, x_i^t, o_i^t, \ldots \text{ if } & y_i^t \, R_i^t \, x_i^t \end{cases}$

and for each $\tau \in T \setminus \{t\}$,

- $\hat{R}_i^\tau : y_i^\tau, o_i^\tau, \ldots$, and
- $\hat{\pi}_i = \pi_i$.

Note that, by Fact 1, $\hat{R}$ is a monotonic transformation of R at y. Since f is *monotonic*, $f(\hat{R}) = y$. Furthermore, since $\hat{R}^t$ is a monotonic transformation of R^t at x^t, *monotonicity* of the TTC mechanism implies $cTTC^t(\hat{R}) = TTC(\hat{R}^t) = x^t$.

Next, consider a preference profile $\bar{R} \in \mathcal{R}_l^N$ such that

for each $i \in N$,

- $\bar{R}_i^t : x_i^t, o_i^t, \ldots$,

for each $\tau \in T \setminus \{t\}$,

- $\bar{R}_i^\tau = \hat{R}_i^\tau$, and
- $\bar{\pi}_i = \pi_i$.

Note that the only relevant difference between $\bar{R}$ and $\hat{R}$ is that under $\bar{R}$, each agent $i \in N$ positions x_i^t as his most preferred type-t object and his endowment o_i^t as his second preferred.

Under $\bar{R}$, each agent i's most preferred allotment is $z_i \equiv (y_i^1, \ldots, y_i^{t-1}, x_i^t, y_i^{t+1}, \ldots, y_i^m)$. Note that $z = (z_i)_{i \in N} \in X$ is an allocation because z is a mixture of x (for type t) and y (for other types). Thus, by *unanimity* of f, $f(\bar{R}) = z$.

Recall that since $(x_1^t, \ldots, x_n^t) = cTTC^t(\hat{R}) = TTC(\hat{R}^t)$, $(x_1^t, \ldots, x_n^t)$ is obtained by applying the TTC algorithm to preference profile $\hat{R}^t$. For each $i \in N$, let s_i be the step of the TTC algorithm at which agent i receives object x_i^t. Without loss of generality, assume that if $i < i'$ then $s_i \leq s_{i'}$.

Next, we will show that $f(\hat{R}) = z$ by using that $f(\bar{R}) = z$ and replacing, step-by-step, each $\bar{R}_i$ with $\hat{R}_i$. More specifically, we will replace the individual preferences in the order $n, n-1, \ldots, 1$.

We first show that $f(\bar{R}_{-n}, \hat{R}_n) = z$. Suppose $x_n^t \, \hat{R}_n^t \, y_n^t$. Then, $(\bar{R}_{-n}, \hat{R}_n)$ is a monotonic transformation of $\bar{R}$ at z. By *monotonicity* of f, $f(\bar{R}_{-n}, \hat{R}_n) = f(\bar{R}) = z$.

Now suppose $y_n^t \, \hat{P}_n^t \, x_n^t$. Let $\tau \in T$ such that $\pi_n(\tau) = 1 < \pi_n(t)$ (if $\pi_n(t) = 1$, then skip this step). Since f is *strategy-proof*, preferences are lexicographic, and τ is the most important type for agent n, we have $f_n^\tau(\bar{R}_{-n}, \hat{R}_n) \, \hat{R}_n^\tau \, f_n^\tau(\bar{R})$. Since $\tau \neq t$, $f_n^\tau(\bar{R}) = z_n^\tau = y_n^\tau$ and $f_n^\tau(\bar{R}_{-n}, \hat{R}_n) \, \hat{R}_n^\tau \, y_n^\tau$.

Since $\tau \neq t$, it follows from the definition of $\hat{R}_n^\tau$ that y_n^τ is the best type-τ object. So, $f_n^\tau(\bar{R}_{-n}, \hat{R}_n) = y_n^\tau$. Now one can, sequentially, from more to less important types, apply similar arguments to show that

$$\text{for each type } t' \in T \text{ with } \pi_n(t') < \pi_n(t),\ f_n^{t'}(\bar{R}_{-n}, \hat{R}_n) = y_n^{t'} = f_n^{t'}(\bar{R}). \tag{1}$$

Since f is *marginally individually rational*, $f_n^t(\bar{R}_{-n}, \hat{R}_n) \in \{x_n^t, y_n^t, o_n^t\}$. Suppose $f_n^t(\bar{R}_{-n}, \hat{R}_n) = o_n^t$ and $o_n^t \neq x_n^t$. Then, $f_n^t(\bar{R}) = z_n^t = x_n^t\ \hat{P}_n^t\ o_n^t = f_n^t(\bar{R}_{-n}, \hat{R}_n)$, which together with (1) would contradict the *strategy-proofness* of f. Hence, $f_n^t(\bar{R}_{-n}, \hat{R}_n) \in \{x_n^t, y_n^t\}$.

Suppose that $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$. By the definition of the TTC algorithm, x_n^t is agent n's most preferred type-t object among the remaining objects at Step s_n of the TTC algorithm at preference profile $\hat{R}^t$. Therefore, object y_n^t is removed (i.e., assigned to some agent) at some Step $s^* < s_n$ of the TTC algorithm at preference profile $\hat{R}^t$.

Let C be the trading cycle of the TTC algorithm at preference profile $\hat{R}^t$ that contains y_n^t. Suppose C only contains one agent, say $j \neq n$. Then, among all objects present at Step s^*, agent j most prefers his own endowment, i.e., $o_j^t = y_n^t$. Hence, $x_j^t = cTTC_j^t(\hat{R}) = TTC_j(\hat{R}^t) = y_n^t = o_j^t$. So, by definition of $\bar{R}$, we have that at $(\bar{R}_{-n}, \hat{R}_n)$ agent j's marginal preferences of type t are given by $\bar{R}_j^t : o_j^t, \ldots$. By *marginal individual rationality* of f, $f_j^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$, which contradicts $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$.

Hence, C consists of agents $i_1, \ldots, i_K$ (with $K \geq 2$) and type-t objects $o_{i_1}^t, \ldots, o_{i_K}^t$ such that $n \notin \{i_1, \ldots, i_K\}$ and $y_n^t \in \{o_{i_1}^t, \ldots, o_{i_K}^t\}$. Note that at $(\bar{R}_{-n}, \hat{R}_n)$, for each $i_k \in \{i_1, \ldots, i_K\}$, agent i_k's marginal preferences of type t are $\bar{R}_{i_k}^t : o_{i_{k+1}}^t (= x_{i_k}^t), o_{i_k}^t, \ldots$ (modulo K). Without loss of generality, assume that $y_n^t = o_{i_1}^t$. It follows from $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t$ and *marginal individual rationality* of f that $f_{i_K}^t(\bar{R}_{-n}, \hat{R}_n) = o_{i_K}^t$. Subsequently, for each agent $i_k \in \{i_2, \ldots, i_K\}$, $f_{i_k}^t(\bar{R}_{-n}, \hat{R}_n) = o_{i_k}^t$. Therefore, $f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n) \neq o_{i_2}^t$. Moreover, $f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n) \neq o_{i_1}^t$ because $f_n^t(\bar{R}_{-n}, \hat{R}_n) = y_n^t = o_{i_1}^t$. Thus, $o_{i_1}^t\ \bar{P}_{i_1}\ f_{i_1}^t(\bar{R}_{-n}, \hat{R}_n)$, which violates *marginal individual rationality* of f. Therefore, $f_n^t(\bar{R}_{-n}, \hat{R}_n) \neq y_n^t$. Hence,

$$f_n^t(\bar{R}_{-n}, \hat{R}_n) = x_n^t = f_n^t(\bar{R}). \tag{2}$$

Having established (1) and (2), one can use arguments similar to those for (1) to show that

$$\text{for each type } t' \in T \text{ with } \pi_n(t') > \pi_n(t),\ f_n^{t'}(\bar{R}_{-n}, \hat{R}_n) = y_n^{t'} = f_n^{t'}(\bar{R}). \tag{3}$$

From (1), (2), and (3) it follows that for each type $\tau \in T$, $f_n^\tau(\bar{R}_{-n}, \hat{R}_n) = f_n^\tau(\bar{R})$. Hence, $f_n(\bar{R}_{-n}, \hat{R}_n) = f_n(\bar{R})$. By *non-bossiness* of f, $f(\bar{R}_{-n}, \hat{R}_n) = f(\bar{R}) = z$.

By applying repeatedly the same arguments for agents $i = n-1, \ldots, 1$, we can sequentially replace each $\bar{R}_i$ with $\hat{R}_i$, and conclude that $f(\hat{R}) = f(\bar{R}) = z$. However, since $(y_1^t, \ldots, y_n^t) \neq (x_1^t, \ldots, x_n^t)$, there exists an agent j such that $y_j^t \neq x_j^t$. Hence, $f_j^t(\hat{R}) = y_j^t \neq x_j^t = z_j^t$, a contradiction. $\square$

A.3 Proof of *strategy-proofness* in Example 5

We show that mechanism f defined in Example 5 for $n = 3$ and $m = 1$ is *strategy-proof*.

***Proof*.** Let $R \in \mathcal{R}^N$. We consider three cases.

Case 1. Preferences of agent 1 are $R_1 : H_1, \ldots$.
By *individual rationality* of f, $f_1(R) = H_1$ and since this is her most preferred object, agent 1 cannot gain by misreporting her preferences.

Let R_2' be some misreport of agent 2. Since agents 1 and 2 are not in conflict at R nor at (R_1, R_2', R_3), mechanism f yields the corresponding TTC allocations at R and (R_1, R_2', R_3). Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R. Similarly, agent 3 does not have a profitable deviation at R.

Case 2. Preferences of agent 1 are $R_1 : H_2, H_1, H_3$. (Since agents 2 and 3 play a symmetric role in the definition of f, similar symmetric arguments work for H_3, H_1, H_2.)
Agents 1 and 2 are not in conflict. Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R.

Next, we verify that agent 1 does not have a profitable deviation at R.

Case 2.a. Preferences of agent 2 are $R_2 : H_2, \ldots$.
Note that by *individual rationality* of f we have $f_2(R) = H_2$. So, $f_1(R) = H_1$. Reporting any other preferences will not give her H_2 either. So, agent 1 does not have a profitable deviation at R.

Case 2.b. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f(R)$ is the no-trade allocation. In particular, agent 1 receives her endowment H_1 at R. Obviously, misreporting $R_1' : H_1, \ldots$ gives her H_1. Any other misreport of agent 1's preferences yields the no-trade allocation. So, agent 1 does not have a profitable deviation at R.

Case 2.c. Preferences of agent 2 are $R_2 : H_1, \ldots$ or preferences of agent 2 are $R_2 : H_3, H_1, H_2$ or [preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are *not* $R_3 : H_2, H_3, H_1$].
It is easy but cumbersome to verify that $f_1(R) = H_2$, i.e., agent 1 receives her most preferred object H_2. So, agent 1 does not have a profitable deviation at R.

Finally, we verify that agent 3 does not have a profitable deviation at R.

Case 2.I. Preferences of agent 3 are $R_3 : H_3, \ldots$.
By *individual rationality* of f, $f_3(R) = H_3$ and since this is her most preferred object, agent 3 cannot gain by misreporting her preferences.

Case 2.II. Preferences of agent 3 are $R_3 : H_1, \ldots$.
Agents 1 and 3 are not in conflict and by *strategy-proofness* of TTC, agent 3 does not have a profitable deviation at R.

Case 2.III. Preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_3$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, if $R_2 : H_1, \ldots$ or $R_2 : H_2, \ldots$, then $f_3(R_1, R_2, R'_3) = H_3$; and if $R_2 : H_3, \ldots$, then $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 2.IV.i. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_1, \ldots$ or $R_2 : H_2, \ldots$.
Agents 1 and 3 are in conflict and for any possible deviation R'_3, $f_3(R_1, R_2, R'_3) = H_3 = f_3(R)$. Hence, agent 3 does not have a profitable deviation at R.

Case 2.IV.ii. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_3, \ldots$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_1$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 3. Preferences of agent 1 are $R_1 : H_2, H_3, H_1$. (Since agents 2 and 3 play a symmetric role in the definition of f, similar symmetric arguments work for H_3, H_2, H_1.)
Agents 1 and 2 are not in conflict. Hence, by *strategy-proofness* of TTC, agent 2 does not have a profitable deviation at R.

Next, we verify that agent 1 does not have a profitable deviation at R.

Case 3.a. Preferences of agent 2 are $R_2 : H_1, \ldots$.
One immediately verifies that $f_1(R) = H_2$, which is her most preferred object. So, agent 1 does not have a profitable deviation at R.

Case 3.b. Preferences of agent 2 are $R_2 : H_2, \ldots$ and preferences of agent 3 are $R_3 : H_1, \ldots$ or $R_3 : H_2, H_1, H_3$.
Then, for any possible deviation R'_1, $f_1(R'_1, R_2, R_3) = H_3 = f_1(R)$. Hence, agent 1 does not have a profitable deviation at R.

Case 3.c. Preferences of agent 2 are $R_2 : H_2, \ldots$ and preferences of agent 3 are $R_3 : H_3, \ldots$ or $R_3 : H_2, H_3, H_1$;

or

Case 3.d. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_3, \ldots$ or $R_3 : H_2, H_3, H_1$.
In cases 3.c and 3.d, we have that for any possible deviation R'_1, $f_1(R'_1, R_2, R_3) = H_1 = f_1(R)$. Hence, agent 1 does not have a profitable deviation at R.

Case 3.e. Preferences of agent 2 are $R_2 : H_3, \ldots$ and preferences of agent 3 are $R_3 : H_1, \ldots$;

or

Case 3.f. Preferences of agent 2 are $R_2 : H_3, H_2, H_1$ and preferences of agent 3 are $R_3 : H_2, H_1, H_3$;

or

Case 3.g. Preferences of agent 2 are $R_2 : H_3, H_1, H_2$ and preferences of agent 3 are $R_3 : H_2, \ldots$.
In cases 3.e, 3.f, and 3.g, $f_1(R) = H_2$, i.e., agent 1 receives her most preferred object H_2. So, agent 1 does not have a profitable deviation at R.

Finally, we verify that agent 3 does not have a profitable deviation at R. Cases 3.I, 3.II, and 3.III below are as 2.I, 2.II, and 2.III. There is a small difference between cases 2.IV and 3.IV.

Case 3.I. Preferences of agent 3 are $R_3 : H_3, \ldots$.
By *individual rationality* of f, $f_3(R) = H_3$ and since this is her most preferred object, agent 3 cannot gain by misreporting her preferences.

Case 3.II. Preferences of agent 3 are $R_3 : H_1, \ldots$.
Agents 1 and 3 are not in conflict and by *strategy-proofness* of TTC, agent 3 does not have a profitable deviation at R.

Case 3.III. Preferences of agent 3 are $R_3 : H_2, H_3, H_1$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_3$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, if $R_2 : H_1, \ldots$ or $R_2 : H_2, \ldots$, then $f_3(R_1, R_2, R'_3) = H_3$; and if $R_2 : H_3, \ldots$, then $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R.

Case 3.IV.i. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_1, \ldots$.
Agents 1 and 3 are in conflict and for any possible deviation R'_3, $f_3(R_1, R_2, R'_3) = H_3 = f_3(R)$. Hence, agent 3 does not have a profitable deviation at R.

Case 3.IV.ii. Preferences of agent 3 are $R_3 : H_2, H_1, H_3$ and preferences of agent 2 are $R_2 : H_2, \ldots$ or $R_2 : H_3, \ldots$.
Agents 1 and 3 are in conflict and one easily verifies that $f_3(R) = H_1$. Any possible profitable misreport of preferences by agent 3 requires that H_2 is acceptable and appears in second position. Hence, the only possible candidate for a profitable deviation is $R'_3 : H_1, H_2, H_3$. However, $f_3(R_1, R_2, R'_3) = H_1$. So, agent 3 does not have a profitable deviation at R. □

A.4 Proof of Theorem 2

We now extend the characterization of the *cTTC* mechanism by *unanimity* (*ontoness*, respectively), *individual rationality*, *strategy-proofness*, and *non-bossiness* (Theorem 1) from the domain of lexicographic preferences to the domain of separable preferences.

Proof. Using the same arguments as in the proof of Theorem 1 for $\mathcal{R}_l^N$, it follows that *cTTC* is *unanimous* (and hence *onto*), *individually rational*, *strategy-proof*, and *non-bossy* on $\mathcal{R}_s^N$ as well.

Next, suppose that mechanism $f : \mathcal{R}_l^N \to X$ satisfies the properties listed in Theorem 2 (by Lemma 1, *ontoness* and *unanimity* can be used interchangeably). We will show that for each $R \in \mathcal{R}_s^N$, $f(R) = cTTC(R)$. We introduce the following notation. For any two separable preferences $R_i, \bar{R}_i \in \mathcal{R}_s$, we write $R_i \sim \bar{R}_i$ if they induce the same marginal preferences, i.e., for each $t \in T$, $R_i^t = \bar{R}_i^t$.

Let $R \in \mathcal{R}_s^N$ such that each agent has lexicographic preferences, i.e., $R \in \mathcal{R}_l^N$. Then it immediately follows from Theorem 1 that $f(R) = cTTC(R)$.

Let $R \in \mathcal{R}_s^N$ such that only one agent does not have lexicographic preferences. We can assume, without loss of generality, that $R_1 \in \mathcal{R}_s \setminus \mathcal{R}_l$ and for each agent $j \neq 1$, $R_j \in \mathcal{R}_l$. Let $y \equiv f(R)$.

For each $t \in T$, let $R_1'(t) \in \mathcal{R}_l$ such that $R_1'(t) \sim R_1$ and the most important type of $R_1'(t)$ is type t. Since $R_1 \sim R_1'(1) \sim R_1'(2) \sim \cdots \sim R_1'(m)$, it follows from the definition of $cTTC$ that $x \equiv cTTC(R) = cTTC(R_1'(1), R_{-1}) = cTTC(R_1'(2), R_{-1}) = \cdots = cTTC(R_1'(m), R_{-1})$. We will show that $y = x$.

Let $t \in T$. It follows from Theorem 1 that $f(R_1'(t), R_{-1}) = cTTC(R_1'(t), R_{-1}) = x$. By *strategy-proofness* of f when moving from $(R_1'(t), R_{-1})$ to (R_1, R_{-1}), $x_1 = f_1(R_1'(t), R_{-1}) \; R_1'(t) \; f_1(R_1, R_{-1}) = y_1$. Then, since $R_1'(t) \sim R_1$ and $R_1'(t)$ is a lexicographic preference relation where t is the most important type, $x_1^t \; R_1^t \; y_1^t$.

Since for each $t \in T$, $x_1^t \, R_1^t \, y_1^t$ and since $R_1 \in \mathcal{R}_s$, we have $x_1 \, R_1 \, y_1$. By *strategy-proofness* of f when moving from (R_1, R_{-1}) to $(R_1'(t), R_{-1})$, we have that $y_1 = f_1(R_1, R_{-1}) \, R_1 \, f_1(R_1'(t), R_{-1}) = x_1$. Hence, $x_1 = y_1$. By *non-bossiness* of f, we have that $y = f(R_1, R_{-1}) = f(R_1'(t), R_{-1}) = x$.

Let $R \in \mathcal{R}_s^N$ such that exactly two agents do not have lexicographic preferences. We can assume, without loss of generality, that $R_1, R_2 \in \mathcal{R}_s \setminus \mathcal{R}_l$ and for each agent $j \neq 1, 2$, $R_j \in \mathcal{R}_l$. Let $y \equiv f(R)$.

For each $t \in T$, let $R_2'(t) \in \mathcal{R}_l$ such that $R_2'(t) \sim R_2$ and the most important type of $R_2'(t)$ is type t. Since $R_2 \sim R_2'(1) \sim R_2'(2) \sim \cdots \sim R_2'(m)$, it follows from the definition of $cTTC$ that $x \equiv cTTC(R) = cTTC(R_2'(1), R_{-2}) = cTTC(R_2'(2), R_{-2}) = \cdots = cTTC(R_2'(m), R_{-2})$. We will show that $y = x$.

Let $t \in T$. At preference profile $(R_2'(t), R_{-2})$, only agent 1 has non-lexicographic preferences. Thus, from the previous case, $f(R_2'(t), R_{-2}) = cTTC(R_2'(t), R_{-2}) = cTTC(R) = x$. By *strategy-proofness* of f when moving from $(R_2'(t), R_{-2})$ to (R_2, R_{-2}), we have that $x_2 = f_2(R_2'(t), R_{-2}) \; R_2'(t) \; f_2(R_2, R_{-2}) = y_2$. Then, since $R_2'(t) \sim R_2$ and $R_2'(t)$ is a lexicographic preference relation where t is the most important type, $x_2^t \; R_2^t \; y_2^t$.

Since for each $t \in T$, $x_2^t \, R_2^t \, y_2^t$ and since $R_2 \in \mathcal{R}_s$, we have $x_2 \, R_2 \, y_2$. By *strategy-proofness* of f when moving from (R_2, R_{-2}) to $(R_2'(t), R_{-2})$, $y_2 = f_2(R_2, R_{-2}) \, R_2 \, f_2(R_2'(t), R_{-2}) = x_2$. Hence, $x_2 = y_2$. By *non-bossiness* of f, we have that $y = f(R_2, R_{-2}) = f(R_2'(t), R_{-2}) = x$.

We can apply repeatedly the same arguments to obtain that for each $k = 0, 1, \ldots, n$ and for each preference profile $R \in \mathcal{R}_s^N$ where exactly k agents have non-lexicographic preferences, $f(R) = cTTC(R)$. Thus, for each $R \in \mathcal{R}_s^N$, $f(R) = cTTC(R)$. □

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