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Time and Dynamics
in General Relativity

Physics and Philosophy of Time
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1 Foreshadowing

NEWTON: "we do not ascribe various durations to the different parts of space, but say that all endure simultaneously. The moment of duration is the same at Rome and at London, on earth and on the stars, and throughout the heavens ... as we understand any moment of duration to be diffused throughout all spaces, according to its kind, without any concept of its parts ...." [*De Gravitatione*]

NEWTON: "Absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external ...." [*Principia*, Scholium to the Definitions]

GEOMETRY & DYNAMICS: In the first quote, Newton tells us that simultaneity is absolute. I (mis)read the second as telling us (in part) that time is homogeneous. *Both* tell us about *both* the geometry of spacetime and about the laws of nature. We'll see that these two aspects can come apart in general relativity.

2 Time(s) in General Relativistic Cosmologies

Minkowski famously proclaimed: "Henceforth space by itself and time by itself are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality." Since Einstein arrived at general relativity by reflecting on how the physical principles underlying the special theory would need to be adapted in order to take account of gravity, it would be natural to guess that Minkowski's proclamation is at least as true of general relativity as of its parent theory. But the situation is not so straightforward—as can be seen from the comments of early cosmologists.

EDDINGTON ON THE EINSTEIN STATIC UNIVERSE (1920): "... we have already urged that the relativity theory is not concerned to deny the possibility of an absolute time, but to deny that that it is concerned in any experimental knowledge yet found; and it need not perturb us if the conception of absolute time turns up in a new form in a theory of phenomena on a cosmical scale, as to which no experimental knowledge is yet available. Just as each limited observer has his own particular separation of space and time, so a being coextensive with the world might well have a special separation of space and time natural to him." [*Space, Time, and Gravitation* 163]

JEANS ON THE FRW MODELS (1936): "Now, the second property [in addition to expansion] which all the mathematical solutions have in common is that every one of them makes a real distinction between space and time. This gives us every justification for reverting to our old intuitional belief that past, present, and future have real objective meanings, and are not mere hallucinations of our individual minds—in brief we are free to believe that time is real. ... we find a distinction between time and space, as soon as we abandon local physics and call the astronomy of the universe to our aid." ["Man and Universe" 23 f.]

Dust Time: The models E&J had in mind were dust cosmologies—in which the matter content is given by a pressure-free perfect fluid—the worldlines of which constitute a timelike geodesic congruence ($\sim$ a freely falling frame of reference, not typically rigid). In Minkowski spacetime, fixing such a congruence suffices to determine a foliation of spacetime by instants of time, via the Einstein simultaneity convention.[1] The same thing happens in a wide array of physically relevant dust cosmologies.[2]

Time from Symmetry: The models that E&J consider are also highly symmetric. All of these models are spatially homogeneous: the group of spacetime symmetries determines a splitting of spacetime into instants of time (and each point on each instant is the same as any other on that instant). In expanding FRW models, these instants correspond to the level sets of matter density. Many well-known models (with all sorts of matter content) have this feature.[3]

Cosmological Time: Recall that in special relativity, timelike geodesics maximize length—the freely falling twin is always older. Essentially the same thing is true in general relativity. So in worlds that, intuitively speaking, have a beginning (typical FRW models—but not the Einstein static universe) we can assign each event a date by looking for the supremum of the lengths of the past-directed timelike curves that emanating from that event. In many well-behaved cosmological spacetimes, the level sets of cosmological time form instants of time.[4]

Examples could be multiplied. But these suffice to raise the following.

— Dust time: note that the spacetime geometry itself knows about the privileged freely falling congruence.[5] As Gödel puts it: in dust cosmology "the world itself has certain distinguished directions, which directly define certain distinguished local times."[6]

— Our three times have over-lapping domains of application. All three are defined and agree in the standard FRW case. Examples in which the first and second both exist but disagree are well known (and called *tilted homogeneous cosmologies*). It is easy to concoct examples in which dust time and cosmological time both exist but disagree.

— Presentism is the doctrine that only the present (or only present things) exist. It is sometimes said that the apparent incompatibility between relativity and presentism is ameliorated in general relativity.[7]

[1] Here and below, unless otherwise noted, *instant of time* means something like *Cauchy surface*.

[2] Sufficent condition for this to work: spacetime is simply connected and the congruence is non-rotating. Necessary condition for this to work: the congruence is non-rotating. See, e.g., Sachs & Wu, *General Relativity for Mathematicans* or O'Neill, *Semi-Riemannian Geometry* on proper time synchronizability.

[3] See the bestiary, Stephani *et al.*, *Exact Solutions of Einstein's Field Equations.*

[4] Let (M, g) be a spacetime in which each event x is assigned a cosmological time $\tau(x) < \infty$ and in which $\tau \to 0$ along each past inextendible causal curve. Then (M, g) is globally hyperbolic and the level sets of τ are (future) Cauchy surfaces and for each x, there is some past-directed geodesic from x whose length is $\tau(x)$. See Andersson *et al.* "The Cosmological Time Function."

[5] See, e.g., Sachs & Wu, *General Relativity for Mathematicans,* §3.14.

[6] "A Remark about the Relationship between Relativity Theory and Idealistic Philosophy," fn. 6.

[7] See, e.g., Saunders, "How Relativity Contradicts Presentism" and Zimmeman, "Presentism and the Space-Time Manifold." For non-presentist theories that take attempt to take tensed facts seriously while respecting the spirit special relativity, see Fine, "Tense and Reality" and Skow, "Relativity and the Moving Spotlight."

3 Interlude: Gödel Strikes Back

(i) "One of the most interesting aspects of relativity theory for the philosophical-minded consists in the fact t that it gave new and surprising insights into the nature of time, of that mysterious and seemingly self-contradictory being which, on the other hand seems to form the very basis of the world's and our own existence. The very starting point of special relativity theory consists in the discovery of a new and very astonishing property of time, namely the relativity of simultaneity . . . ."

(ii) "There exist cosmological solutions of another kind than those known at present, to which the aforementioned procedure of defining an absolute time is not applicable, because the local times of the special observers used above cannot be fitted together into one world time. Nor can any other procedure which would accomplish this purpose exist for them; i.e., those worlds possess such properties of symmetry, that for any possible concept of simultaneity and succession there exist others that cannot be distinguished from it by any intrinsic properties, only by reference to individual objects, such as, e.g., a particular galactic system."

(iii) "Of what use is it if such conditions prevail in certain *possible* worlds? Does it mean anything for the question interesting us whether in *our* world there exists an objective lapse of time? I think it does. . . . The mere compatibility with the laws of nature of worlds in which there is no distinguished absolute time, and, therefore, no objective lapse of time can exist, throws some light on the meaning of time also in those worlds in which an absolute time *can* be defined. For if someone asserts that this absolute time is lapsing, he accepts as a consequence that, whether or not an objective lapse of time exists . . . depends on the particular way in which matter and its motion are arranged in the world. This is not a straightforward contradiction; nevertheless, a philosophical view leading to such consequences can hardly be considered as satisfactory."[8]

REGARDING (ii). In order to show that a partition of Minkowski spacetime by instants fails to be invariant under the full group of spacetime symmetries, act on it by a boost—each instant in the given partition will be mapped to a distinct instant that intersects it—and so the partition is not invariant under boosts. Gödel observes that a variant on this trick will work in any rotating dust solution with the feature that each point is fixed by a one-dimensional family of rotations—and he constructs a famous solution with this feature (here candidate-instants are hypersurfaces transverse to the dust worldlines).

REGARDING (iii).[9]

∗ On the one hand: the considerations offered here strike many as being disappointingly weak. After all, one accepts that in general relativity, structural features of time (such as whether or not it is infinitely extended) depend on the arrangement of matter and its motion—why should absolute simultaneity, or even objective becoming, be different?[10]

∗ On the other hand—consider a question like whether there is a preferred parity in nature. This could be taken to be a straightforward question about the distribution of matter and motion.[11] But more likely, it is to be understood as a question about the structure of the laws.

[8] "A Remark about the Relationship between Relativity Theory and Idealistic Philosophy," 557, 560, 561 f.

[9] For further discussion and references, see Belot, "Dust, Time, and Symmetry."

[10] For this objection, see Earman, *Bangs, Crunches, Whimpers, and Shrieks,* 198.

[11] In which case we might want to heed the following: "In the case of human beings, the hair on the crown of the head grows in a spiral from the left to the right. All hops wind around their poles from left to right, whereas beans wind in the opposite direction. Almost all snails, with the exception of perhaps, only three species, have shells which, when viewed from above, that is to say, when their curvature is traced from the apex to the embouchure, coil from left to right"; and, furthermore, "all the peoples of the world are right-handed (apart from a few exceptions which, like that of squinting, do not upset the universality of the regular natural order)" [Kant, "Concerning the Ultimate Grounds of the Differentiation of Directions in Space"].

- $*$ But—on *another* other hand—the considerations offered in (iii) *still* seem disappointingly weak: if, e.g., we wanted to know whether the dynamics of classical mechanics single out a preferred temporal origin, we would not be satisfied with the observation that there exist time-translation-invariant solutions—we would, rather, want to know something about the invariance of the differential equation or the Hamiltonian of the theory under time-translation.

4 Time-Dependence and Time-Independence in General Relativity

Suppose we ask whether the dynamics of general relativity is a time-dependent or time-independent. What would this mean?

In the more straightforward setting of classical mechanics, the dynamics of a system is time-independent is to say something like: the equations of motion retain their form under $t \mapsto t + a$ (time translation); the time-translation, $\phi(t + a)$, of a solution, $\phi(t)$, is another solution; energy is a conserved quantity that generates time translation. Roughly speaking: this holds for closed systems, but fails for typical open systems.

Now: general relativity is a generally covariant theory. That is: its equations of motion apply in arbitrary coordinate systems.

> To get a feeling for how this works, suppose we begin with Newton's $F = ma$ where F is the gravitational force. We can use this in any inertial frame—but not in others. But if we throw in pseudo-force terms (Coriolis force, centrifugal force, etc.) we can work in rotating frames. If we are willing to countenance weird enough extra terms, we can work in arbitrary coordinates.

Let's see what say our tests for time-independence in this context. To keep things simple, let's start with a copy $(\mathbb{R}^4, \eta)$ of Minkowski spacetime with fixed inertial coordinates (t, x, y, z) and restrict attention to general relativistic spacetimes $(\mathbb{R}^4, g)$ with this same set of points. So we will in effect have a fixed, dynamcially irrelevant, background Minkowski metric—and in the first instance, when we ask about time translation, we will be asking about the transformation $(t, x, y, z) \mapsto (t + a, x, y, z)$.

(a) The field equations of general relativity are covariant under all coordinate transformations (including $t \mapsto t + a$).

(b) The space of solutions is likewise closed under all transformations of the pointset of spacetime manifold (including $t \mapsto t + a$).

(c) In its standard Hamiltonian formulation, the Hamiltonian of the theory is identically zero—and so is invariant under any transformation you like.[12]

(d) The Lagrangian of the theory is invariant under $t \mapsto t+a$. So we can use Noether's theorem to construct a conserved quantity associated with this symmetry. But: the result depends on the background metric—so this is unphysical.[13]

Victory? No. Each of the above points is pretty nearly a direct consequence of the general covariance of general relativity. But just about any theory can be given a generally covariant formulation.[14] In particular,

[12]For discussion and references, see Belot and Earman, "Presocratic Quantum Gravity." For a rousing donnybrook over the import of the vanishing of the Hamiltonian, see Earman, "Thoroughly Modern McTaggart" and Maudlin, "Thoroughly Muddled McTaggart." For a frustratingly irenic approach, see Belot, "The Representation of Time and Change in Mechanics."

[13]See Christodoulou, *Mathematical Problems of General Relativity. I,* 48–54. NB: the Langrangian used here is nonstandard but gives the usual equations of motion.

[14]For techniques, see, e.g., Lee and Wald, "Local Symmetries and Constraints"; Sorkin, "An Example Relevant to the Kretschmann–Einstein Debate"; and Torre, Covariant Phase Space Reformulation of Parameterized Field Theory." For discussion, see Earman, "Two Challenges for Substantive General Covariance."

we could start out with a patently time-dependent theory, give it a generally covariant reformulation, and end up with the same sort of reasons to think that it is generally covariant that we have here for general relativity.

Let us start again. Rather than asking whether general relativity as a whole is time-independent, let us proceed by singling out sectors of the theory that are time-independent in one or another sense (note that in a sense, this is how we have to proceed in classical mechanics as well).

Stationary Spacetimes. In stationary spacetime, there is a family of observers, each of whom reports that there the geometry in their immediate neighbourhood is unchanging.[15] It seems natural to say that we have time-indepdencne of (gravitational) physics in the stationary sector of general relativity.

Asymptotically Flat Spacetimes. If (M, g) is a relativistic spacetime and Σ is a Cauchy surface (a well-behaved instant of time in the given spacetime), then Σ inherits some geometric structure from its embedding in (M, g).Let $\mathcal{F}$ be a family of freely falling observers who agree that Σ is a simultaneity slice (so $\mathcal{F}$ is a timelike geodesic congruence orthogonal to Σ).

(a) g induces a Riemannian metric h on Σ—this encodes the spatial relations between the members of $\mathcal{F}$ as they pass through Σ.

(b) g also induces on Σ a (two-index covariant) tensor k, the *extrinsic curvature*—this is $\frac{1}{2}\dot{h}$, as calculated by the members of $\mathcal{F}$ (each using their own proper time).

(c) g induces on Σ a real-valued function K, the *mean extrinsic curvature*—this is the rate at which the members of $\mathcal{F}$ see space expanding/contracting as they pass through Σ. (K is the trace of k.)

Now: suppose we let $\Sigma = \mathbb{R}^3$ and let h and k look pretty similar to what we would get if Σ were an instant of time in Minkowski spacetime—with any dissimilarities going to zero as we go out to infinity along Σ. Then a monumental theorem of Christidoulou and Klainerman concerning the nonlinear stability of Minkowski spacetime tells us that there is a unique spacetime geometry with an instant of time that looks like (Σ, h, k)—and that this spacetime geometry is asymptotic to Minkowski spacetime, as you go to infinity along any geodesic.

We call any (Σ, h, k) of the form just described an *asymptotically flat initial data set* and call the (M, g) with instants of this form *asymptotically flat spacetimes.* Then we have the following facts.

- Let (M, g) be asymptotically flat. Embed it in our background copy of Minkowski spacetime $(\mathbb{R}^4, \eta)$ with $g - \eta \to 0$ suitably quickly as we go out to spatial infinity. We now find that the conserved quantity E associated with the symmetry $t \to t + a$ is geometric—it is independent of the details of the embedding.[16]

- This quantity can be thought of as the generator of time translation in this setting, without reference to our hokey nondynamical background metric η. Start with the space of all asymptotically flat spacetimes. Quotient this space by the set of diffeormorphisms asymptotic to the identity at spatial infinity. The resulting space has a symplectic structure. So the energy E gives us a dynamical flow—which corresponds to implementing a time translation at spatial infinity.[17]

[15] There are competing conventions in the literature; I follow those—like Wald, *General Relativity*—who define stationary spacetimes to be those featuring timelike Killing fields.

[16] Using Cartesian coordinates on Σ, find, for each $r > 0$ the integral over the sphere or radius r of the quantity $\Sigma_{i,j}\partial_i h_{ij} - \partial_j h_{ij}$. See Christodoulou, *Mathematical Problems of General Relativity. I,* 35 f. and 53 f.

[17] For a detailed version of this story, see, e.g., Ashtekar *et al.* "The Covariant Phase Space of Asymptotically Flat Gravitational Fields."

* Just like we can study time translation relative to η, we can also study spatial translations relative to η. This gives us, for any instant in an asymptotically flat spacetime, a conserved quantity P, the momentum.[18]

* Modulo some technicalities: Minkowski spacetime aside, each asymptotically flat spacetime admits a unique decomposition into asymptotically flat instants, subject to the conditions that $K \equiv 0$ and $P \equiv 0$.[19] Further: there is a preferred prameterization (corresponding to the condition that the lapse be asymptotic to one at spatial infinity). So we can think of asymptotically flat spacetimes as curves in the space of (diffeomorphism equivalence classes of) asymptotically flat initial data surfaces that arise in this way—with E corresponding to the freedom to select an arbitrary origin for the parameterization of time.[20]

NB: typical asymptotically flat spacetimes have no symmetries. So we the time translation symmetry we are talking about here is a symmetry of the dynamics, rather than of the individual geometries under consideration.

The Spatially Compact Case—Very Roughly Speaking. Consider now the case of spatially compact globally hyperbolic spacetimes. In a wide range of cases, it is possible to foliate such spacetimes in a unique way by instants of time on which K is constant. So each spacetime can be thought of as a curve in the space of (diffeomorphism equivalence classes of) constant mean extrinsic curvature initial data. This space carries a symplectic form. And there is a Hamiltonian that generates the family of curves that we want! But: when written in terms of the mean curvature time, this Hamiltonian is time-dependent (except in certain special cases).[21]

So this looks like it may be a sector of general relativity which is not time-translation invariant. Unless there turns out to be a better time and a better Hamiltonian . . . .[22]

5 Queries

1) What exactly is required in order for general relativity to provide a more hospitable environment for presentism than does special relativity? How are presentists to choose between the various competing times on offer? Does being very ‘global’ count against candidate?

2) Questions about the nature of time in general relativity often depend on how precisely we draw the borders around the theory (e.g., whether we allow in solutions with closed time like curves). Is there a sharp, objective distinction in the offing here?

3) In interpreting a physical theory, we often adopt the fiction that it is true. But we know that general relativity has its limitations. Do the preceding queries take on a different hue if we attend to this?

4) How should we understand asymptotic symmetries? Physicists tend to distinguish them from gauge symmetries, holding that the latter but not the former generate physically distinct states.

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[18]Substitute into the definition of E the integrand $2(k_{ij} - h_{ij}K)$. See Christodoulou, *Mathematical Problems of General Relativity. I,* 35 f. and 53 f.

[19]See Christodoulou, *Mathematical Problems of General Relativity. I,* 58 f.

[20]Or, rather, that this is true at the heuristic level follows from the above.

[21]For references, see Belot, “Time and Change in Mechanics,” §7.

[22]It is known that the mean extrinsic curvature time and the cosmological time discussed above part ways in some cases in which they are both defined. See Andersson *et al.* “Cosmological Time Versus CMC Time in Spacetimes of Constant Curvature.”